Reading §2.4 & 2.5

§2.4: QED

This week we looked at scattering amplitudes in QED & YM theory.

We learned how to write the polarization vectors for spin-1 particles in spinor helicity variables.

$$\varepsilon_-^{\mu}(p;q) = \frac{-\langle p|\gamma^{\mu}|q]}{\sqrt{2}\,[qp]} \quad , \quad \varepsilon_+^{\mu} = \frac{-\langle q|\gamma^{\mu}|p]}{\sqrt{2}\,\langle qp\rangle}$$

It is important to realize that that these polarization vectors are defined using a reference spinor $q \neq p$. This is a reflection of the extra gauge sym of a spin-1 particle:

$$\varepsilon_{\pm}^{\mu} \to \varepsilon_{\pm}^{\mu} + c\,p^{\mu}$$

We also explored the special kinematics of 3pt amplitudes of massless particles where either of the conditions must be met:

1) $|1] \propto |2] \propto |3]$
2) $|1\rangle \propto |2\rangle \propto |3\rangle$

These facts imply that:

1) a non-vanishing on-shell 3pt amplitude to only massless particles either contains only square or angle brackets
2) 3pt on-shell amplitudes to only massless particles are non-zero for only complex momenta

§2.5: Yang-Mills

express amplitudes in the trace basis. For tree level diagrams we have

$$A_n = g^{n-2} \sum_{\text{perms}\,\sigma} A_n[1\,\sigma(23\cdots n)]\;\mathrm{Tr}(T^{a_1} T^{\sigma(a_1} T^{a_2} \cdots T^{a_n)})$$

color ordered amplitude

inherits properties from the trace such as cyclicity

n-point gluon amplitudes:

1) $A_n = 0$ if all helicities are the same
2) $A_n = 0$ if all helicities are the same except for one
3) $A_n \neq 0$ if all helicities are the same except for two:

$$A_n[1^+ \cdots i^- (i+1)^+ \cdots j^- (j+1)^+ \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots\cdots \langle n1\rangle}$$

the Parke-Taylor formula ...

Color ordered amplitudes have the following properties

1) Cyclic – $A_n[12\cdots n] = A_n[n12\cdots(n-1)]$
2) Reflection – $A_n[12\cdots n] = A_n[n\,(n-1)\cdots 21]$
3) $U(1)$ decoupling identity: $A_n[123\cdots n] + A_n[231\cdots n] + \cdots + A_n[23\cdots 1n]$

The trace basis is overcomplete and there are further linear relations among the color ordered tree-level amplitudes called Kleiss-Kuijf relations. This reduces the # of independent tree level amplitudes to $(n-2)!$

Furthermore, the BCJ relations reduce the # of ind tree level amplitudes to $(n-3)!$

Exercises: 2.10 (along z), 2.14, 2.15, 2.21, 2.22, 2.23, 2.26 of EH

2.10

We set $\vec{p}$ to be along $\hat{z}$ for simplicity $p^\mu = (E, 0, 0, E) \Rightarrow \theta = 0$

$$\tilde{\varepsilon}^\mu_\pm(p) = \pm \frac{e^{\mp i\phi}}{\sqrt{2}}\left(0, \cos\phi \pm i \sin\phi, \sin\phi \mp i \cos\phi, 0\right)$$

$$= \pm \frac{e^{\mp i\phi}}{\sqrt{2}}\left(0, e^{\pm i\phi}, \mp i\, e^{\pm i\phi}, 0\right)$$

$$= \pm \frac{1}{\sqrt{2}}\left(0, 1, \mp i, 0\right)$$

a) show that $\tilde{\varepsilon}_\pm^2 = 0$ and $\tilde{\varepsilon}_\pm \cdot p = 0$.

$$\hat{\varepsilon}_\pm^2 = \vec{\tilde{\varepsilon}}_\pm \cdot \vec{\tilde{\varepsilon}}_\pm = \frac{1}{2}(1, \mp i, 0) \cdot (1, \mp i, 0) = 0.$$

$$p \cdot \hat{\varepsilon}_\pm = \vec{p} \cdot \vec{\tilde{\varepsilon}}_\pm \sim (\#, \#, 0) \cdot (0, 0, \#) = 0.$$

