Phys 731: String Theory - Assignment 2

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Problem 1

Part (a)

We are given the action

$$S = S_P + S_J,$$

where

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \delta^{ab}\partial_a X(\sigma)\partial_b X(\sigma)$$
$$S_J = \int d^2\sigma\, J(\sigma)X(\sigma).$$

Here, $\sigma^a \in (-\infty, \infty)$ like in a regular QFT. First, we write the action in terms of Fourier space

$$\begin{aligned}
S &= \int d^2\sigma \left[\frac{1}{4\pi\alpha'}\delta^{ab}\partial_a X(\sigma)\partial_b X(\sigma) + J(\sigma)X(\sigma)\right] \\
&= \int d^2\sigma \left[\frac{-1}{4\pi\alpha'}X(\sigma)\partial^2 X(\sigma) + J(\sigma)X(\sigma)\right] \\
&= \int d^2\sigma \int \frac{d^2\kappa}{(2\pi)^2}\frac{d^2\kappa'}{(2\pi^2)}\left[\frac{\kappa'^2}{4\pi\alpha'}\tilde{X}(\kappa)\tilde{X}(\kappa') + \frac{1}{2}\tilde{J}(\kappa)\tilde{X}(\kappa') + \frac{1}{2}\tilde{J}(\kappa')\tilde{X}(\kappa)\right]e^{i\sigma\cdot(\kappa+\kappa')} \\
&= \int \frac{d^2\kappa}{(2\pi)^2}\left[\frac{\kappa^2}{4\pi\alpha'}\tilde{X}(\kappa)\tilde{X}(-\kappa) + \frac{1}{2}\tilde{J}(\kappa)\tilde{X}(-\kappa) + \frac{1}{2}\tilde{J}(-\kappa)\tilde{X}(\kappa)\right].
\end{aligned}$$

The generating functional is

$$\begin{aligned}
Z[J] &= \int \mathcal{D}X\, e^{-S} \\
&= \int \mathcal{D}X\, \exp\left[-\int \frac{d^2\kappa}{(2\pi)^2}\left[\frac{\kappa^2}{4\pi\alpha'}\tilde{X}(\kappa)\tilde{X}(-\kappa) + \frac{1}{2}\tilde{J}(\kappa)\tilde{X}(-\kappa) + \frac{1}{2}\tilde{J}(-\kappa)\tilde{X}(\kappa)\right]\right] \\
&= \int \mathcal{D}X\, \exp\left[-\int \frac{d^2\kappa}{(2\pi)^2}\frac{\kappa^2}{4\pi\alpha'}\left[\tilde{X}(\kappa)\tilde{X}(-\kappa) + \frac{2\pi\alpha'}{\kappa^2}\tilde{J}(\kappa)\tilde{X}(-\kappa) + \frac{2\pi\alpha'}{\kappa^2}\tilde{J}(-\kappa)\tilde{X}(\kappa)\right]\right].
\end{aligned}$$

We can change integration variables to put this is the form of a Gaussian integral. If we let

$$\tilde{X}(\kappa) = \tilde{\chi}(\kappa) - \frac{2\pi\alpha'}{\kappa^2}\tilde{J}(\kappa),$$

