Problem 10.1: One-loop structure of QED

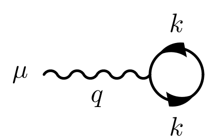
In Section 10.1 we argued form general principles that the photon one-point and three-point functions vanish, while the four-point function is finite.

(a) Verify directly that the one-loop diagram contributing to the one-point function vanishes. There are two Feynman diagrams contributions to the three-point function at one-loop order. Show that these cancel. Show that the diagrams contributing to any n-point photon amplitude, for n odd, cancel in pairs.

(b) The photon four-point amplitude is a sum of six diagrams. Show explicitly that the potential logarithmic divergences of these diagrams cancel.

10.1 part (a)

The one-point function is given by the diagram

$$i\Pi^{\mu}_{\text{1-pt}}(q^2) = $$ 

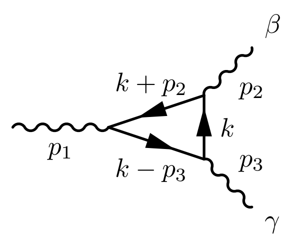
(1)

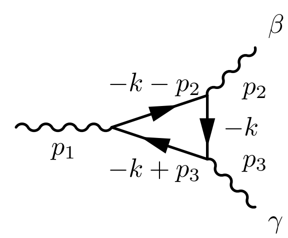
Applying the Feynman rules we have

$$i\Pi^{\mu}_{\text{1-pt}}(q^2) = \text{Tr}\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4}(-ie)\gamma^{\mu}\frac{i(\not{k}+m_{\text{e}})}{k^2-m_{\text{e}}^2+i\epsilon}\right] = 0. \tag{2}$$

The trace of $m_{\text{e}}\gamma^{\mu}$ and the integral $\int \mathrm{d}^4k\frac{k^{\nu}}{f(k^2)}$ vanish giving the result (2).

At one-loop, the three-point function is

$$i\Pi^{\alpha\beta\gamma}_{\text{3-pt}} = $$ 

$+$ 

(3)

where the momentum flow in the loop is counter-clockwise and the momentum of the external photons is directed inward towards the loop. The first graph in equation (4) contains electrons while the second graph contains positrons. In the second graph we use minus the momentum of the first graph so that k is the same for both graphs – this is due to the fact that the momentum flow is in the same direction as in the first diagrams and that the trace is taken against the fermion flow, which is reversed relative to the first diagram.

Applying the QED Feynman rules we find

$$\begin{aligned} i\Pi^{\alpha\beta\gamma}_{\text{3-pt}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} & \frac{(-ie)^3 i^3}{((k+p_2)^2-m_{\text{e}}^2)\,(k^2-m_{\text{e}}^2)\,((k-p_3)^2-m_{\text{e}}^2)} \\ & \times \text{Tr}\left[\gamma^{\alpha}\left(\not{k}+\not{p}_2+m_{\text{e}}\right)\gamma^{\beta}\left(\not{k}+m_{\text{e}}\right)\gamma^{\gamma}\left(\not{k}-\not{p}_3+m_{\text{e}}\right)\right. \\ & \quad \left.-\gamma^{\alpha}\left(\not{k}-\not{p}_3-m_{\text{e}}\right)\gamma^{\gamma}\left(\not{k}-m_{\text{e}}\right)\gamma^{\beta}\left(\not{k}+\not{p}_2-m_{\text{e}}\right)\right]. \end{aligned} \tag{4}$$

The first term in the trace of (4) is

$$\mathrm{Tr}\left[\gamma^\alpha(\not{k}+\not{p}_2)\gamma^\beta\not{k}\gamma^\gamma(\not{k}-\not{p}_3)\right]+m_\mathrm{e}^2\mathrm{Tr}\left[\gamma^\alpha(\not{k}+\not{p}_2)\gamma^\beta\gamma^\gamma+\gamma^\alpha\gamma^\beta\not{k}\gamma^\gamma+\gamma^\alpha\gamma^\beta\gamma^\gamma(\not{k}-\not{p}_3)\right] \tag{5}$$

while the second term is

$$\begin{aligned}
&-\mathrm{Tr}\left[\gamma^\alpha(\not{k}-\not{p}_3)\gamma^\gamma\not{k}\gamma^\beta(\not{k}+\not{p}_2)\right]-m_\mathrm{e}^2\mathrm{Tr}\left[\gamma^\alpha(\not{k}-\not{p}_3)\gamma^\gamma\gamma^\beta+\gamma^\alpha\gamma^\gamma\not{k}\gamma^\beta+\gamma^\alpha\gamma^\gamma\gamma^\beta(\not{k}+\not{p}_2)\right]\\
&=-\mathrm{Tr}\left[(\not{k}+\not{p}_2)\gamma^\beta\not{k}\gamma^\gamma(\not{k}-\not{p}_3)\gamma^\alpha\right]-m_\mathrm{e}^2\mathrm{Tr}\left[\gamma^\beta\gamma^\gamma(\not{k}-\not{p}_3)\gamma^\alpha+\gamma^\beta\not{k}\gamma^\gamma\gamma^\alpha+(\not{k}+\not{p}_2)\gamma^\beta\gamma^\gamma\gamma^\alpha\right]\\
&=-\mathrm{Tr}\left[\gamma^\alpha(\not{k}+\not{p}_2)\gamma^\beta\not{k}\gamma^\gamma(\not{k}-\not{p}_3)\right]-m_\mathrm{e}^2\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\gamma(\not{k}-\not{p}_3)+\gamma^\alpha\gamma^\beta\not{k}\gamma^\gamma+\gamma^\alpha(\not{k}+\not{p}_2)\gamma^\beta\gamma^\gamma\right].
\end{aligned} \tag{6}$$

