# 1 Problem 1

a) Find the open string amplitude for one tachyon and two massless vectors. Show that it is non-vanishing even without Chan-Paton degrees of freedom, and also give the form with Chan-Paton wavefunctions included.

b) Find a gauge invariant Lagrangian that produces such an amplitude.

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## 1.1 Part (a)

The open string tachyon and massless vector states are given by

$$\begin{aligned} \alpha_0^\mu \left|0;k\right\rangle &\quad \text{with} \quad m_0^2 = -\alpha' \\ e_\mu \alpha_1^\mu \left|0;k\right\rangle &\quad \text{with} \quad m_1^2 = 0 \end{aligned}$$

where the polarizations of the massless vector states satisfy $e \cdot k = 0$. The vertex operators for the tachyon and two massless vectors are, respectively,

$$\begin{aligned} \mathcal{V}_1 &= g_o : e^{ik_1 \cdot X(y_1, y_1)} : \\ \mathcal{V}_2 &= -\frac{i g_o}{\sqrt{2\alpha'}} e_{2\mu} : \dot{X}^\mu e^{ik_2 \cdot X(y_2,y_2)} : \\ \mathcal{V}_3 &= -\frac{i g_o}{\sqrt{2\alpha'}} e_{3\mu} : \dot{X}^\mu e^{ik_3 \cdot X(y_3,y_3)} : \end{aligned}$$

where $y_i$ are the insertion points of the operators on the boundary.

For three particle scattering the $S$-matrix is

$$\begin{aligned} S\left(k_1, k_2, k_3\right) = & \sum_{\substack{\text{compact} \\ \text{topologies}}} \int \frac{[dX\, dg]}{V_{\text{diff}\times\text{Weyl}}} \exp\left(-S_X - \lambda\chi\right) \prod_{i=1}^{3} \int d^2\sigma_i \sqrt{g(\sigma_i)} \mathcal{V}_i\left(k_i, \sigma_i\right) \\ & \underset{\text{at tree level}}{\longrightarrow} -\frac{g_o^3}{2\alpha'} e^{-\lambda} \text{Jacobian}(g) \left\langle {}^{\star}_{\star} e^{ik_1\cdot X_1} {}^{\star\star}_{\star\star} e_2 \cdot \dot{X}_2 e^{ik_2 \cdot X_2} {}^{\star\star}_{\star\star} e_3 \cdot \dot{X}_3 e^{ik_3\cdot X_3} {}^{\star}_{\star} \right\rangle \\ & \quad + \left(y_2 \leftrightarrow y_3\right) \end{aligned}$$

where $X_i = X(y_i, y_i)$. Only the fully contracted terms in of the OPE for ${}^{\star}_{\star} e^{ik_1\cdot X_1} {}^{\star\star}_{\star\star} e_2 \cdot \dot{X}_2 e^{ik_2 \cdot X_2} {}^{\star\star}_{\star\star} e_3 \cdot \dot{X}_3 e^{ik_3\cdot X_3} {}^{\star}_{\star}$ survive the expectation value. Therefore, the relevant terms in the OPE are

$$\begin{aligned} {}^{\star}_{\star} e^{ik_1\cdot X_1} {}^{\star\star}_{\star\star} e_2 \cdot \dot{X}_2 e^{ik_2 \cdot X_2} {}^{\star\star}_{\star\star} e_3 \cdot \dot{X}_3 e^{ik_3\cdot X_3} {}^{\star}_{\star} \sim 2\alpha' \Bigg[ 2\alpha' \Bigg( & \frac{\left(k_3\cdot e_2\right)\left(k_1 \cdot e_3\right)}{y_{23} y_{13}} + \frac{\left(k_3\cdot e_2\right)\left(k_2 \cdot e_3\right)}{y_{23}^2} - \frac{\left(k_1\cdot e_2\right)\left(k_1 \cdot e_3\right)}{y_{12} y_{13}} \\ & - \frac{\left(k_1\cdot e_2\right)\left(k_2 \cdot e_3\right)}{y_{12} y_{23}} \Bigg) - \frac{\left(e_2 \cdot e_3\right)}{y_{23}^2} \Bigg] + y_{12}^{2\alpha' k_1\cdot k_2} y_{13}^{2\alpha' k_1\cdot k_2} y_{23}^{2\alpha' k_2 \cdot k_3} \end{aligned}$$

