PHYS 432 Physics of Fluids

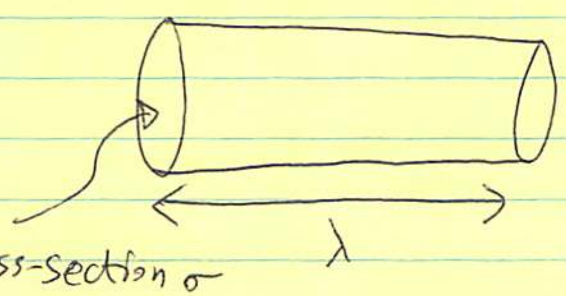
We start by asking "what is a fluid?"

The obvious answer is "something that flows" such as a liquid or gas. A solid has a non-zero shear modulus and can statically support a shear stress, and so we don't think of it as a fluid. But we'll see that solids can be handled by adding a shear modulus to the fluid equations.

In fact, by fluid we mean a material that we can treat as a continuous substance – ie. we don't have to worry about the fact that it is made up of atoms.

The requirement is that the mean free path λ is $\ll L$, the scale on which macroscopic properties such as velocity or temperature vary.

eg. in air



$n\sigma\lambda = 1$ defines the mean free path

density of air

$$\sigma \sim 10^{-20}\,\mathrm{m}^2 \qquad n = \frac{1\,\mathrm{kg/m^3}\ \rho}{28 \times 1.6\times10^{-27}\,\mathrm{kg}} \approx 3\times10^{25}\,\mathrm{m}^{-3}$$

N_2 molecules $\quad m_p$

$$\Rightarrow \quad \lambda = \frac{1}{3\times10^{25}\times10^{-20}} \approx 3\times10^{-6}\,\mathrm{m} = \text{few } \mu\mathrm{m}$$

$\ll$ macroscopic lengthscales

So the flow of air at atmospheric pressure can be studied by treating the air as a fluid, a continuum. Locally, at any given point in space, the particles are in local thermodynamic equilibrium so that we can write for example $P = nk_BT$ for an ideal gas. The temperature at that location measures the random velocities of the particles; we will track the bulk velocity or the average velocity of the particles as the vector field $\underline{u}(\underline{r})$

(at location $\underline{r}$)

bulk velocity of the fluid

Similarly the density, temperature, pressure are functions of position $\underline{r}$, ie. $T(\underline{r})$, $\rho(\underline{r})$, $P(\underline{r})$.

The fluid treatment requires that, for example, $\dfrac{T}{dT/dx} \gg \lambda$.

The Fluid Equations

One route to the fluid equations is via statistical mechanics in which we start with the microscopic description of the material and average over lengthscales $< L$ (expand in the small parameter λ/L).

[The names are Liouvilles theorem $\rightarrow$ Boltzmann equation $\rightarrow$ moments of the Boltzmann equation]

Instead, we'll take a short cut and use conservation laws to derive the fluid equations.

1. <u>Continuity Equation</u> (mass conservation)

$$M = \int \rho \, dV$$

fluid element

fixed boundary in space

$$\frac{dM}{dt} = \frac{d}{dt}\int \rho \, dV = \int \frac{\partial \rho}{\partial t} \, dV$$

$$= -\int \rho \underline{u} \cdot d\underline{S}$$

the mass can change because i) the internal density changes

or 2) because mass flows across the surface of the fluid element

The quantity $\rho \underline{u}$ is the <u>mass flux</u> units $g/cm^2/s$

Now apply the divergence theorem:

$$\int \frac{\partial \rho}{\partial t} \, dV = -\int \underline{\nabla} \cdot (\rho \underline{u}) \, dV$$

but the choice of volume V is arbitrary $\Rightarrow$

$$\boxed{\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{u})}$$

This is the continuity equation, a local expression of mass conservation.