Since (6) is the exact negative of (5), the three-point function (4) vanishes. Note that in the second line of (6) we have used the fact that the trace of a product of gamma matrices is the same as the trace of the reversed product: Recall that the charge conjugation matrix, $C=\gamma^0\gamma^2$, satisfies

$$C^2=1 \quad \text{and} \quad C\gamma^\mu C=-(\gamma^\mu)^T. \tag{7}$$

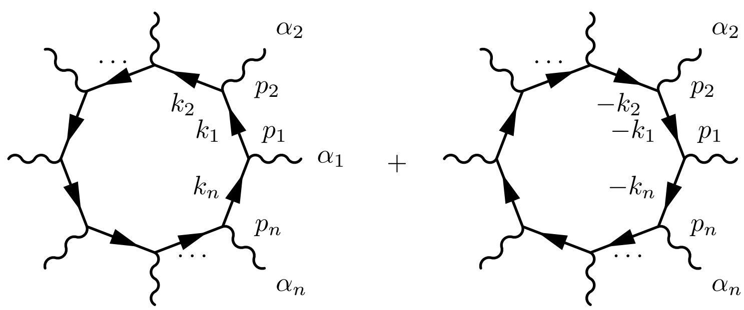
Therefore,

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^{\alpha_1}\gamma^{\alpha_2}\ldots\gamma^{\alpha_n}\right]&=\mathrm{Tr}\left[CC\gamma^{\alpha_1}CC\gamma^{\alpha_2}CC\ldots CC\gamma^{\alpha_n}CC\right]\\
&=-(1)^n\mathrm{Tr}\left[C(\gamma^{\alpha_1})^T(\gamma^{\alpha_2})^T\ldots(\gamma^{\alpha_n})^TC\right]\\
&=-(1)^n\mathrm{Tr}\left[(\gamma^{\alpha_n}\ldots\gamma^{\alpha_2}\gamma^{\alpha_1})^T\right]\\
&=-(1)^n\mathrm{Tr}\left[\gamma^{\alpha_n}\ldots\gamma^{\alpha_2}\gamma^{\alpha_1}\right]\\
&=\mathrm{Tr}\left[\gamma^{\alpha_n}\ldots\gamma^{\alpha_2}\gamma^{\alpha_1}\right]
\end{aligned} \tag{8}$$

where in the last line we have used the fact that n must be even to get a non-zero trace.

The n-point function for n odd is given by

$$i\Pi^{\alpha_1\ldots\alpha_n}_{n\text{-pt}} =$$



$+\ldots \quad (9)$

where $k_n\equiv k$ and $k_i=k+\sum_{l=1}^{i}p_l$ for $1\leq i\leq n-1$. Applying the Feynman rules we obtain

$$\begin{aligned}
i\Pi^{\alpha_1\ldots\alpha_n}_{n\text{-pt}}=\int\frac{\mathrm{d}^4k}{(2\pi)^4}&\frac{(-ie)^ni^n}{(k_1^2-m_\mathrm{e}^2)\ldots(k_n^2-m_\mathrm{e}^2)}\\
&\times\mathrm{Tr}\left[(\not{k}_n+m_\mathrm{e})\gamma^{\alpha_n}\ldots(\not{k}_1+m_\mathrm{e})\gamma^{\alpha_1}+\gamma^{\alpha_1}(-\not{k}_1+m_\mathrm{e})\ldots\gamma^{\alpha_n}(-\not{k}_n+m_\mathrm{e})\right]+\ldots.
\end{aligned} \tag{10}$$

We can show, for odd n, that the trace from the second diagram of (9) is the negative of the trace from the first diagram in (9):

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^{\alpha_1}(-\not{k}_1+m_\mathrm{e})\ldots\gamma^{\alpha_n}(-\not{k}_n+m_\mathrm{e})\right]&=\mathrm{Tr}\left[CC\gamma^{\alpha_1}CC(-\not{k}_1+m_\mathrm{e})CC\ldots CC\gamma^{\alpha_n}CC(-\not{k}_n+m_\mathrm{e})CC\right]\\
&=\mathrm{Tr}\left[C(-\gamma^{\alpha_1})^T(\not{k}_1^T+m_\mathrm{e})\ldots(-\gamma^{\alpha_n})^T(\not{k}_n^T+m_\mathrm{e})C\right]\\
&=(-1)^n\mathrm{Tr}\left[\left((\not{k}_n+m_\mathrm{e})\gamma^{\alpha_n}\ldots(\not{k}_1+m_\mathrm{e})\gamma^{\alpha_1}\right)^T\right]\\
&=-\mathrm{Tr}\left[(\not{k}_n+m_\mathrm{e})\gamma^{\alpha_n}\ldots(\not{k}_1+m_\mathrm{e})\gamma^{\alpha_1}\right].
\end{aligned} \tag{11}$$

Therefore the first two terms in the n-point function vanish for odd n.

There are more diagrams other than those shown explicitly in equation (9). However, each diagram has a pair where the particles in the loop are antiparticles and vice versa. In exactly the same way that the first two diagrams of (9) cancel, each other pair of diagrams cancel. Thus, the n-point function for odd n vanishes.