(see Appendix for the details). Simplify using momentum conservation $k_1 + k_2 + k_3 = 0$ and $e_i \cdot k_i = 0$, we obtain

$${}^{\star}_{\star} e^{ik_1\cdot X_1} {}^{\star\star}_{\star\star} e_2 \cdot \dot{X}_2 e^{ik_2 \cdot X_2} {}^{\star\star}_{\star\star} e_3 \cdot \dot{X}_3 e^{ik_3\cdot X_3} {}^{\star}_{\star} \sim \frac{2\alpha'}{y_{23}^2} \left[2\alpha' \left(k_3 \cdot e_2\right)\left(k_2 \cdot e_3\right) - \left(e_2 \cdot e_3\right)\right] \frac{|y_{23}|}{|y_{12}||y_{13}|}.$$

The other cyclic ordering yield the same result. Therefore, the scattering amplitude is

$$2\left(2\pi\right)^D \delta^D\left(\sum k\right) \left(i g_o \left[\left(e_2\cdot e_3\right) - 2\alpha' \left(k_3 \cdot e_2\right)\left(k_2 \cdot e_3\right)\right]\right)$$

where we have used

$$e^{-\lambda} = \frac{i}{\alpha' g_o^2}$$
$$\text{Jacobian}(g) = |y_{12} y_{13} y_{23}| \,.$$

To include the Chan-Paton factor we multiply by $\frac{1}{2}\text{Tr}\left[\lambda_a \left\{\lambda_b, \lambda_c\right\}\right]$

$$(2\pi)^D \delta^D \left(\sum k\right) \left(i g_o \left[(e_2 \cdot e_3) - 2\alpha' (k_3 \cdot e_2)(k_2 \cdot e_3)\right]\right) \text{Tr}\left[\lambda_a \left\{\lambda_b, \lambda_c\right\}\right].$$

## 1.2 Part (b)

The trace part looks like the trace over non-Abelian group generators. Since the tachyon is a scalar, we guess that the Lagrangian should be a scalar field coupled to a non-abelian gauge field

$$\begin{aligned} L &= -\frac{1}{4}F^2 - \frac{1}{2}\left((D_\mu \phi)^2 - \frac{1}{\alpha'}\phi^2\right) + \text{Tr}\left[-\frac{1}{4}F^2 - \frac{1}{2}\left((D_\mu \phi^a)^2 - \frac{1}{\alpha'}\phi^2\right) + g_o F^a_{\mu\nu} F^{b\mu\nu} \phi^c \lambda^a \lambda^b \lambda^c\right] \\ &= -\frac{1}{4}F^2 - \frac{1}{2}\left((D_\mu \phi)^2 - \frac{1}{\alpha'}\phi^2\right) + g_o F^a_{\mu\nu} F^{b\mu\nu} \phi^c \text{Tr}\left[\lambda^a \lambda^b \lambda^c\right] \end{aligned}$$

where $\phi$ has a gauge index. Here

$$\begin{aligned} -\frac{1}{4}F^2 &\quad \text{gauge boson kinetic term,} \\ -\frac{1}{2}(D_\mu \phi)^2 &\quad \text{tachyon kinetic and 4pt. interaction,} \\ \frac{1}{2\alpha'}\phi^2 &\quad \text{tachyon mass term } \left(m_{\text{tachyon}} = -\frac{1}{\alpha'}\right), \\ g_o F^2 &\quad \text{3pt. interaction term needed to get } A - A - \phi \text{ scattering.} \end{aligned}$$