then the measure of the path integral

$$\mathcal{D}X(x) \propto \prod_x dX(x)$$

does not change under such a constant shift

$$\mathcal{D}X(x) = \mathcal{D}\chi(x)$$

and the path integral becomes Gaussian

$$\begin{aligned}
Z[\tilde{J}] &= \int \mathcal{D}X \, \exp\left[-\int \frac{d^2\kappa}{(2\pi)^2} \frac{\kappa^2}{4\pi\alpha'} \left[\tilde{X}(\kappa)\tilde{X}(-\kappa) + \frac{2\pi\alpha'}{\kappa^2}\tilde{J}(\kappa)\tilde{X}(-\kappa) + \frac{2\pi\alpha'}{\kappa^2}\tilde{J}(-\kappa)\tilde{X}(\kappa)\right]\right] \\
&= \int \mathcal{D}\chi \, \exp\left[-\int \frac{d^2\kappa}{(2\pi)^2} \frac{\kappa^2}{4\pi\alpha'} \left[\tilde{\chi}(\kappa)\tilde{\chi}(-\kappa) + \left(\frac{2\pi\alpha'}{\kappa^2}\right)^2 \tilde{J}(-\kappa)\tilde{J}(\kappa)\right]\right] \\
&= \exp\left[-\int \frac{d^2\kappa}{(2\pi)^2} \frac{\pi\alpha'}{\kappa^2}\tilde{J}(-\kappa)\tilde{J}(\kappa)\right] \int \mathcal{D}\chi \, \exp\left[-\frac{\int d^2\kappa}{(2\pi)^2} \frac{\kappa^2}{4\pi\alpha'}\tilde{\chi}(\kappa)\tilde{\chi}(-\kappa)\right] \\
&= Z_0 \exp\left[-\pi\alpha' \int \frac{d^2\kappa}{(2\pi)^2} \tilde{J}(\kappa)\frac{1}{\kappa^2}\tilde{J}(-\kappa)\right] \\
&= Z_0 \exp\left[-\pi\alpha' \int \frac{d^2\kappa}{(2\pi)^2} \tilde{J}(\kappa)\tilde{\Delta}(\kappa)\tilde{J}(-\kappa)\right]
\end{aligned}$$

where

$$Z_0 = Z[\tilde{J}]|_{\tilde{J}=0} = \int \mathcal{D}\chi \, \exp\left[-\int \frac{d^2\kappa}{(2\pi)^2} \frac{\kappa^2}{4\pi\alpha'}\tilde{\chi}(\kappa)\tilde{\chi}(-\kappa)\right]$$

is a (possibly infinite) constant and

$$\tilde{\Delta}(\kappa) = \frac{1}{\kappa^2}$$

is the massless propagator. This can be recast as a functional of J

$$Z[J] = Z_0 \exp\left[-\pi\alpha' \int d^2\sigma d^2\sigma' \, J(\sigma)\Delta(\sigma - \sigma')\tilde{J}(\sigma')\right]$$

where

$$\Delta(\sigma - \sigma') = \int \frac{d^2\kappa^2}{(2\pi)^2} \frac{e^{i\kappa\cdot(\sigma-\sigma')}}{\kappa^2}$$

is the position space massless propagator.

Part (b)

We are asked to find the momentum space propagator by expanding $Z[\tilde{J}]/Z_0$ to second order in $\tilde{J}$. Expanding,

$$\begin{aligned}
Z[\tilde{J}] = \exp&\left[-\pi\alpha' \int \frac{d^2\kappa}{(2\pi)^2} \tilde{J}(\kappa)\frac{1}{\kappa^2}\tilde{J}(-\kappa)\right] \\
1 - \pi\alpha' &\int \frac{d^2\kappa}{(2\pi)^2} \tilde{J}(\kappa)\frac{1}{\kappa^2}\tilde{J}(-\kappa) + \dots
\end{aligned}$$

we see that

$$\langle 0|T\,\tilde{X}(\kappa)\tilde{X}(\kappa')|0\rangle = \frac{\delta^2(\kappa-\kappa')}{\kappa^2}.$$

Part (c)

In this problem we are asked to explicitly evaluate the position space propagator

$$\begin{aligned}
\Delta(\sigma-\sigma') &= \int \frac{d^2\kappa^2}{(2\pi)^2}\frac{e^{i\kappa\cdot(\sigma-\sigma')}}{\kappa^2} \\
&= \int \frac{d\kappa}{(2\pi)}\frac{1}{\kappa}\int\frac{d\theta}{(2\pi)}e^{i\kappa|\sigma-\sigma'|\cos\theta} \\
&= \int \frac{d\kappa}{(2\pi)}\frac{J_0\left(\kappa\left|\sigma-\sigma'\right|\right)}{\kappa}
\end{aligned}$$

where the integral representation for BesselJ is

$$J_{n\in\mathbb{N}}(x) = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{i(n\theta-x\sin\theta)} = e^{i\frac{n\pi}{2}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(n\theta-x\cos\theta)}.$$