b) $\tilde{\varepsilon}^\mu_\pm$ is null $\Rightarrow$ we can write $(\tilde{\varepsilon}^\mu_\pm(p))_{a\dot{b}} = (\sigma_\mu)_{a\dot{b}}\, \tilde{\varepsilon}^\mu_\pm(p)$. Calculate $(\tilde{\varepsilon}^\mu_\pm(p))_{a\dot{b}}$ and find $|r\rangle$ s.t. $(\tilde{\varepsilon}^\mu_\pm)_{a\dot{b}} = -|p]_a \langle r|_{\dot{b}}$

$$(\hat{\varepsilon}^\mu_\pm)_{a\dot{b}} = \tilde{\varepsilon}^\mu_\pm (\sigma_\mu)_{a\dot{b}}$$

$$= -\cancel{\tilde{\varepsilon}^0_\pm}(\sigma^0_\mu)_{a\dot{b}} + \tilde{\varepsilon}^1_\pm (\sigma^1)_{a\dot{b}} + \varepsilon^2_\pm (\sigma^2)_{a\dot{b}} + \cancel{\varepsilon^3_\pm}(\sigma^3)_{a\dot{b}}$$

$$= (\pm \sigma^1_{a\dot{b}} \pm \mp i\, \sigma^2_{a\dot{b}})/\sqrt{2}$$

$$= \frac{\pm}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{cases} \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{for } + \\ \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{for } - \end{cases}$$

now from prob. 2.1 we know:

$$|p\rangle^{\dot{a}} = \sqrt{2E}\left(\cos\theta/2 \quad \sin\theta/2\, e^{i\phi}\right)^T \to \sqrt{2E}\,(1, 0)^T$$

$$\langle p|_{\dot{a}} = \sqrt{2E}\left(-\sin\tfrac{\theta}{2} e^{I\phi} \quad \cos\theta/2\right) \to \sqrt{2E}\,(0, 1)$$

$$[p|^a = \sqrt{2E}\left(\cos\tfrac{\theta}{2} \quad \sin\tfrac{\theta}{2}\, e^{-i\phi}\right) \to \sqrt{2E}\,(1, 0)$$

$$|p]_a = \sqrt{2E}\left(-\sin\tfrac{\theta}{2} e^{-i\phi} \quad \cos\theta/2\right)^T \to \sqrt{2E}\,(0, 1)^T$$

breaking $\tilde{\mathcal{E}}^{\mu}_{a\dot{b}}$ into a tensor prod we can find r s.t. $\tilde{\mathcal{E}}^{\mu}_{a\dot{b}} = -|p]\langle r|$

$$(\tilde{\mathcal{E}}^{\mu}_{+})_{a\dot{b}} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sqrt{2}\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\frac{1}{\sqrt{2E}}|p]} \otimes \underbrace{(1\ 0)}_{\frac{-1}{\sqrt{2E}}\langle r'|}$$

where $r' = (E, 0, 0, -E)$

$$\langle r| = \sqrt{2E}\,(-1, 0) \qquad \langle r'|p\rangle = \sqrt{2E}\,\sqrt{2E}\,(-1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2E$$

we can renormalize the $\langle r'$ st. $\langle r|p\rangle = -\sqrt{2}$ as the problem hints at. Therefore defining

$$\langle r = \frac{\langle r'}{\sqrt{E}}$$

we have:

$$\langle r|p\rangle = \sqrt{2}$$

and

$$\tilde{\mathcal{E}}_{+a\dot{a}} = -|p]_a \langle r|_{\dot{a}}$$

c) Show that $(\mathcal{E}^{\mu}_{+})_{a\dot{b}} = \sqrt{2}\,|p]\langle q| / \langle qp\rangle$ follows from $\mathcal{E}^{\mu}_{+} = \dfrac{\langle q|\gamma^{\mu}|p]}{\sqrt{2}\langle qp\rangle}$

$$\gamma^{\mu} = \begin{pmatrix} 0 & (\sigma^{\mu})_{a\dot{b}} \\ (\bar{\sigma}^{\mu})^{\dot{a}b} & \end{pmatrix} \qquad |p] = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \qquad \langle p| = (0, \langle p|_{\dot{a}})$$