10.1 part (b)

The diagrams for the 4-pt function are given in Fig. 1. To calculate the divergent part of each diagram we only need to retain the terms in the numerator with highest power of loop momentum, k. Furthermore, the trace of the diagram with particles in the loop is the same as the trace for antiparticles in the loop and therefore add. Therefore, the total amplitude is

$$\mathcal{M} = 2(\mathcal{M}_1 + \mathcal{M}_3 + \mathcal{M}_5). \tag{12}$$

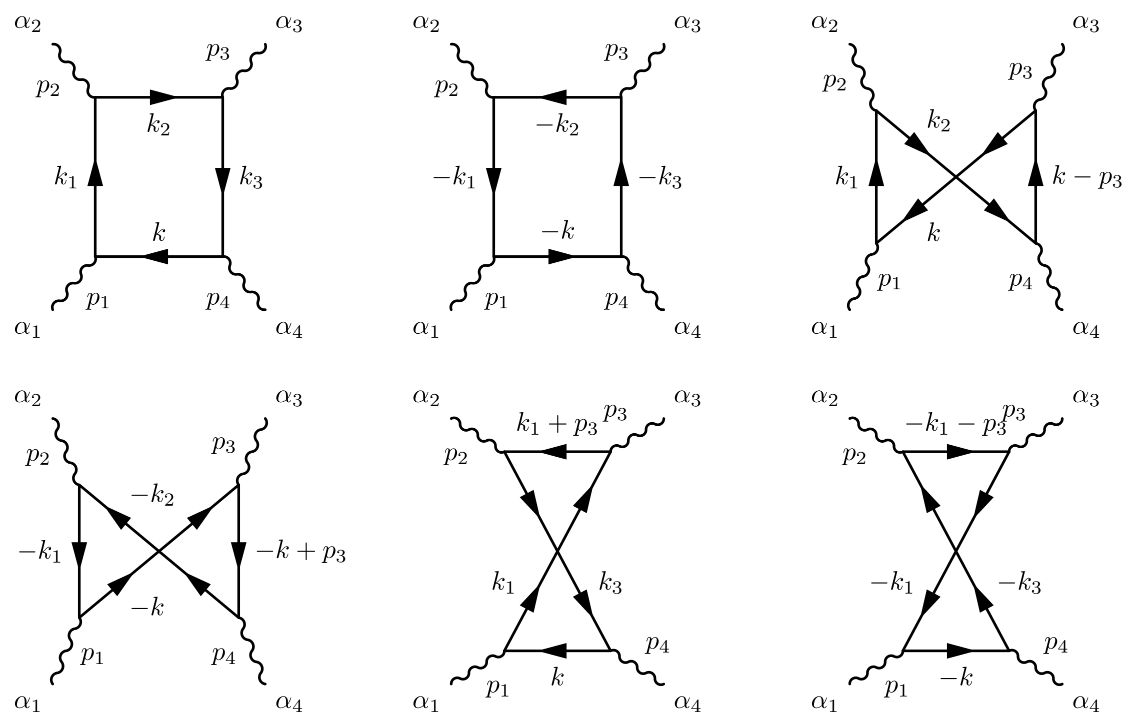


Figure 1: Diagrams contributing to the 4-pt function. To keep track of the diagrams let us label them by $\mathcal{M}_i$ where $i = 1, \ldots, 6$.

We begin by evaluating the divergent part of $\mathcal{M}_1$ by series expanding the denominators and keeping the $\mathcal{O}(k^{-4})$ term

$$\mathcal{M}_1 = e^4 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\not{k}\gamma^{\alpha_4}\not{k}\gamma^{\alpha_3}\not{k}\gamma^{\alpha_2}\not{k}\gamma^{\alpha_1}\right]}{(k^2 - m_e^2)^4} + \text{finite terms}. \tag{13}$$

Note that because we know that there will be no divergence we can work in 4 dimensions to make the Dirac algebra in the trace simpler

$$\begin{aligned}
\mathrm{Tr}\left[\not{k}\gamma^{\alpha_4}\not{k}\gamma^{\alpha_3}\not{k}\gamma^{\alpha_2}\not{k}\gamma^{\alpha_1}\right] &\to \frac{k^4}{24}\left(g_{\beta_1\beta_2}g_{\beta_3\beta_4} + g_{\beta_1\beta_3}g_{\beta_2\beta_4} + g_{\beta_1\beta_4}g_{\beta_2\beta_3}\right)\mathrm{Tr}\left[\gamma^{\beta_4}\gamma^{\alpha_4}\gamma^{\beta_3}\gamma^{\alpha_3}\gamma^{\beta_2}\gamma^{\alpha_2}\gamma^{\beta_1}\gamma^{\alpha_1}\right] \\
&= \frac{k^4}{3}\left(\mathrm{Tr}\left[\gamma^{\alpha_4}\gamma^{\alpha_3}\gamma^{\alpha_2}\gamma^{\alpha_1}\right] - g^{\alpha_4\alpha_2}\mathrm{Tr}\left[\gamma^{\alpha_3}\gamma^{\alpha_1}\right]\right) \\
&= \frac{4k^4}{3}\left(g^{\alpha_4\alpha_3}g^{\alpha_2\alpha_1} - 2g^{\alpha_4\alpha_2}g^{\alpha_3\alpha_1} + g^{\alpha_4\alpha_1}g^{\alpha_3\alpha_2}\right). \qquad (14)
\end{aligned}$$

The trace in $\mathcal{M}_3$ is the same as the trace in $\mathcal{M}_1$ with $\alpha_4 \leftrightarrow \alpha_3$ and the trace in $\mathcal{M}_5$ is the same as the trace in $\mathcal{M}_1$ with $\alpha_3 \leftrightarrow \alpha_2$. Therefore,

$$\begin{aligned}
\mathcal{M} \propto\ & g^{\alpha_4\alpha_3}g^{\alpha_2\alpha_1} - 2g^{\alpha_4\alpha_2}g^{\alpha_3\alpha_1} + g^{\alpha_4\alpha_1}g^{\alpha_3\alpha_2} \\
& + g^{\alpha_4\alpha_3}g^{\alpha_2\alpha_1} - 2g^{\alpha_3\alpha_2}g^{\alpha_4\alpha_1} + g^{\alpha_3\alpha_1}g^{\alpha_4\alpha_2} \\
& + g^{\alpha_4\alpha_2}g^{\alpha_3\alpha_1} - 2g^{\alpha_4\alpha_3}g^{\alpha_2\alpha_1} + g^{\alpha_4\alpha_1}g^{\alpha_2\alpha_3} \\
=\ & 0 \qquad (15)
\end{aligned}$$

Problem 10.2: Renormalization of the Yukawa Lagrangian

Consider the pseudo scalar Yukawa Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\not{\partial} - m_\psi)\psi - ig\bar{\psi}\gamma^5\psi\phi \qquad (16)$$

where ϕ is a real scalar field and ψ is a Dirac fermion. Notice that this Lagrangian is invariant under the parity transformation $\psi(t,\mathbf{x}) \to \gamma^0\psi(t,-\mathbf{x}), \phi(t,\mathbf{x}) \to -\phi(t,-\mathbf{x})$, in which the field ϕ carries odd parity.