The continuity equation can be rewritten as

$$\underbrace{\left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)}_{\text{this derivative comes up a lot}}\rho = -\rho\,\underline{\nabla}\cdot\underline{u}$$

we write $\boxed{\dfrac{D\rho}{Dt} = -\rho\,\underline{\nabla}\cdot\underline{u}}$

where $\dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}$ is the advective derivative or Lagrangian derivative

We distinguish two different points of view:

EULERIAN and LAGRANGIAN

EULERIAN: describe the fluid properties at fixed points in space

LAGRANGIAN: describe fluid properties following a fluid element

– Check that $\dfrac{D}{Dt}$ is indeed the Lagrangian derivative:

Consider a quantity f (eg. density or temperature or a component of velocity)

write the path of a fluid element as $\underline{r}(t) = \big(x(t), y(t), z(t)\big)$

the velocity of the fluid element is

$$\underline{u} = \frac{d\underline{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

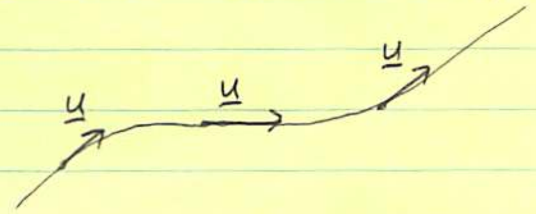
The rate of change of f following the fluid is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z}$$

$$= \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right) f \quad \checkmark$$

– A brief aside on streamlines:

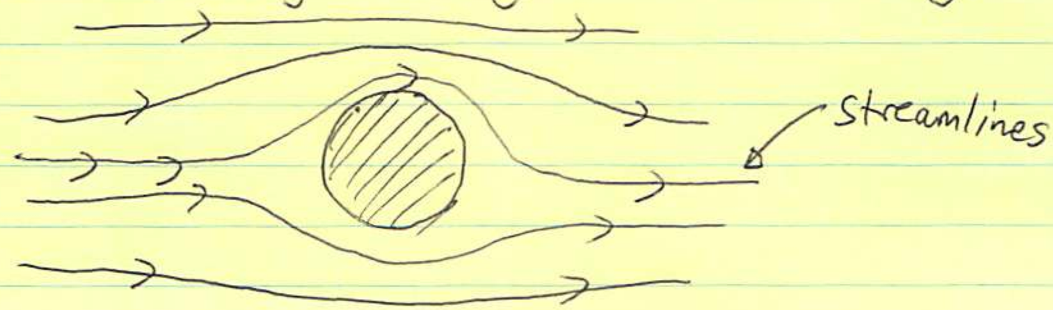
A curve that follows the direction of $\underline{u}$ at a given time is a STREAMLINE



(the tangent to the streamline is always in the direction $\hat{\underline{u}}$)

[These are equivalent to magnetic field lines for a magnetic field $\underline{B}$]

For a steady flow $\left(\frac{\partial}{\partial t} = 0\right)$ the fluid elements follow the streamlines. E.g. steady flow around a cylinder



In that case a quantity f that is constant along a streamline $(\underline{u}\cdot\underline{\nabla}) f = 0$ is also constant for a fluid element, since

$$\frac{Df}{Dt} = \underbrace{\cancel{\frac{\partial f}{\partial t}}}_{0} + \underline{u}\cdot\underline{\nabla} f = 0.$$

2. Momentum Equation

Now consider momentum of our fluid element

$$\frac{d}{dt}\left(\int dV \rho \underline{u}\right) = \underbrace{-\int \rho \underline{u}\, \underline{u}.d\underline{S}}_{} + (\text{forces})$$

momentum flux across the boundary

eg. $\rho u_x u_z$ = flux of x-momentum in the z-direction

This is a vector equation: in component form

$$\frac{d}{dt}\int \rho u_i \, dV = -\int \rho u_i u_j \, dS_j + (\text{forces})_i$$

rate of change of the i-th component of momentum.