$$\begin{aligned}
(\mathcal{E}^{\mu}_{+})_{a\dot{a}} &= \mathcal{E}^{\mu}_{+}(\sigma_{\mu}) \\
&= \frac{\langle q|\gamma^{\mu}|p]}{\sqrt{2}\langle qp\rangle}(\sigma_{\mu})_{a\dot{a}} \\
&= \frac{1}{\sqrt{2}\langle qp\rangle}\left(\langle q|_{\dot{b}}(\bar{\sigma}^{\mu})^{\dot{b}b}|p]_b\right)(\sigma_{\mu})_{a\dot{a}} \\
&= \frac{1}{\sqrt{2}\langle qp\rangle}|p]_b\langle q|_{\dot{b}}\,\varepsilon^{bc}\varepsilon^{\dot{b}\dot{c}}\underbrace{(\sigma^{\mu})_{c\dot{c}}(\sigma_{\mu})_{a\dot{a}}}_{-2\varepsilon_{ca}\varepsilon_{\dot{c}\dot{a}}} \\
&= \frac{-\sqrt{2}\,|p]_b\langle q|_{\dot{b}}}{\langle qp\rangle}\varepsilon^{bc}\varepsilon^{\dot{b}\dot{c}}\varepsilon_{ca}\varepsilon_{\dot{c}\dot{a}} \\
&= \frac{-\sqrt{2}\,|p]_b\langle q|_{\dot{b}}}{\langle qp\rangle}\delta_a{}^b\,\delta_{\dot{a}}{}^{\dot{b}} \\
&= -\sqrt{2}\frac{|p]_a\langle q|_{\dot{a}}}{\langle qp\rangle}
\end{aligned}$$

missing a factor of (-1) to cancel this (-1)?

d) suppose that $\forall\, c_+$ st. $\mathcal{E}^{\mu}_{+}(p;q) = \tilde{\mathcal{E}}^{\mu}_{+}(p) + c_+ p^{\mu}$. Show that this relationship requires $\langle rp\rangle = -\sqrt{2}$ and $c_+ = -\langle rq\rangle / \langle pq\rangle$

$$\begin{aligned}
\mathcal{E}^{\mu}_{+}(p;q) &= \tilde{\mathcal{E}}^{\mu}_{+}(p) + c_+ p^{\mu} \\
P_2\,\not{\mathcal{E}}_{+}(p;q) &= P_2\left(\tilde{\not{\mathcal{E}}}_{+}(p) + c_+\not{p}\right) \\
\sqrt{2}\frac{|p]\langle q|}{\langle qp\rangle} &= -|p]\langle r| + c_+\,p]\langle p|
\end{aligned}$$

$$\Rightarrow |p]\left(\frac{-\sqrt{2}\langle q|}{\langle qp\rangle} = -\langle r| + c_+\langle p|\right)$$

$$\Rightarrow \frac{\sqrt{2}}{\langle qp\rangle}\langle q| + \langle r| - c_+\langle p| = 0$$

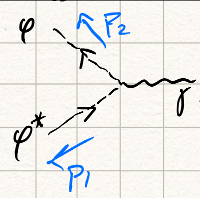
dotting in $|p\rangle$ we find that

$$\frac{\sqrt{2}\,\langle q p\rangle}{\langle q p\rangle} + \langle r|p\rangle = 0 \quad \Rightarrow \quad \langle r|p\rangle = -\sqrt{2}$$ as we saw before.

dotting in bra $\langle r|$ we find

$$\frac{\sqrt{2}\,\langle q r\rangle}{\langle q p\rangle} - C_+ \underbrace{\langle p r\rangle}_{+\sqrt{2}} = 0 \quad \Rightarrow \quad C_+ = \frac{\langle q r\rangle}{\langle q p\rangle} = -\frac{\langle r q\rangle}{\langle q p\rangle}$$

scalar QED : $\mathcal{L} = -\frac{1}{4} F^2 - |\partial\varphi|^2 + ieA^\mu[(\partial_\mu\varphi^*)\varphi - \varphi^*\partial_\mu\varphi] - e^2 A^2|\varphi|^2 - \frac{1}{4}\lambda(|\varphi|^2)^2$



$= ie(p_2 - p_1)^\mu$

$= -2ie^2\eta_{\mu\nu}$

$= -i\lambda$

2.14 : compute $A_3(\varphi\varphi^*\gamma^-)$. show that it is ind of the reference spinor and write A_3 in terms of only angle brackets. Use complex conj to write down $A_3(\varphi\varphi^*\gamma^+)$

$$iA_3(\varphi\varphi^*\gamma^-) = ie(p_2-p_1)^\mu\, \mathcal{E}_\mu(p_3;q)$$

$$= -ie\,(p_2-p_1)^\mu \frac{\langle 3|\gamma_\mu|q]}{\sqrt{2}\,[q3]}$$

$$= \frac{-ie}{\sqrt{2}} \frac{\langle 3|\,2-1\,|q]}{[q3]}$$

$$= \frac{-ie}{\sqrt{2}} \frac{\langle 3|2|q] - \langle 3|1|q]}{[q3]}$$

$$= -\frac{ie}{\sqrt{2}} \frac{-\langle 32\rangle[2q] + \langle 31\rangle[1q]}{[q3]}$$