(a) Determine the superficially divergence amplitudes and work out the Feynman rules for renormalized perturbation theory for Lagrangian. Include all necessary counter term vertices. Show that the theory contains a superficially divergent ϕ^4. This means that the theory cannot be renormalized unless one includes a scalar self-interaction,

$$\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4, \qquad (17)$$

and a counter term of the same form. It is of course possible to set the renormalized value of this coupling to zero, but that is not a natural choice, since the counter term will still be nonzero. Are there any further interactions required?

(b) Compute the divergent part (the pole as $d \to 4$) of each counter term, to the one-loop order of perturbation theory, implementing a sufficient set of renormalization condition. You need not worry about finite parts of the counter terms. Since the divergent parts must have a fixed dependence on the external momenta, you can simplify this calculation by choosing the momenta in the simplest possible way.

Part (a)

The Feynman rules for the pseudo scalar Yukawa theory are listed in Figs. 2 and 3.

Figure 2: Feynman propagators for the Lagrangian (16).

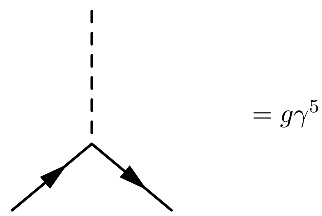


Figure 3: Feynman vertices for the Lagrangian (16).

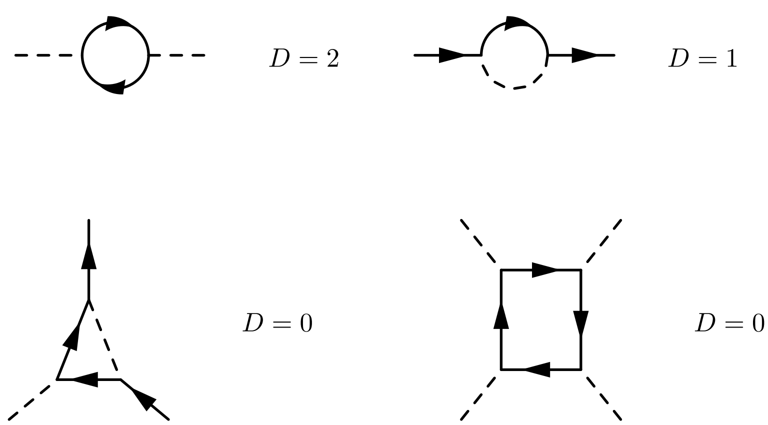


Figure 4: All one-loop diagrams along with their superficial divergence.

Our first task is to determine all the superficially divergent diagrams at one-loop (Fig. 4). Note that since the Lagrangian is invariant under parity, the interaction cannot change the parity of the initial state. This means that scattering amplitudes with an odd number of external pseudo scalars and no external fermions must vanish. Furthermore, since there is a divergent ϕ^4 diagram in Fig. 4, we must include a ϕ^4 interaction,

$$\delta\mathcal{L} = -\frac{\lambda}{4!}\phi^4, \tag{18}$$

and a corresponding counter term to renormalize the theory. The new Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\not{\partial} - m_\psi)\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4. \tag{19}$$

Next, we rescale the fields by the field strength

$$\begin{aligned} \psi &\to \sqrt{Z_\psi}\psi \\ \phi &\to \sqrt{Z_\phi}\phi \end{aligned} \tag{20}$$

to get

$$\mathcal{L} = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - \frac{1}{2}Z_\phi m_\phi^2\phi^2 - \frac{\lambda}{4!}Z_\phi^2\phi^4 + Z_\psi\bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi - igZ_\psi\sqrt{Z_\phi}\bar{\psi}\gamma^5\psi\phi. \tag{21}$$

Defining

$$\begin{aligned} Z_\psi &= 1 + \delta_{Z_\psi} \\ Z_\phi &= 1 + \delta_{Z_\phi} \\ m_\psi^2 Z_\psi &\to m_\psi^2 + \delta_{m_\psi} \\ m_\phi^2 Z_\phi &\to m_\phi^2 + \delta_{m_\phi} \\ Z_\phi^2\lambda &\to \lambda + \delta\lambda, \\ g\sqrt{Z_\phi}Z_\psi &\to g + \delta g. \end{aligned} \tag{22}$$

where m_ψ, m_ϕ, λ and g are now the physical masses and coupling constants, the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \delta\mathcal{L}, \\ \mathcal{L}_0 &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4, \\ \delta\mathcal{L} &= \frac{1}{2}\delta_{Z_\phi}(\partial_\mu\phi)^2 - \frac{1}{2}\delta_{m_\phi}\phi^2 - \frac{1}{4!}\delta_\lambda\phi^4 + \delta_{Z_\psi}\bar{\psi}(i\partial\!\!\!/)\psi - \delta_{m_\psi}\bar{\psi}\psi - i\delta_g\bar{\psi}\gamma^5\psi\phi. \end{aligned} \tag{23}$$

In addition to the Feynman rules of Figs. 2 and 3, the counter term Lagrangian adds the vertices of Fig. 5.