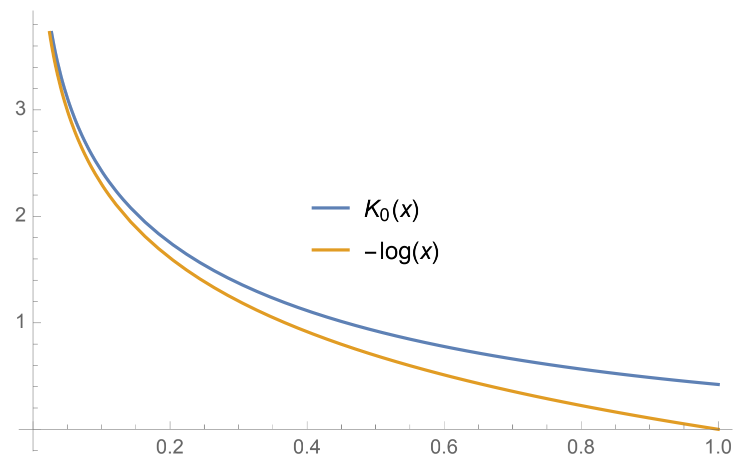
As expected, we see that this integral diverges. The divergence comes from the IR part of the integral. In the IR region, $\frac{J_0\left(\kappa\left|\sigma-\sigma'\right|\right)}{\kappa} \approx \frac{1}{\kappa} - \frac{\kappa\left|\sigma-\sigma'\right|^2}{4} + \ldots$ which diverges as $k\to 0$. This integral is UV finite. For $\kappa \gg 1$, $J_0 \sim \sin \sim \cos$ and the integrand drops off like $1/\kappa \to 0$.

Part (d)

Adding an extra mass to the propagator we obtain

$$\begin{aligned}
\Delta_m(\sigma-\sigma') &= \int \frac{d^2\kappa^2}{(2\pi)^2}\frac{e^{i\kappa\cdot(\sigma-\sigma')}}{\kappa^2+m^2} \\
&= \int \frac{d\kappa}{(2\pi)}\frac{1}{\kappa}\int\frac{d\theta}{(2\pi)}e^{i\kappa|\sigma-\sigma'|\cos\theta} \\
&= \int \frac{d\kappa}{(2\pi)}\frac{\kappa}{\kappa^2+m^2}J_0\left(\kappa\left|\sigma-\sigma'\right|\right) \\
&= K_0\left(m\left|\sigma-\sigma'\right|\right)
\end{aligned}$$

where K_0 is a modified Bessel function of the second kind. This last integral was performed in Mathematica; I have found no such integral tabulated in references such as Abramowitz and Stegun. Now in the limit of small separation, $|\sigma-\sigma'| \ll 1$, the propagator is logarithmic $K_0\left(m\left|\sigma-\sigma'\right|\right) \sim -\log\left(m\left|\sigma-\sigma'\right|\right)$.



Part (e)

Differentiating the above result we obtain

$$\begin{aligned}
\langle 0|T\partial_a X(\sigma)\partial_b' X(\sigma')|0\rangle &= \partial_a\partial_b'\langle 0|TX(\sigma)X(\sigma')|0\rangle = \partial_a\partial_b' K_0\left(m\left|\sigma-\sigma'\right|\right) \\
\langle 0|T\partial_1 X(\sigma)\partial_1' X(\sigma')|0\rangle &= -\frac{m^2\left(\Sigma^1\right)^1}{|\Sigma|^2}K_0\left(m\left|\Sigma\right|\right) - \frac{\left(\Sigma^1-\Sigma^2\right)\left(\Sigma^1+\Sigma^2\right)}{|\Sigma|^{3/2}}K_1\left(m\left|\Sigma\right|\right) \\
\langle 0|T\partial_1 X(\sigma)\partial_2' X(\sigma')|0\rangle &= -\frac{m^2\Sigma^1\Sigma^2}{|\Sigma|^2}K_2\left(m\left|\Sigma\right|\right) \\
\langle 0|T\partial_2 X(\sigma)\partial_1' X(\sigma')|0\rangle &= -\frac{m^2\Sigma^1\Sigma^2}{|\Sigma|^2}K_2\left(m\left|\Sigma\right|\right) \\
\langle 0|T\partial_2 X(\sigma)\partial_2' X(\sigma')|0\rangle &= -\frac{m^2\left(\Sigma^2\right)^1}{|\Sigma|^2}K_0\left(m\left|\Sigma\right|\right) + \frac{\left(\Sigma^1-\Sigma^2\right)\left(\Sigma^1+\Sigma^2\right)}{|\Sigma|^{3/2}}K_1\left(m\left|\Sigma\right|\right)
\end{aligned}$$

where $\Sigma^a = \sigma^a - \sigma^{a\prime}$.