$\sum_i i\rangle[i = 0 = \sum_i i]\langle i$:

1) $\langle 1|(1\rangle[1 + 2\rangle[2 + 3\rangle[3)|q] = 0$

$\Rightarrow \langle 12\rangle[2q] + \langle 13\rangle[3q] = 0.$

$\Rightarrow [2q] = -\frac{\langle 13\rangle[3q]}{\langle 12\rangle} = +\frac{\langle 13\rangle[q3]}{\langle 12\rangle}$

2) $\langle 2|(1\rangle[1 + 2\rangle[2 + 3\rangle[3)|q] = 0$

$\Rightarrow \langle 21\rangle[1q] + \langle 23\rangle[3q] = 0$

$\Rightarrow [1q] = -\frac{\langle 23\rangle[3q]}{\langle 21\rangle} = -\frac{\langle 23\rangle[q3]}{\langle 12\rangle}$

$$\Rightarrow iA_3(\varphi\varphi^*\gamma^-) = \frac{+ie}{\sqrt{2}}\left[\frac{\langle 32\rangle}{[q3]}\frac{\langle 13\rangle[q3]}{\langle 12\rangle} + \frac{\langle 31\rangle}{[q3]}\frac{\langle 23\rangle[q3]}{\langle 12\rangle}\right]$$

$$= \frac{ie}{\sqrt{2}}\left[-\frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle} - \frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle}\right]$$

$$= \sqrt{2}\,ie\,\frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle}$$

* only depends on angle

taking the complex conj, we get:

$$iA_3(\varphi\varphi^*\gamma^+) = \sqrt{2}\,ie\,\frac{[13][23]}{[12]}$$

* only depends on square

2.15 Consider $A_4(\varphi\varphi^* \gamma\gamma)$. Show that the $|\varphi|^2 A^2$ contact term gives a vanishing contribution to the on shell amp A_4 no matter what helicity the γ's are.

$iA_4^{h_1 h_2} \equiv$ 

$= -2ie^2 \eta_{\mu\nu}\, \varepsilon^\mu(p_1;q_1)\, \varepsilon^\nu(p_2;q_2)$

$$A_4^{--} = -2e^2 \frac{\langle 1|\gamma^\mu|q_1] \langle 2|\gamma_\mu|q_2]}{2\,[q_1 1]\,[q_2 2]}$$

$$= -e^2 \frac{\langle 12\rangle [q_1 q_2]}{[q_1 1][q_2 2]}$$

$$= 0$$ choosing $q_1 = q_2 \neq p_1$ or p_2

$$A_4^{-+} = -\frac{2e^2}{2} \frac{\langle 1|\gamma^\mu|q_1] \langle q_2|\gamma_\mu|2]}{[q_1 1] \quad \langle q_2 2\rangle}$$

$$= +e^2 \frac{\langle 1 q_2\rangle [q_1 2]}{\langle 2 q_2\rangle [q_1 1]}$$

$$= 0.$$ choosing $q_2 = p_1$, $q_1 = p_2$

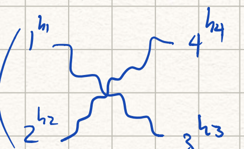
since $|p\rangle^* = [p|$, $|p]^* = \langle p|$, ...

$$(\varepsilon_+^\mu)^* = \left(\frac{-\langle q|\gamma^\mu|p]}{\sqrt{2}\langle q p\rangle}\right)^* = \frac{-\langle q|\gamma^\mu|p]^*}{\sqrt{2}\langle q p\rangle^*} = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}\,[q p]} = \varepsilon_-^\mu$$

$\Rightarrow A^{++} = (A^{--})^* = 0$ and $A^{+-} = (A^{-+})^* = 0.$

2.21 Show that it is always possible to choose the polarization vectors such that the contribution from the 4pt contact term to the 4pt amplitude vanishes.