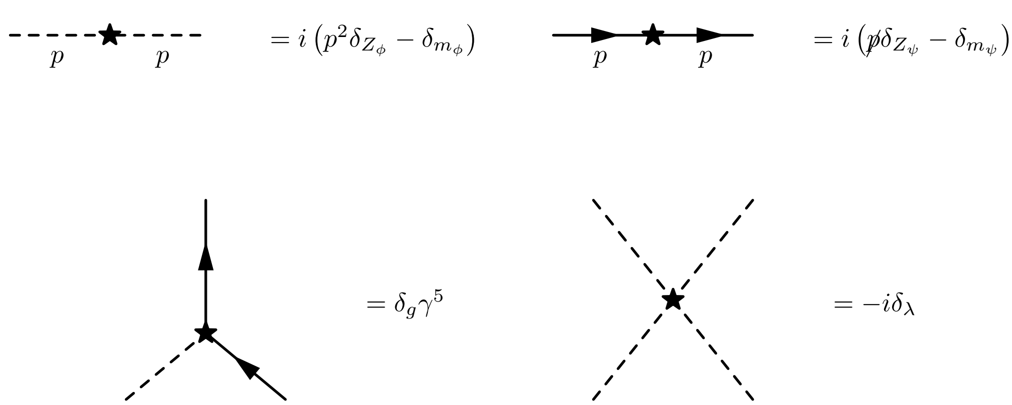
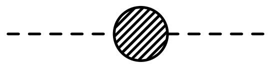


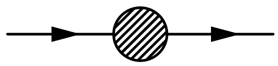
Figure 5: Counter term vertices.

Part (b)

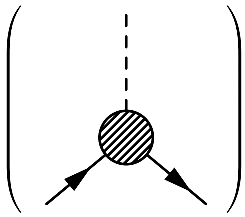
Now we compute the infinite part of the counter terms. We will use an on-shell scheme. We rescale the of the fields by the field strength in such away that we set the reside of the Fourier transformed propagator to 1. Furthermore, to give meaning to the mass and coupling counter terms we must define the physical mass and physical coupling constants. The physical mass is defined as the location of the pole of the Fourier transformed propagator and the physical coupling constants are defined to the magnitude of the scattering amplitude at zero momentum



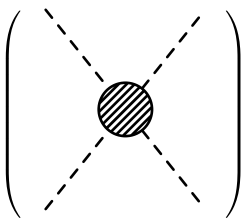
$$= \frac{i}{p^2 - m_\phi} + \text{terms regular at } p^2 = m_\phi^2, \tag{24}$$



$$= \frac{i}{\not p - m_\psi} + \text{terms regular at } \not p = m_\psi, \tag{25}$$

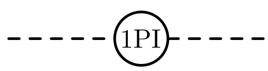


$$= g\gamma^5, \tag{26}$$

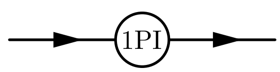


$$= -i\lambda \text{ at } s = 4m_\phi^2, t = u = 0. \tag{27}$$

The renormalization conditions (24) and (25) can be stated more clearly by defining the self energies

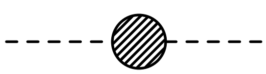


$$= -iM^2(p^2), \tag{28}$$



$$= -i\Sigma(\not p). \tag{29}$$

Then the full two-point function is given by the geometric series



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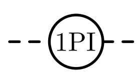

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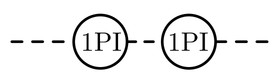
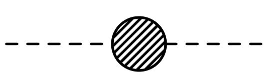


$$+ \ldots \tag{30}$$

$$= \frac{i}{p^2 - m_\phi^2 - M^2(p^2)}, \tag{31}$$



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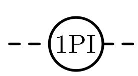

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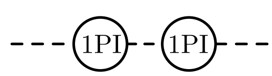


$$+ \ldots \tag{32}$$

$$= \frac{i}{\not p - m_\psi - \Sigma(\not p)}. \tag{33}$$

The renormalization conditions (24) and (25) can then be stated as

$$M^2(p^2)|_{p^2=m_\phi^2} = 0, \tag{34}$$

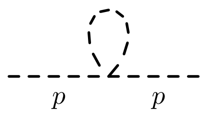
$$\Sigma(\not{p})|_{\not{p}=m_\psi} = 0, \tag{35}$$

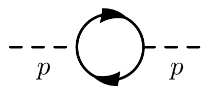
$$\left.\frac{\mathrm{d}M^2(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_\phi^2} = 0, \tag{36}$$

$$\left.\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\right|_{\not{p}=m_\psi} = 0. \tag{37}$$

Computation of δ_{Z_ϕ} and δ_{m_ϕ}

At one-loop the scalar self-energy is

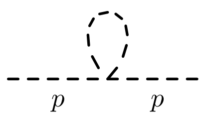
$-iM^2(p^2)$ = 

+ 

+ 

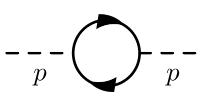
. (38)

We evaluate the first two diagrams above, keeping only the divergent terms, and then determine the divergent parts of the counter terms. The first diagram is



$$\begin{aligned} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-i\lambda)\frac{i}{k^2-m_\phi^2} \\ &= \lambda\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-m_\phi^2} \\ &= \frac{-i\lambda}{(4\pi)^{\frac{d}{2}}}\left(\frac{1}{m_\phi^2}\right)^{1-\frac{d}{2}}\Gamma(1-d/2) \\ &\sim \frac{-i\lambda m_\phi^2}{16\epsilon\pi^2}. \end{aligned} \tag{39}$$

where $\sim$ denotes the equality of the divergent terms. The second diagram is



$$\begin{aligned} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{Tr}\left[(g\gamma^5)\frac{i(\not{k}+m_\psi)}{k^2-m_\psi^2}(g\gamma^5)\frac{i\left((\not{k}+\not{p})+m_\psi\right)}{(k+p)^2-m_\psi^2}\right] \\ &= dg^2\int \mathrm{d}x\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^2+k\cdot p-m_\psi^2}{(k^2+2xk\cdot p+xp^2-m_\psi^2)^2} \\ &\to dg^2\int \mathrm{d}x\int\frac{\mathrm{d}^d \ell}{(2\pi)^d}\frac{\ell^2-x(1-x)p^2-m_\psi^2}{(\ell^2-\Delta)^2} \end{aligned} \tag{40}$$