Apply the divergence theorem

$$\int \rho u_i u_j \, dS_j = \int dV \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right)$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \left(\begin{matrix}\text{force}\\ \text{term}\end{matrix}\right)}$$

We can simplify this using the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_j}(\rho u_j)$$

cancel because of continuity

$$\Rightarrow \quad \rho \frac{\partial u_i}{\partial t} + \cancel{u_i \frac{\partial \rho}{\partial t}} = \cancel{-u_i \frac{\partial}{\partial x_j}(\rho u_j)} - \rho u_j \frac{\partial}{\partial x_j} u_i$$

$$+ \text{(force term)}$$

$$\Rightarrow \quad \rho \left(\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla} \right) u_i = \text{(force term)}$$

or

$$\boxed{\rho \frac{D u_i}{D t} = (\text{forces})_i}$$

This is just Newton's law $F = ma$ written for the fluid element.

We see here the NON-LINEARITY of the fluid equations in the term $(\underline{u} \cdot \underline{\nabla})\underline{u}$

eg. if we expand in Fourier modes e^{ikx}, this term is

$$(\underline{u} \cdot \underline{\nabla})\underline{u} \propto e^{i2kx}$$

so different spatial modes are coupled to each other, they don't evolve independently as in linear systems. We see this clearly in turbulence, where a large scale stirring (eg. coffee cup) generates a lot of small scale fluid motion.

As we emphasized in the first lecture, qualitatively different flows arise as we change the size of the velocity $\underline{u}$.

Now let's think about what the force term might look like. There are two kinds of forces that could act on the fluid element:

<u>body forces</u> act on each particle in the fluid element

total force $\int \underline{f}\, dV$ $\left(\underline{f} = \text{force per unit volume}\right)$

eg. gravity $\underline{f} = \rho \underline{g}$

<u>surface stress</u> a force acting on the surface of the fluid element

total force $\int \underline{\underline{T}} \cdot d\underline{S} = \int T_{ij}\, dS_j$

where T_{ij} = stress tensor

eg. pressure $T_{ij} = -P\, \delta_{ij}$

$\Rightarrow$ eg. the force in the x-direction is then

$$\int T_{xj}\, dS_j = -\int P\, \hat{\underline{x}} \cdot d\underline{S}$$

the external pressure pushes inwards in a direction opposite to the normal to the surface, so the force at each location on the surface is oppositely directed to the normal. If the ~~force~~ normal is in the x-direction, then the force is in the x-direction, and so on. This means that T_{ij} must be diagonal (non-zero only when $i=j$). Other kinds of forces will not be diagonal (eg. a sideways shearing force on the surface), we'll see examples later.

Using the divergence theorem again,

$$\int T_{ij}\, dS_j = \int \frac{\partial}{\partial x_j} T_{ij}\, dV$$

for pressure this is $-\int \delta_{ij} \frac{\partial P}{\partial x_j}\, dV = -\int \frac{\partial P}{\partial x_i}\, dV$

$\Rightarrow$ the differential form of the momentum equation is

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + f_i + \frac{\partial}{\partial x_j} T_{ij}}$$

or

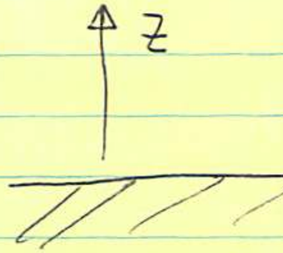
$$\boxed{\rho \frac{D\underline{u}}{Dt} = \underline{f} + \underline{\nabla}\cdot\underline{\underline{T}}}$$

If the forces acting are pressure and gravity only

$$\boxed{\rho \frac{D\underline{u}}{Dt} = \rho \underline{g} - \underline{\nabla} P}$$

Example: Hydrostatic atmosphere

Consider a plane-parallel, static, isothermal atmosphere.