We need to calculate the 3pt interaction term to get the $A - A - \phi$ vertex

$$\begin{aligned} g_o \phi^a F^b_{\mu\nu} F^{c\mu\nu} \text{Tr}\left[\lambda^a \lambda^b \lambda^c\right] &= \frac{g_o}{2} \phi^a F^b_{\mu\nu} F^{c\mu\nu} \text{Tr}\left[\lambda^a \left\{\lambda^b, \lambda^c\right\}\right] \\ &= \frac{g_o}{2}\phi^a \left(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu + g_o f^{bde} A^d_\mu A^e_\nu\right)\left(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu} + g_o f^{cfg} A^{f\mu} A^{g\nu}\right) \text{Tr}\left[\lambda^a \left\{\lambda^b, \lambda^c\right\}\right] \\ &= g_o \phi^a \Big[\partial_\mu A^b_\nu \partial^\mu A^{c\nu} - \partial_\mu A^b_\nu \partial^\nu A^{c\mu} + \frac{g_o}{2}\left(f^{cde}\partial_\mu A^b_\nu + f^{bde}\partial_\mu A^c_\nu\right) A^{d\mu} A^{e\nu} \\ &\quad - \frac{g_o}{2}\left(f^{cde}\partial_\nu A^b_\mu - f^{bde}\partial_\nu A^c_\mu\right) A^{d\mu} A^{e\nu} + \frac{g_o^2}{2} f^{bde} f^{cfg} A^d_\mu A^e_\nu A^{f\mu} A^{g\nu}\Big] \text{Tr}\left[\lambda^a \left\{\lambda^b, \lambda^c\right\}\right]. \end{aligned}$$

From the above equation we can read off the $A - A - \phi$ vertex

$$-2 i g_o \left(-(k_2 \cdot k_3)(e_2 \cdot e_3) + (k_2 \cdot e_3)(e_2 \cdot k_3)\right) \text{Tr}\left[\lambda^a \left\{\lambda^b, \lambda^c\right\}\right]$$

where all momentum are incoming and the factor of 2 comes from permuting the legs of the diagram. For on-shell scattering,

$$k_1 = -k_2 - k_3$$
$$-k_1^2 = -2k_2 \cdot k_3 = -\frac{1}{\alpha'}$$

and the amplitude becomes

$$\frac{i g_o}{\alpha'}\left((e_2 \cdot e_3) - 2(k_2 \cdot e_3)(e_2 \cdot k_3)\right) \text{Tr}\left[\lambda^a \left\{\lambda^b, \lambda^c\right\}\right]$$

which matches the earlier result.

### 1.3 Appendix for 1(a)

- We need to calculate the full OPE for ${}^\star_\star e^{ik_1\cdot X_1}{}^{\star\star}_{\star\star} e_2\cdot \dot{X}_2 e^{ik_2\cdot X_2}{}^\star_\star$. In general, the normal ordered product of two operators is

$$:\mathcal{F}::\mathcal{G}:=\exp\left(\int d^2z\, d^2\omega\, G\left(z,\bar{z};\omega\bar{\omega}\right)\frac{\delta}{\delta X^\mu_{\mathcal{F}}\left(z,\bar{z}\right)}\frac{\delta}{\delta X_{\mathcal{G}\mu}\left(\omega,\bar{\omega}\right)}\right):\mathcal{F}\mathcal{G}:$$

where

$$G_{\text{open}}\left(z,\bar{z};\omega,\bar{\omega}\right)=-\frac{\alpha'}{2}\ln\left|z-\omega\right|^2-\frac{\alpha'}{2}\ln\left|z-\bar{\omega}\right|^2,$$
$$G_{\text{closed}}\left(z,\bar{z};\omega,\bar{\omega}\right)=-\frac{\alpha'}{2}\ln\left|z-\omega\right|^2.$$