where $\Delta = -x(1-x)p^2 + m_\psi^2$. Performing the momentum integration yields

$$
\begin{aligned}
&-\frac{idg^2}{4\pi^2}\int \mathrm{d}x \left[\frac{-2\Delta + x(1-x)p^2 + m_\psi^2}{\epsilon} - \Delta + \left(-2\Delta + x(1-x)p^2 + m_\psi^2\right)\log\left(\frac{4\pi e^{-\gamma_E}}{\Delta}\right) + \mathcal{O}(\epsilon)\right] \\
&= -\frac{ig^2}{16\pi^2}\int \mathrm{d}x \left[\frac{-2\Delta + x(1-x)p^2 + m_\psi^2}{\epsilon} + 3\Delta - 2x(1-x)p^2 - 2m_\psi^2\right. \\
&\qquad\qquad \left. + \left(-2\Delta + x(1-x)p^2 + m_\psi^2\right)\log\left(\frac{4\pi e^{-\gamma_E}}{\Delta}\right) + \mathcal{O}(\epsilon)\right] \\
&\sim -\frac{ig^2}{4\pi^2\epsilon}\int \mathrm{d}x \left(3x(1-x)p^2 - m_\psi^2\right) \\
&= -\frac{ig^2}{4\pi^2\epsilon}\left(\frac{p^2}{2} - m_\psi^2\right) \qquad (41)
\end{aligned}
$$

Therfore, the self energy is

$$
-iM^2(p^2) \sim -\frac{i\lambda m_\phi^2}{16\epsilon\pi^2} - \frac{ig^2}{4\pi^2\epsilon}\left(\frac{p^2}{2} - m_\psi^2\right) + i\left(p^2\delta_{Z_\phi} - \delta_{m_\phi}\right). \qquad (42)
$$

Next we evaluate the derivative of the self-energy

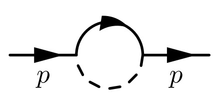
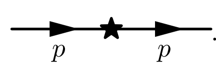
$$
-i\frac{\mathrm{d}M^2(p^2)}{\mathrm{d}p^2} \quad \sim \quad -\frac{ig^2}{8\pi^2\epsilon} + i\delta_{Z_\phi}. \qquad (43)
$$

Then, the renormalization conditions imply

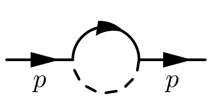
$$
\begin{aligned}
\delta_{Z_\phi} &\sim \frac{g^2}{8\pi^2\epsilon}, \\
\delta_{m_\phi} &\sim -\frac{1}{16\epsilon\pi^2}\left(\lambda m_\phi^2 + 4g^2\left(m_\phi^2 - m_\psi^2\right)\right). \qquad (44)
\end{aligned}
$$

Computation of $\boldsymbol{\delta_{Z_\psi}}$ and $\boldsymbol{\delta_{m_\psi}}$

At one-loop the fermion self-energy is

$-i\Sigma(\not p) \quad = \quad$  $+$  . (45)

We start by evaluating the first diagram



$$
\begin{aligned}
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{ig\gamma^5 i(\not k + \not p + m_\psi)g\gamma^5}{((k+p)^2 - m_\psi^2)(k^2 - m_\phi^2)} \\
&= g^2\int \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\not k + \not p - m_\psi}{((k+xp)^2 + x(1-x)p^2 - xm_\psi^2 - (1-x)m_\phi^2)^2} \\
&\rightarrow g^2\int \mathrm{d}x\,[(1-x)\not p - m_\psi]\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 - \Delta')^2} \qquad (46)
\end{aligned}
$$

where $\Delta' = -x(1-x)p^2 + xm_\psi^2 + (1-x)m_\phi^2$. Performing the momentum integral we obtain

$$\begin{aligned}
&\frac{ig^2}{16\pi^2}\left(\frac{4\pi}{\Delta'}\right)\Gamma(\epsilon)\int \mathrm{d}x\,[(1-x)\not{p} - m_\psi] \\
&= \frac{ig^2}{16\pi^2}\left(\frac{\not{p}}{2} - m_\psi\right)\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma_E}}{\Delta'}\right) + \mathcal{O}(\epsilon)\right) \\
&\sim \frac{ig^2}{16\epsilon\pi^2}\left(\frac{\not{p}}{2} - m_\psi\right). \qquad (47)
\end{aligned}$$

Therefore, up to finite terms, the fermion self energy is

$$-i\Sigma(\not{p}) \sim \frac{ig^2}{16\epsilon\pi^2}\left(\frac{\not{p}}{2} - m_\psi\right) + i\left(\not{p}\delta_{Z_\psi} - \delta_{m_\psi}\right). \qquad (48)$$

Its derivative is then

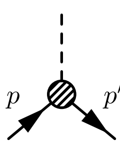
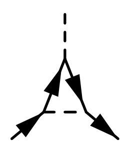
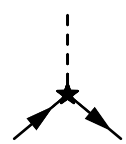
$$-i\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}} \quad \sim \quad \frac{ig^2}{32\epsilon\pi^2} + i\delta_{Z_\psi}. \qquad (49)$$

The renormalization conditions then imply

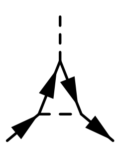
$$\begin{aligned}
\delta_{Z_\psi} &\sim -\frac{g^2}{32\epsilon\pi^2}, \\
\delta_{m_\psi} &\sim \frac{g^2 m_\psi}{16\epsilon\pi^2}. \qquad (50)
\end{aligned}$$