Part (f)

The complex coordinates are defined to be

$$\begin{aligned}
z &= \sigma^1 + i\sigma^2 \\
\bar{z} &= \sigma^1 - i\sigma^2.
\end{aligned}$$

The scalar product in complex coordinates is

$$(\sigma-\sigma')\cdot(\sigma-\sigma') = (z-z')(\bar{z}-\bar{z}') = (z-z')\overline{(z-z')} = \left|z-z'\right|^2.$$

With the above, we can rewrite the results of part (d) and (e):

$$\langle 0|TX(\sigma)X(\sigma')|0\rangle = K_0\left(m\left|z-z'\right|\right),$$

$$\langle 0|T\partial_a X(\sigma)\partial'_b X(\sigma')|0\rangle = -\frac{m^2}{4}\begin{pmatrix} \frac{\bar{z}-\bar{z}'}{z-z'}K_2\left(m\left|z-z'\right|\right) & K_0\left(m\left|z-z'\right|\right) \\ K_0\left(m\left|z-z'\right|\right) & \frac{z-z'}{\bar{z}-\bar{z}'}K_2\left(m\left|z-z'\right|\right) \end{pmatrix}.$$

Part (g)

Expanding the generating functional to 4^{th} order in J yields

$$\begin{aligned}\frac{Z[J]}{Z_0} &= \exp\left[-\pi\alpha'\int d^2\sigma d^2\sigma'\, J(\sigma)\Delta(\sigma-\sigma')J(\sigma')\right] \\ &= 1-\pi\alpha'\int d^2\sigma d^2\sigma'\, J(\sigma)\Delta(\sigma-\sigma')J(\sigma') \\ &\quad + \frac{(\pi\alpha')^2}{2}\int d^2\sigma_1 d^2\sigma_2 d^2\sigma_3 d^2\sigma_4\, J(\sigma_1)\Delta(\sigma_1-\sigma_2)J(\sigma_2)J(\sigma_3)\Delta(\sigma_3-\sigma_4)J(\sigma_4)\end{aligned}$$

The four point correlation function is related to the third term above

$$\begin{aligned}&\frac{(\pi\alpha')^2}{2}\int d^2\sigma_1 d^2\sigma_2 d^2\sigma_3 d^2\sigma_4\, J(\sigma_1)J(\sigma_2)J(\sigma_3)J(\sigma_4)\Delta(\sigma_1-\sigma_2)\Delta(\sigma_3-\sigma_4) \\ &= \frac{(\pi\alpha')^2}{2}\int d^2\sigma_1 d^2\sigma_2 d^2\sigma_3 d^2\sigma_4\, J(\sigma_1)J(\sigma_2)J(\sigma_3)J(\sigma_4) \\ &\quad \times \frac{1}{3}\left(\Delta(\sigma_1-\sigma_2)\Delta(\sigma_3-\sigma_4)+\Delta(\sigma_1-\sigma_3)\Delta(\sigma_2-\sigma_4)+\Delta(\sigma_1-\sigma_4)\Delta(\sigma_2-\sigma_3)\right)\end{aligned}$$

which implies

$$\langle 0|X(\sigma_1)X(\sigma_2)X(\sigma_3)X(\sigma_4)|0\rangle = \Delta(\sigma_1-\sigma_2)\Delta(\sigma_3-\sigma_4)+\Delta(\sigma_1-\sigma_3)\Delta(\sigma_2-\sigma_4)+\Delta(\sigma_1-\sigma_4)\Delta(\sigma_2-\sigma_3).$$