$$A_4^{h_1h_2h_3h_4} = -i \left(\right) \propto (\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)$$



So the amplitude only depends on dot products of polarization vectors. There are only so many dot products that we can form:

$$\{\varepsilon_\pm\cdot\varepsilon_\pm,\ \varepsilon_\pm\cdot\varepsilon_\mp\}.$$

$$\varepsilon_-(p_1;q_1)\cdot\varepsilon_-(p_2;q_2) = \frac{\langle p_1|\gamma^\mu|q_1]\langle p_2|\gamma_\mu|q_2]}{2\,[q_1p_1]\,[q_2p_2]} = \frac{\langle p_1p_2\rangle[q_1q_2]}{[q_1p_1][q_2p_2]} = 0 \text{ for } q_1=q_2$$

$$\varepsilon_+(p_1;q_1)\cdot\varepsilon_+(p_2;q_2) = \frac{\langle q_1q_2\rangle[p_1p_2]}{\langle q_1 1\rangle\langle q_2 2\rangle} = 0 \quad\text{for } q_1=q_2$$

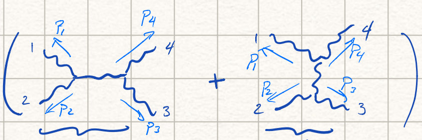
$$\varepsilon_-(p_1;q_1)\cdot\varepsilon_+(p_2;q_2) = \frac{\langle p_1q_2\rangle[q_1p_2]}{[p_1q_1][p_2q_2]} = 0 \quad\text{for } q_1=p_2 \text{ or } q_2=p_1$$

$\Rightarrow$ it is clear that we can always pick the q_i's such that A_4 vanishes for all choices of $\{h_1,h_2,h_3,h_4\}$.

2.22 show that choosing a good set of $\{q_i\}$ that the full 4pt gluon amp. vanishes when all 4 helicities are the same.

we have already shown that $(\text{contact diagram})=0$, so we need to calculate:

$$A_4^{----} = -i\Big(\ \underbrace{\text{s-channel}}_{iA_s^{----}} + \underbrace{\text{t-channel}}_{iA_t^{----}}\ \Big)$$



$$A_s^{----} = (-i)(-\sqrt{2})\, \varepsilon_-^{\mu_1}(p_1;q_1)\, \varepsilon_-^{\mu_2}(p_2;q_2)\,\big(\eta_{\mu_1\mu_2}\, p_{1\alpha} + \eta_{\mu_2\alpha}\, p_{2\mu_1} - \eta_{\alpha\mu_1}(p_1+p_2)_{\mu_2}\big)$$
$$\times(-\sqrt{2})\, \varepsilon_-^{\mu_3}(p_3;q_3)\, \varepsilon_-^{\mu_4}(p_4;q_4)\,\big(-\eta_{\mu_3\mu_4}(p_1+p_2)_\beta + \eta_{\mu_3\beta}\, p_{3\mu_4} + \eta_{\mu_4\beta}\, p_{4\mu_3}\big)$$
$$\times(-i)\quad g^{\alpha\beta}/(p_1+p_2)^2$$

$$= -2\,\big(\varepsilon_{1-}\cdot\varepsilon_{2-}\, p_{1\alpha} + \varepsilon_{2-\alpha}\, \varepsilon_{1-}\cdot p_2 - \varepsilon_{1-\alpha}\,(p_1+p_2)\cdot\varepsilon_{2-}\big)$$
$$\times\big(-\varepsilon_{3-}\cdot\varepsilon_{4-}\,(p_1+p_2)_\beta + \varepsilon_{3-\beta}\, \varepsilon_{4-}\cdot p_3 + \varepsilon_{4-\beta}\, \varepsilon_{3-}\cdot p_4\big)$$
$$\times\ g^{\alpha\beta}/(p_1+p_2)^2$$

$$= -2\,\big(\varepsilon_{1-}\cdot\varepsilon_{2-}\, p_{1\alpha} + \varepsilon_{2-\alpha}\, \varepsilon_{1-}\cdot p_2 - \varepsilon_{1-\alpha}\, p_1\cdot\varepsilon_{2-}\big)$$
$$\times\big(-\varepsilon_{3-}\cdot\varepsilon_{4-}\,(p_1+p_2)_\beta + \varepsilon_{3-\beta}\, \varepsilon_{4-}\cdot p_3 + \varepsilon_{4-\beta}\, \varepsilon_{3-}\cdot p_4\big)\quad g^{\alpha\beta}/(p_1+p_2)^2$$