Computation of δ_{δ_g}

Since the tree level amplitude already satisfies the renormalization condition, we require the one-loop contribution to the 3-point $\psi\psi\phi$ amplitude to vanish

$\left(\right.$  $\left.\right)_{\text{one-loop, amputated}}$ =  +  = 0. (51)

We start by evaluating the following loop diagram



$$\begin{aligned}
&= -ig^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma^5(\not{k}+\not{p}+m_\psi)\gamma^5(\not{k}-\not{p}'+m_\psi)\gamma^5}{((k+p)^2-m_\psi^2)((k-p')^2-m_\psi^2)(k^2-m_\phi^2)} \\
&= -ig^3 2!\int\mathrm{d}x\int\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(-\not{k}\not{k}+\not{k}\not{p}'-\not{p}\not{k}+\not{p}\not{p}'-2m_\psi\not{k}-m_\psi(\not{p}-\not{p}')-m_\psi^2)}{((k+xp-yp')^2-\Delta(p,p';x,y))^3}\gamma^5 \qquad (52)
\end{aligned}$$

where $\Delta(p,p';x,y) = -2xyp\cdot p' - x(1-x)p^2 - y(1-y)p'^2 + (x+y)m_\psi^2 + (1-x-y)m_\phi^2$. Shifting the loop momentum to $\ell = k + xp - yp'$ we obtain

$$-ig^3 2!\int\mathrm{d}x\int\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\ell^2+f(p,p';x,y)}{(\ell^2-\Delta(p,p';x,y))^3}\gamma^5 \qquad (53)$$

where $f(p,p';x,y) = x(1+x)p^2+y(1+y)p'^2+2xyp\cdot p'+(1-x-y)\not{p}\not{p}'+m_\psi(2x-1)\not{p}-m_\psi(2y-1)\not{p}'-m_\psi^2$. Since the renormalization conditions are specified for on-shell fermions, f is to be sadwitched between the fermion spinors,

$$\bar{u}(p')f(p,p';x,y)u(p) = \bar{u}(p')\left(\left(x(4+x)+y^2-2\right)m_\psi^2 + 2(1-x-y+xy)p\cdot p'\right)u(p). \tag{54}$$

Furthermore, we are only interested in the limit where $p = p'$. Therefore, we take

$$\begin{aligned}\bar{u}(p')f(p,p';x,y)u(p) &\to \bar{u}(p)f(p,p;x,y)u(p)\\ &= \bar{u}(p)\left(x(2+x)-y(2-y)\right)m_\psi^2 u(p)\end{aligned} \tag{55}$$

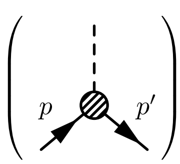
and

$$\begin{aligned}\Delta(p,p';x,y) &\to \Delta(p,p;x,y)\\ &= (x-y)^2 m_\psi^2 + (1-x-y)m_\phi^2.\end{aligned} \tag{56}$$

With these considerations we obtain

$$\begin{aligned}&-2ig^3\int \mathrm{d}x\int\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\ell^2+f(p,p;x,y)}{(\ell^2-\Delta(p,p;x,y))^3}\gamma^5\\ &= -2ig^3\int\mathrm{d}x\int\mathrm{d}y\left[-\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\left(\frac{1}{\Delta}\right)^{2-d/2}\Gamma(2-d/2)-\frac{i}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{3-d/2}\Gamma(3-d/2)f\right]\gamma^5\\ &= -2g^3\int\mathrm{d}x\int\mathrm{d}y\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-d/2}\left[\frac{d}{2}-\frac{f}{\Delta}(2-d/2)\right]\gamma^5\\ &= -\frac{2g^3}{16\pi^2}\int\mathrm{d}x\int\mathrm{d}y\left[\frac{2}{\epsilon}+2\log\left(\frac{4\pi e^{-\gamma_E}}{\Delta}\right)-\frac{f}{\Delta}+\mathcal{O}(\epsilon)\right]\gamma^5\\ &\sim -\frac{g^3}{4\pi^2\epsilon}\gamma^5\int\mathrm{d}x\int\mathrm{d}y\\ &= -\frac{g^3}{4\pi^2\epsilon}\gamma^5.\end{aligned} \tag{57}$$

Thus, the divergent part of the one-loop contributions to the total amplitude, evaluated for on-shell fermions at zero momentum, becomes



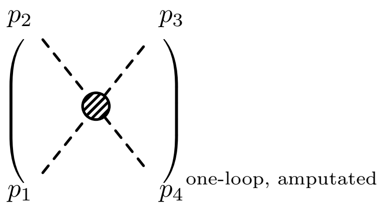
$$\left(\ \cdots\ \right)_{\text{one-loop, amputated}} \sim -\frac{g^3}{4\pi^2\epsilon}\gamma^5 + \delta_g\gamma^5. \tag{58}$$

The renormalization conditions then imply

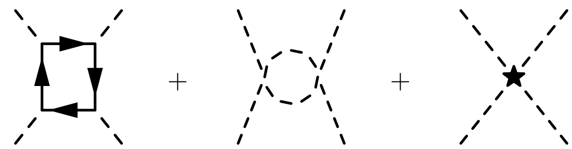
$$\delta_g \sim \frac{g^3}{4\pi^2\epsilon}. \tag{59}$$

Computation of δ_{δ_λ}

Since the tree level amplitude already satisfies the renormalization condition, we require the one-loop contribution to the 4-point scalar amplitude to vanish