Thus,

$$\begin{aligned}
&:e^{ik_z\cdot X(z,\bar{z})}::\partial_\omega X^\mu\left(\omega,\bar{\omega}\right)e^{ik_\omega\cdot X(\omega,\bar{\omega})}:\\
&=:\exp\left(\int d^2z_1\,d^2\omega_1\,G\left(z_1,\bar{z}_1;\omega_1,\bar{\omega}_1\right)\frac{\delta}{\delta X^\mu_{\mathcal{F}}\left(z_1,\bar{z}_1\right)}\frac{\delta}{\delta X_{\mathcal{G}\mu}\left(\omega_1,\bar{\omega}_1\right)}\right)e^{ik_z\cdot X_{\mathcal{F}}(z,\bar{z})}\partial_\omega X^\mu_{\mathcal{G}}\left(\omega,\bar{\omega}\right)e^{ik_\omega\cdot X_{\mathcal{G}}(\omega,\bar{\omega})}:\\
&=:\sum_{n=0}^{\infty}\frac{1}{n!}\int d^2z_1\ldots d^2z_n\,d^2\omega_1\ldots d^2\omega_n\,G\left(z_1,\bar{z}_1;\omega_1,\bar{\omega}_1\right)\cdots G\left(z_n,\bar{z}_n;\omega_n,\bar{\omega}_n\right)\\
&\quad\times\left[\left(\prod_{i=1}^{n}\frac{\delta}{\delta X_{\mathcal{F}\mu_i}\left(z_i,\bar{z}_i\right)}\right)e^{ik_\omega\cdot X_{\mathcal{F}}(z,\bar{z})}\right]\left[\left(\prod_{i=1}^{n}\frac{\delta}{\delta X_{\mathcal{G}\mu_i}\left(\omega_i,\bar{\omega}_i\right)}\right)\partial_\omega X^\mu_{\mathcal{G}}\left(\omega,\bar{\omega}\right)e^{ik_\omega\cdot X_{\mathcal{G}}(\omega,\bar{\omega})}\right]:\\
&=ik_z^\mu\left(\sum_{n=1}^{\infty}\frac{n}{n!}G^{n-1}\left(z,\bar{z};\omega,\bar{\omega}\right)\left(-k_z\cdot k_\omega\right)^{n-1}\right)\partial_\omega G\left(z,\bar{z};\omega,\bar{\omega}\right):e^{ik_z\cdot X(z,\bar{z})}e^{ik_\omega\cdot X(\omega,\bar{\omega})}:\\
&\quad+\left(\sum_{n=0}^{\infty}\frac{\left(-k_z\cdot k_\omega\right)^n G^n\left(z,\bar{z};\omega,\bar{\omega}\right)}{n!}\right):\partial_\omega X^\mu_{\mathcal{G}}\left(\omega,\bar{\omega}\right)e^{ik_z\cdot X(z,\bar{z})}e^{ik_\omega\cdot X(\omega,\bar{\omega})}:\\
&=ik_z^\mu\exp\left(-k_z\cdot k_\omega G\left(z,\bar{z};\omega,\bar{\omega}\right)\right)\partial_\omega G\left(z,\bar{z};\omega,\bar{\omega}\right):e^{ik_z\cdot X(z,\bar{z})}e^{ik_\omega\cdot X(\omega,\bar{\omega})}:\\
&\quad+\exp\left(-k_z\cdot k_\omega G\left(z,\bar{z};\omega,\bar{\omega}\right)\right):\partial_\omega X^\mu_{\mathcal{G}}\left(\omega,\bar{\omega}\right)e^{ik_z\cdot X(z,\bar{z})}e^{ik_\omega\cdot X(\omega,\bar{\omega})}:
\end{aligned}$$