$\downarrow \underline{g} = -g\hat{\underline{z}}$ $\quad\Rightarrow \frac{\partial}{\partial t} = 0, \ \underline{u} = 0$

$$g = \frac{GM}{R^2} = \text{constant}$$

$$\Rightarrow \quad \boxed{\frac{dP}{dz} = -\rho g} \quad \text{equation of hydrostatic balance}$$

To solve this, we need the "equation of state" - the relationship between P and ρ.
For an isothermal ideal gas $\quad P = \frac{\rho k_B T}{\mu m_p}$

$$\Rightarrow \quad \frac{\partial P}{\partial z} = \frac{k_B T}{\mu m_p g} \frac{\partial \rho}{\partial z} = -\rho g$$

$$\Rightarrow \quad \frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{\mu m_p g}{k_B T} = -\frac{1}{H}$$

defines the "scale height" $\boxed{H = \frac{k_B T}{\mu m_p g}}$

the solution is $\boxed{\rho = \rho_0 e^{-z/H}}$

For Earth
$$\left.\begin{aligned} g &= 10\ m/s^2 \\ T &= 300K \\ \mu &= 28 \end{aligned}\right\} \Rightarrow H \simeq 10km.$$

This sounds about right — the height of Everest is $\sim H$
Atmospheric pressure changes by order unity on this scale.

Example: temperature at the center of the Sun

The Sun is a spherical ball of (almost) ideal gas with mass $M = 2\times10^{30}$ kg and $R = 7\times10^8$ m. in hydrostatic balance.

P, ρ, T decrease from large central values P_c, ρ_c, T_c to much smaller values at the surface

Therefore, roughly, $$\frac{dP}{dr} \sim \frac{P_c}{R} \sim \frac{k_B \rho_c T_c}{\mu m_p R}$$

and $$\rho g \simeq \rho_c \frac{GM}{R^2}$$

(we're dropping factors of order unity here, let's try to get the order of magnitude)

$$\Rightarrow \quad \frac{k_B T_c}{\mu m_p} \sim \frac{GM}{R}$$

$$T_c \sim \frac{GM\mu m_p}{R k_B} \sim \underline{10^7 K}.$$

about right! (detailed models give 1.5×10^7K)

Example: the ocean

The key difference between atmosphere and ocean is that the water is incompressible, $\rho =$ constant

Defining z now as increasing downwards into the ocean,

$$\downarrow z \qquad \downarrow \underline{g} = g\hat{z} \qquad\qquad \frac{dP}{dz} = \rho g \quad \Rightarrow \quad P = P_{atm} + \rho g z$$

atmospheric pressure at the top of the ocean

Because water is 1000 times more dense than air, the scale height is not 10 km but instead 10 m!

$$H = \frac{P}{dP/dz} = \frac{P}{\rho g} = \underbrace{\frac{P_{atm}}{\rho g}}_{10\,m} + z$$

(every 10 m depth we gain 1 atm of pressure).

If the depth of the ocean is $\simeq$ 3 km,

the pressure at the ocean floor is $\rho g z \simeq 1000\,\mathrm{kg/m^3} \times 10\,\mathrm{ms^{-2}} \times 3000\,\mathrm{m}$

$= 30$ MPa

$= 300$ atm!

We might worry that the density of water would not be constant when subject to such great pressure, but in fact its compressibility is so small, $-\dfrac{\partial \ln V}{\partial P} \simeq 4 \times 10^{-10}\,\mathrm{Pa^{-1}}$ that it changes its density by only ~~a fraction of a~~ 1%: $4 \times 10^{-10} \times 3 \times 10^{7}\,\mathrm{Pa} \simeq 10^{-2}$.

Summary so far: the fluid equations

We've seen that the fluid motion is described by

continuity $$\frac{D\rho}{Dt} = -\rho \underline{\nabla} \cdot \underline{u}$$

momentum $$\rho \frac{D\underline{u}}{Dt} = \underbrace{-\underline{\nabla} P + \rho \underline{g}}_{\text{more generally this is } \underline{\nabla} \cdot \underline{\underline{T}}}$$

Together with an energy equation, these are the "fluid equations". We'll leave the energy equation for later, often we don't need it.

eg. if (time for heat transfer) $\ll$ (flow time)
we can treat the fluid as isothermal and use the equation of state $P = \rho k_B T / \mu m_p$ to close the equations.