choose $q_1=q_2$ and $q_3=q_4$ so that $\varepsilon_{1-}\cdot\varepsilon_{2-} = \varepsilon_{3-}\cdot\varepsilon_{4-}=0$. (see 2.21)

$$A_4^{----} = \frac{-2}{(p_1+p_2)^2}\Big\{\big(\varepsilon_{2-}(\varepsilon_{1-}\cdot p_2) - \varepsilon_{1-}(p_1\cdot\varepsilon_{2-})\big)\cdot\big(\varepsilon_{3-}(\varepsilon_{4-}\cdot p_3) + \varepsilon_{4-}(\varepsilon_{3-}\cdot p_4)\big)\Big\}$$
$$= \frac{-2}{(p_1+p_2)^2}\Big\{(\varepsilon_{2-}\cdot\varepsilon_{3-})(\varepsilon_{1-}\cdot p_2)(\varepsilon_{4-}\cdot p_3) + (\varepsilon_{2-}\cdot\varepsilon_{4-})(\varepsilon_{1-}\cdot p_2)(\varepsilon_{3-}\cdot p_4)$$
$$- (\varepsilon_{1-}\cdot\varepsilon_{3-})(\varepsilon_{4-}\cdot p_3)(\varepsilon_{2-}\cdot p_1) - (\varepsilon_{1-}\cdot\varepsilon_{4-})(p_1\cdot\varepsilon_{2-})(\varepsilon_{3-}\cdot p_4)\Big\}$$

choosing $q_1=q_3=q_4=q_2$ we find that $(\varepsilon_{2-}\cdot\varepsilon_{3-}) = (\varepsilon_{2-}\cdot\varepsilon_{4-}) = (\varepsilon_{1-}\cdot\varepsilon_{3-}) = (\varepsilon_{1-}\cdot\varepsilon_{4-}) = 0$ and the whole amplitude vanishes.

the t-and u-channel amplitudes are related to the s-channel amplitude by crossing

$$A_t^{----} = \big[A_s^{----}\big]_{p_2\leftrightarrow p_4}$$

$$A_u^{----} = \big[A_s^{----}\big]_{p_2\leftrightarrow p_3}$$

therefore the whole 4pt amplitude vanishes when all helicities are the same.

note that the amplitude where all but one of the helicities are the same also vanishes. This follows simply from noting that the choice $q_i=q$ for all gluons that have the same helicity: $\varepsilon_\pm(p_i;q)\cdot\varepsilon_\pm(p_j;q)=0$. Therefore the amplitude can only depend on $\varepsilon_\pm(p_i;q)\cdot\varepsilon_\mp(p';q')$ where p' and q' denote the momentum and reference momentum of the single gluon with opposite helicity. Since,

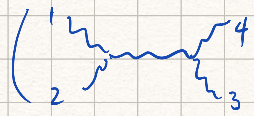
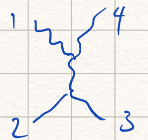
$$\varepsilon_\pm(p_i;q)\cdot\varepsilon_\mp(p';q') = \frac{\langle q\,p'\rangle[p_i\,q']}{\langle q\,p_i\rangle[q'\,p']}$$

we can set $q=p'$ to get $\varepsilon_\pm(p_i;q)\cdot\varepsilon_\mp(p';q')=0$. Thus, the amplitude vanishes b/c it is proportional to either $\varepsilon_\pm\cdot\varepsilon_\pm$ or $\varepsilon_\mp\cdot\varepsilon_\pm$ which vanish.

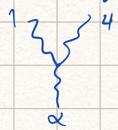
2.23 calculate $A_4[1^-2^-3^+4^+]$ using Feynman rules and a smart choice of reference spinors. Show that it can be brought into the form

$$A_4[1^-2^-3^+4^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

lets start off by choosing $q_1 = q_2 = q$ and $q_3 = q_4 = q'$ so that all $\varepsilon_\pm \cdot \varepsilon_\pm = 0$. Then,

$$A_4[1^-2^-3^+4^+] = (-i)\Big($$  $+$  $\Big)$

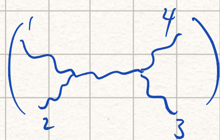
Lets choose q & q' so that the t-channel diagram vanishes.