where we have used the result

$$\begin{aligned}
&\left(\prod_{i=1}^{n}\frac{\delta}{\delta X_{\mathcal{G}\mu_i}\left(\omega_i,\bar{\omega}_i\right)}\right)\partial_\omega X^\mu_{\mathcal{G}}\left(\omega,\bar{\omega}\right)e^{ik_\omega\cdot X_{\mathcal{G}}(\omega,\bar{\omega})}\\
&=\left(\prod_{i=1}^{n}ik_{\omega\mu_i}\delta\left(\omega_i-\omega,\bar{\omega}_i-\bar{\omega}\right)\right)\partial_\omega X^\mu_{\mathcal{G}}\left(\omega,\bar{\omega}\right)e^{ik_\omega\cdot X_{\mathcal{G}}(\omega,\bar{\omega})}\\
&\quad+\sum_{i=1}^{n}\eta^\mu_{\mu_j}\partial_\omega\delta\left(\omega_i-\omega,\bar{\omega}_i-\bar{\omega}\right)\left(\prod_{j\neq1}^{n}ik_{\omega\mu_i}\delta\left(\omega_j-\omega,\bar{\omega}_j-\bar{\omega}\right)\right)e^{ik_\omega\cdot X_{\mathcal{G}}(\omega,\bar{\omega})}.
\end{aligned}$$

- Specifically, we obtain

$${}^\star_\star e^{ik_1\cdot X(y_1,y_1)}{}^{\star\star}_{\star\star}\partial_{y_2}X^\mu\left(y_2,y_2\right)e^{ik_2\cdot X(y_2,y_2)}{}^\star_\star=2i\alpha'\frac{\left(k_1\cdot e_2\right)}{y_{12}}y_{12}^{2\alpha' k_1\cdot k_2}{}^\star_\star e^{ik_1\cdot X_1}e^{ik_2\cdot X_2}{}^\star_\star+y_{12}^{2\alpha' k_1\cdot k_2}{}^\star_\star e_2\cdot\dot{X}_2 e^{ik_1\cdot X_1}e^{ik_2\cdot X_2}{}^\star_\star$$

- The OPE for the full operator is

$$\begin{aligned}
{}^\star_\star e^{ik_1\cdot X_1}{}^{\star\star}_{\star\star}e_2\cdot\dot{X}_2 e^{ik_2\cdot X_2}{}^{\star\star}_{\star\star}e_3\cdot\dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star&=2i\alpha'\frac{\left(k_1\cdot e_2\right)}{y_{12}}y_{12}^{2\alpha' k_1\cdot k_2}{}^\star_\star e^{ik_1\cdot X_1}e^{ik_2\cdot X_2}{}^{\star\star}_{\star\star}e_3\cdot\dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star\\
&\quad+y_{12}^{2\alpha' k_1\cdot k_2}{}^\star_\star e_2\cdot\dot{X}_2 e^{ik_1\cdot X_1}e^{ik_2\cdot X_2}{}^{\star\star}_{\star\star}e_3\cdot\dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star.
\end{aligned}$$

– We need to OPE's for

$$\mathcal{O}_1 = {}^\star_\star e^{ik_1\cdot X_1} e^{ik_2\cdot X_2}{}^{\star\star}_{\star\star} e_3\cdot \dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star$$
$$\mathcal{O}_2 = {}^\star_\star e_2\cdot \dot{X}_2 e^{ik_1\cdot X_1} e^{ik_2\cdot X_2}{}^{\star\star}_{\star\star} e_3\cdot \dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star.$$

At this point we can drop all terms that are not fully contracted. This will make the calculation of the above OPE's much easier.