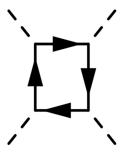


$=$



$$= 0. \tag{60}$$

We start by evaluating the following box diagram



$$= i^4 g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^5(\not{k}+m_\psi)\gamma^5(\not{k}+\not{p}_1+\not{p}_2+\not{p}_3+m_\psi)\gamma^5(\not{k}+\not{p}_1+\not{p}_2+m_\psi)\gamma^5(\not{k}+\not{p}_1+m_\psi)\right]}{(k^2-m_\psi^2)((k+p_1+p_2+p_3)^2-m_\psi^2)((k+p_1+p_2)^2-m_\psi^2)((k+p_1)^2-m_\psi^2)}$$

$$= g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[(-\not{k}+m_\psi)(\not{k}+\not{p}_1+\not{p}_2+\not{p}_3+m_\psi)(-\not{k}-\not{p}_1-\not{p}_2+m_\psi)(\not{k}+\not{p}_1+m_\psi)\right]}{(k^2-m_\psi^2)((k+p_1+p_2+p_3)^2-m_\psi^2)((k+p_1+p_2)^2-m_\psi^2)((k+p_1)^2-m_\psi^2)} \tag{61}$$

Following the advice given in the question we specify to a specific frame to simplify the momentum. Let us choose $p_1+p_2=0=p_3+p_4$. Then this diagram becomes

$$\begin{aligned}
&g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[(-\not{k}+m_\psi)(\not{k}+\not{p}_3+m_\psi)(-\not{k}+m_\psi)(\not{k}+\not{p}_1+m_\psi)\right]}{(k^2-m_\psi^2)^2((k+p_3)^2-m_\psi^2)((k+p_1)^2-m_\psi^2)} \\
&g^4 3! \int \mathrm{d}x\,(1-x) \int \mathrm{d}y \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{f(k,p_1,p_3)}{(k^2+2(yp_3+(x-y)p_1)\cdot k+yp_3^2+(x-y)p_1^2-m_\psi^2)^4} \\
&6g^4 \int \mathrm{d}x\,(1-x) \int \mathrm{d}y \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{f(k,p_1,p_3)}{\left((k+yp_3+(x-y)p_1)^2-\Delta(p_1,p_3;x,y)\right)^4}
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
f(k,p_1,p_3) = 4k^4 &+ 4k^2\left(k\cdot(p_1+p_3)-p_1\cdot p_3-2m_\psi^2\right) \\
&+ 8k\cdot p_1 k\cdot p_3 - 4m_\psi^2\left(k\cdot(p_1+p_3)-p_1\cdot p_3\right)+m_\psi^4,
\end{aligned} \tag{63}$$

and

$$\Delta(p_1,p_3;x,y) = -y(1-y)p_3^2-(x-y)\left(1-(x-y)\right)p_1^2+2y(x-y)p_3\cdot p_1+m_\psi^2. \tag{64}$$

Shifting the loop-momentum to $\ell = k+yp_3+(x-y)p_1$, imposing the on-shell condition, and setting $p_3=p_1$ as required of the renormalization condition, we obtain

$$6g^4 \int \mathrm{d}x\,(1-x)\int \mathrm{d}y \int \frac{\mathrm{d}^d \ell}{(2\pi)^d}\frac{f(\ell,p_1,p_1;x,y)}{(\ell^2-\Delta(p_1,p_3;x,y))^4} \tag{65}$$

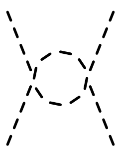
where

$$f(\ell,p_1,p_1;x,y) = 4\ell^4+m_\psi^2\ell^2\left(\frac{8-16x+16x^2}{d}-12-8x+8x^2\right)+m_\psi^4\left(5+8x-4x^2-8x^3+4x^4\right) \tag{66}$$

Only the first term of f contributes to the divergent part of the amplitude,

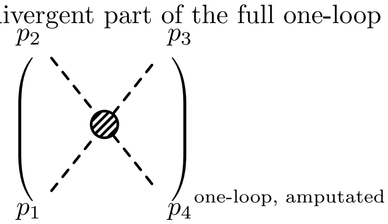
$$
\begin{aligned}
&\sim 24g^4 \int \mathrm{d}x\,(1-x) \int \mathrm{d}y \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^4}{(\ell^2-\Delta(p_1,p_3;x,y))^4} \\
&= 24g^4 \int \mathrm{d}x\,(1-x) \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^4}{(\ell^2-\Delta(p_1,p_3;x,y))^4} \\
&= 24g^4 \int \mathrm{d}x\,(1-x) \frac{i}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2} \Gamma(2-d/2) \\
&\sim \frac{3ig^4}{4\pi^2\epsilon}
\end{aligned}
\tag{67}
$$

Next we evaluate the scalar bubble diagram



$$
\begin{aligned}
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(-i\lambda)i(-i\lambda)i}{(k^2-m_\phi^2)((k+p_1+p_2)^2-m_\phi^2)} \\
&= \lambda^2 \int \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2+2xk\cdot(p_1+p_2)+xp_1^2+xp_2^2-m_\phi^2)^2} \\
&= \lambda^2 \int \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2+x(p_1+p_2))^2+x(1-x)(p_1^2+p_2^2)-2x^2p_1\cdot p_2-m_\phi^2)^2} \\
&= \lambda^2 \int \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{(\ell^2+x(1-x)(p_1^2+p_2^2)-2x^2p_1\cdot p_2-m_\phi^2)^2} \\
&= \lambda^2 \int \mathrm{d}x \frac{i}{(4\pi)^{d/2}} \left(\frac{1}{-x(1-x)(p_1^2+p_2^2)+2x^2p_1\cdot p_2+m_\phi^2}\right)^{2-d/2} \Gamma(2-d/2) \\
&\sim \frac{i\lambda^2}{16\pi^2}.
\end{aligned}
\tag{68}
$$

Finally, the divergent part of the full one-loop contribution to the amplitude is



$$
\sim \frac{3ig^4}{4\pi^2\epsilon} + \frac{i\lambda^2}{16\pi^2} - i\delta_\lambda. \tag{69}
$$

The renormalization conditions then imply

$$
\delta_\lambda \sim \frac{3g^4}{4\pi^2\epsilon} + \frac{\lambda^2}{16\pi^2}. \tag{70}
$$