 $\propto (\varepsilon_{1-}\cdot\varepsilon_{4+})\,p_{1\alpha} + (p_4\cdot\varepsilon_{1-})\,\varepsilon_{4+\alpha} - \underbrace{(p_1+p_4)\cdot\varepsilon_{4+}}_{p_1\cdot\varepsilon_{4+}}\,\varepsilon_{1-\alpha}$

choosing $q = p_4$ then $\varepsilon_{1-}\cdot\varepsilon_{4+} = 0$. also note that

$$p_4\cdot\varepsilon_{1-} = p_4\cdot\varepsilon_-(p_1;q) = p_4\cdot\varepsilon_-(p_1;p_4) = 0.$$

Thus we just have to fix q' such that $p_1\cdot\varepsilon_4(p_4;q') = 0$. This is achieved by letting $q' = p_1$.

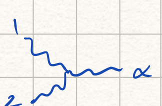
$\Rightarrow A_4[1^-2^-3^+4^+] = -i\Big($  $\Big)\Big|_{\substack{q_1=q_2=q=p_4\\ q_3=q_4=q'=p_1}} = (-i)\, iA_L^\alpha \frac{(-ig_{\alpha\beta})}{(p_1+p_2)}\, iA_R^\beta\Big|_{\substack{q_1=q_2=q=p_4\\ q_3=q_4=q'=p_1}}$

note that with this choice of q_i's all $\varepsilon_i\cdot\varepsilon_j = 0$ with the exception of.

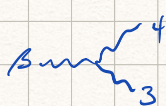
$$\begin{aligned}
\varepsilon_2\cdot\varepsilon_3 &= \varepsilon_-(p_2;p_4)\cdot\varepsilon_+(p_3;p_1)\\
&= \frac{\langle 2|\gamma^\mu|4]\,\langle 1|\gamma_\mu|3]}{2\,[42]\,\langle 13\rangle}\\
&= \frac{\langle 21\rangle[43]}{\langle 13\rangle[42]}\\
&= -\frac{\langle 12\rangle[34]}{\langle 13\rangle[24]}
\end{aligned}$$

what about the $p_i\cdot\varepsilon_j$?

$$p_1\cdot\varepsilon_3 = p_1\cdot\varepsilon_4 = 0$$
$$p_4\cdot\varepsilon_1 = p_4\cdot\varepsilon_2 = 0.$$

$iA_L^\alpha =$  $= -i\sqrt{2}g\Big(\cancel{(\varepsilon_1\cdot\varepsilon_2)}\,p_1^\alpha + (\varepsilon_1\cdot p_2)\,\varepsilon_2^\alpha - ((p_1+p_2)\cdot\varepsilon_2)\,\varepsilon_1^\alpha\Big)$

$$= -i\sqrt{2}\,g\Big((\varepsilon_1\cdot p_2)\,\varepsilon_2^\alpha - (p_1\cdot\varepsilon_2)\,\varepsilon_1^\alpha\Big)$$

$iA_R^\beta =$  $= -i\sqrt{2}g\Big(\cancel{(\varepsilon_3\cdot\varepsilon_4)}\,p_3^\beta + (\varepsilon_3\cdot p_4)\,\varepsilon_4^\beta - (p_3+p_4)\cdot\varepsilon_4\,\varepsilon_3^\beta\Big)$

$$= -i\sqrt{2}g\Big((\varepsilon_3\cdot p_4)\,\varepsilon_4^\beta - (p_3\cdot\varepsilon_4)\,\varepsilon_3^\beta\Big)$$

$$A_4 = \frac{(-i)^2 \, iA_L^{\alpha} \, iA_{R\alpha}}{(p_1+p_2)^2}$$

$$= \frac{(-i\sqrt{2}g)^2 \left((\varepsilon_1\cdot p_2)\,\varepsilon_2^{\alpha} - (p_1\cdot\varepsilon_2)\,\varepsilon_1^{\alpha} \right)\left((\varepsilon_3\cdot p_4)\,\varepsilon_{4\alpha} - (p_3\cdot\varepsilon_4)\,\varepsilon_{3\alpha} \right)}{(p_1+p_2)^2}$$

$$= \frac{-2g^2}{(p_1+p_2)^2} \left\{ -(\varepsilon_1\cdot p_2)(\varepsilon_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_4) \right\}$$

$$= \frac{-2g^2}{\langle 12\rangle [12]} \; \frac{\langle 12\rangle [34]}{\langle 13\rangle [24]} \; \underbrace{(\varepsilon_1\cdot p_2)} \; \underbrace{(p_3\cdot\varepsilon_4)}$$