– The OPE for $\mathcal{O}_1$:

$$\mathcal{O}_1 = {}^\star_\star e^{ik_1\cdot X_1} e^{ik_2\cdot X_2} e_3\cdot \dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star + {}^\star_\star \sum_{CC}\left[e^{ik_1\cdot X_1} e^{ik_2\cdot X_2}, e_3\cdot \dot{X}_3 e^{ik_3\cdot X_3}\right]_{CC}{}^\star_\star$$
$$\sim {}^\star_\star \sum_{m,n,l=0}^{\infty}\frac{1}{m!n!l!}\sum_{CC}\left[(ik_1\cdot X_1)^m (ik_2\cdot X_2)^n, \left(e_3\cdot\dot{X}_3\right)(ik_3\cdot X_3)^l\right]_{CC}{}^\star_\star$$

where $\sum_{CC}[A,B]_{CC}$ is the sum over all cross-contractions of $A$ and $B$ and $\sim$ means equivalent up to terms that are normal ordered. Keeping only fully contracted terms we obtain

$$\begin{aligned}
\mathcal{O}_1 \sim & \left[(ik_1\cdot X_1), \left(e_3\cdot\dot{X}_3\right)\right]_{CC}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m-1)!n!}\left[(ik_1\cdot X_1),(ik_3\cdot X_3)\right]_{CC}^{m-1}\left[(ik_2\cdot X_2),(ik_3\cdot X_3)\right]_{CC}^{n} \\
& + \left[(ik_2\cdot X_2), \left(e_3\cdot\dot{X}_3\right)\right]_{CC}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n-1)!m!}\left[(ik_1\cdot X_1),(ik_3\cdot X_3)\right]_{CC}^{m}\left[(ik_2\cdot X_2),(ik_3\cdot X_3)\right]_{CC}^{n-1} \\
& = \left(\left[(ik_1\cdot X_1), \left(e_3\cdot\dot{X}_3\right)\right]_{CC} + \left[(ik_2\cdot X_2), \left(e_3\cdot\dot{X}_3\right)\right]_{CC}\right) e^{[(ik_1\cdot X_1),(ik_3\cdot X_3)]_{CC}} e^{[(ik_2\cdot X_2),(ik_3\cdot X_3)]_{CC}} \\
& = \left(-2i\alpha'(k_1\cdot e_3)\frac{\partial \ln(y_1-y_3)}{\partial y_3} - 2i\alpha'(k_2\cdot e_3)\frac{\partial\ln(y_2-y_3)}{\partial y_3}\right) e^{2\alpha' k_1\cdot k_3\ln(y_1-y_3)} e^{2\alpha' k_2\cdot k_3 \ln(y_2-y_3)} \\
& = \left(2i\alpha'\frac{(k_1\cdot e_3)}{y_{13}} + 2i\alpha'\frac{(k_2\cdot e_3)}{y_{23}}\right) y_{13}^{2\alpha' k_1\cdot k_3} y_{23}^{2\alpha' k_2\cdot k_3}.
\end{aligned}$$

- The OPE for $\mathcal{O}_2$:

$$
\begin{aligned}
\mathcal{O}_2 &= {}^\star_\star e_2 \cdot \dot{X}_2 e^{ik_1\cdot X_1} e^{ik_2\cdot X_2} e_3 \cdot \dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star + {}^\star_\star \sum_{CC} \left[ e_2 \cdot \dot{X}_2 e^{ik_1\cdot X_1} e^{ik_2\cdot X_2}, e \cdot \dot{X}_3 e^{ik_3\cdot X_3} \right]_{CC} {}^\star_\star \\
&\sim {}^\star_\star \sum_{m,n,l=0}^{\infty} \frac{1}{m!n!l!} \sum_{CC} \left[ \left(e_2 \cdot \dot{X}_2\right) (ik_1 \cdot X_1)^m (ik_2 \cdot X_2)^n , \left(e \cdot \dot{X}_3\right) (ik_3 \cdot X_3)^l \right]_{CC} {}^\star_\star \\
&\sim \left[ \left(e_2 \cdot \dot{X}_2\right), \left(e \cdot \dot{X}_3\right) \right]_{CC} \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \left[(ik_1 \cdot X_1), (ik_3 \cdot X_3)\right]^m_{CC} \left[(ik_2 \cdot X_2), (ik_3 \cdot X_3)\right]^n_{CC} \\
&\quad + \sum_{m=1,n=0}^{\infty} \frac{1}{(m-1)!n!} \left[(ik_1 \cdot X_1), (ik_3 \cdot X_3)\right]^{m-1}_{CC} \left[(ik_2 \cdot X_2), (ik_3 \cdot X_3)\right]^n_{CC} \\
&\qquad \times \left[ \left(e_2 \cdot \dot{X}_2\right), (ik_3 \cdot X_3) \right]_{CC} \left[ \left(e_3 \cdot \dot{X}_3\right), (ik_1 \cdot X_1) \right]_{CC} \\
&\quad + \sum_{m=0,n=1}^{\infty} \frac{1}{m!\,(n-1)!} \left[(ik_1 \cdot X_1), (ik_3 \cdot X_3)\right]^{m}_{CC} \left[(ik_2 \cdot X_2), (ik_3 \cdot X_3)\right]^{n-1}_{CC} \\
&\qquad \times \left[ \left(e_2 \cdot \dot{X}_2\right), (ik_3 \cdot X_3) \right]_{CC} \left[ \left(e_3 \cdot \dot{X}_3\right), (ik_2 \cdot X_2) \right]_{CC} \\
&= \Bigg( \left[ \left(e_2 \cdot \dot{X}_2\right), \left(e_3 \cdot \dot{X}_3\right) \right]_{CC} + \left[ \left(e_2 \cdot \dot{X}_2\right), (ik_3 \cdot X_3) \right]_{CC} \left[ \left(e_3 \cdot \dot{X}_3\right), (ik_1 \cdot X_1) \right]_{CC} \\
&\quad + \left[ \left(e_2 \cdot \dot{X}_2\right), (ik_3 \cdot X_3) \right]_{CC} \left[ \left(e_3 \cdot \dot{X}_3\right), (ik_2 \cdot X_2) \right]_{CC} \Bigg) e^{[(ik_1\cdot X_1),(ik_3\cdot X_3)]_{CC}} e^{[(ik_2\cdot X_2),(ik_3\cdot X_3)]_{CC}} \\
&= \left( -2\alpha' \frac{(e_2 \cdot e_3)}{y_{23}^2} + (2\alpha')^2 \frac{(k_3 \cdot e_2)(k_1 \cdot e_3)}{y_{23} y_{13}} + (2\alpha')^2 \frac{(k_3 \cdot e_2)(k_2 \cdot e_3)}{y_{23}^2} \right) y_{13}^{2\alpha' k_1 \cdot k_2} y_{23}^{2\alpha' k_2 \cdot k_3}
\end{aligned}
$$

- Now we can write the full OPE

$$
\begin{aligned}
{}^\star_\star e^{ik_1\cdot X_1}{}^{\star\star}_{\star\star} e_2 \cdot \dot{X}_2 e^{ik_2\cdot X_2}{}^{\star\star}_{\star\star} e_3 \cdot \dot{X}_3 e^{ik_3\cdot X_3}{}^\star_\star &\sim 2\alpha' \Bigg[ 2\alpha' \Bigg( \frac{(k_3 \cdot e_2)(k_1 \cdot e_3)}{y_{23} y_{13}} + \frac{(k_3 \cdot e_2)(k_2 \cdot e_3)}{y_{23}^2} - \frac{(k_1 \cdot e_2)(k_1 \cdot e_3)}{y_{12} y_{13}} \\
&\quad - \frac{(k_1 \cdot e_2)(k_2 \cdot e_3)}{y_{12} y_{23}} \Bigg) - \frac{(e_2 \cdot e_3)}{y_{23}^2} \Bigg] + y_{12}^{2\alpha' k_1 \cdot k_2} y_{13}^{2\alpha' k_1 \cdot k_2} y_{23}^{2\alpha' k_2 \cdot k_3}
\end{aligned}
$$