$$\varepsilon_-(p_1;p_4)\cdot p_2 = -\frac{\langle 1|2|4]}{\sqrt{2}\,[14]} = +\frac{\langle 12\rangle [24]}{\sqrt{2}\,[14]}$$

$$\varepsilon_+(p_4;p_1)\cdot p_3 = -\frac{\langle 1|3|4]}{\sqrt{2}\,\langle 14\rangle} = +\frac{\langle 13\rangle [34]}{\sqrt{2}\,\langle 14\rangle}$$

$$= \frac{-2g^2}{\langle 12\rangle [12]} \; \frac{\langle 12\rangle [34]}{\cancel{\langle 13\rangle} \cancel{[24]}} \; \frac{\langle 12\rangle \cancel{[24]}}{\sqrt{2}\,[14]} \; \frac{\cancel{\langle 13\rangle} [34]}{\sqrt{2}\,\langle 14\rangle}$$

$$= -g^2 \; \frac{\langle 12\rangle^2}{\langle 12\rangle \; \langle 14\rangle} \; \frac{[34]^2}{[12]\,[14]}$$

$$[34) = \frac{[34]\langle 34\rangle}{\langle 34\rangle} = s_{34} = s_{12} = \frac{[12]\langle 12\rangle}{\langle 34\rangle}$$

$$= -g^2 \; \frac{\langle 12\rangle^3}{\langle 12\rangle\langle 14\rangle\langle 34\rangle} \; \frac{[34]\cancel{[12]}}{\cancel{[12]}\,[14]}$$

Momentum Cons. $|1\rangle[1| + |2\rangle[2| + |3\rangle[3| + |4\rangle[4| = 0.$

$$\Rightarrow \langle 21\rangle [14] + \langle 23\rangle [34] = 0$$

$$\Rightarrow \frac{[34]}{[14]} = \frac{\langle 12\rangle}{\langle 23\rangle}$$

$$= -g^2 \; \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 14\rangle \langle 34\rangle \langle 23\rangle}$$

$$= g^2 \; \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 41\rangle \langle 34\rangle \langle 23\rangle}$$

2.26 use $A_4[1^- 2^- 3^+ 4^+]$ to show that the $U(1)$ decoupling formula holds.

$$A_4[1^-2^-3^+4^+] + A_4[2^-1^-3^+4^+] + A_4[2^-3^+1^-4^+] \stackrel{?}{=} 0.$$

$$A_4[1^-2^-3^+4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A_4[2^-1^-3^+4^+] = \frac{\langle 21\rangle^4}{\langle 21\rangle\langle 13\rangle\langle 34\rangle\langle 42\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 24\rangle}$$

$$A_4[2^-3^+1^-4^+] = \frac{\langle 12\rangle^4}{\langle 32\rangle\langle 31\rangle\langle 14\rangle\langle 42\rangle} = \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 14\rangle\langle 24\rangle\langle 23\rangle}$$

applying perm to denom only to preserve little group scaling

$$\begin{aligned}
A_4[1^-2^-3^+4^+] + A_4[2^-1^-3^+4^+] &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 34\rangle}\left(\frac{1}{\langle 13\rangle\langle 24\rangle} + \frac{1}{\langle 23\rangle\langle 41\rangle}\right)\\
&= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 34\rangle}\;\frac{\langle 23\rangle\langle 41\rangle + \langle 13\rangle\langle 24\rangle}{\langle 13\rangle\langle 24\rangle\langle 23\rangle\langle 41\rangle}\\
&= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 34\rangle}\;\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle\langle 23\rangle\langle 41\rangle}\\
&= \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 24\rangle\langle 23\rangle\langle 41\rangle}\\
&= -A_4[2^-3^+1^-4^+]
\end{aligned}$$

where we have used the schouten identity with $i=1, j=2, k=3, r=4$

$$\langle 23\rangle\langle 41\rangle + \langle 13\rangle\langle 24\rangle + \langle 12\rangle\langle 43\rangle = 0.$$

$$\Rightarrow \langle 23\rangle\langle 41\rangle\langle 24\rangle = +\langle 12\rangle\langle 34\rangle$$

Therefore, $A_4[1^-2^-3^+4^+] + A_4[2^-1^-3^+4^+] + A_4[2^-3^+1^-4^+] = 0$