if (time for heat transfer) $\gg$ (flow time)
then the flow is isentropic $P \propto \rho^\gamma$

For an incompressible fluid, we have

$$\left.\begin{aligned} &\underline{\nabla} \cdot \underline{u} = 0 \\ &\frac{D\underline{u}}{Dt} = -\frac{\underline{\nabla} P}{\rho} + \underline{g} \end{aligned}\right\} \text{ these are known as the Euler equations}$$

Proving vector identities

Four things: 1) Einstein summation convention

$$\underline{A} \cdot \underline{B} = A_i B_i$$

2) $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

eg. $\underline{A} \cdot \underline{B} = \delta_{ij} A_i B_j$

3) $\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ even permutation} \quad 123,\ 231,\ 312 \\ -1 & \text{" odd} \quad 132,\ 213,\ 321 \\ 0 & \text{otherwise} \end{cases}$

a way to represent cross-products,

eg. $(\underline{A} \times \underline{B})_i = \varepsilon_{ijk} A_j B_k$

4) identity $\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Examples: 1) $\left[\underline{A} \times (\underline{B} \times \underline{C})\right]_i = \varepsilon_{ijk} A_j \varepsilon_{klm} B_l C_m$

$$= \varepsilon_{ijk}\varepsilon_{klm} A_j B_l C_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m$$
$$= A_j B_i C_j - A_j B_j C_i$$

$$\Rightarrow \quad \underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$

2) $\left[\underline{u} \times (\underline{\nabla} \times \underline{u})\right]_i = \varepsilon_{ijk} u_j \varepsilon_{klm} \partial_l (u_m)$

$= \ldots \quad = u_j \partial_i u_j - u_j \partial_j u_i = \left[\underline{\nabla} \tfrac{1}{2} u^2 - \underline{u} \cdot \underline{\nabla} \underline{u}\right]_i$

Bernoulli's principle

If we write the gravity as the gradient of the gravitational potential $\underline{g} = -\underline{\nabla}\chi$

and if we have a constant density fluid so that

$$\frac{\underline{\nabla} p}{\rho} = \underline{\nabla}\left(\frac{p}{\rho}\right)$$

then the right-hand side of the momentum equation can be written as a gradient:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{u} = -\underline{\nabla}\left(\frac{p}{\rho} + \chi\right)$$

On the left hand side, we can use the identity

$$(\underline{u}.\underline{\nabla})\underline{u} = -\underline{u}\times(\underline{\nabla}\times\underline{u}) + \underline{\nabla}\left(\tfrac{1}{2}u^2\right)$$

$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} - \underline{u}\times(\underline{\nabla}\times\underline{u}) = -\underline{\nabla}\left(\frac{p}{\rho} + \chi + \frac{1}{2}u^2\right) \qquad (*)$$

$$= -\underline{\nabla} H$$

where we define $\boxed{H = \frac{p}{\rho} + \chi + \frac{1}{2}u^2}$

For a steady flow, $\frac{\partial}{\partial t} = 0$ and then equation (*)

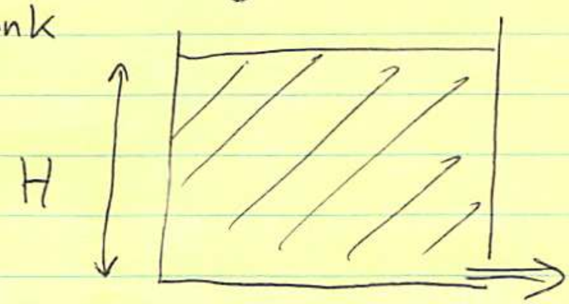
$\Rightarrow \quad \boxed{\underline{u}.\underline{\nabla} H = 0}$ $\quad$ $\boxed{\text{H is constant along streamlines in a steady flow}}$

This is known as Bernoulli's theorem.

If the flow is <u>irrotational</u> $\nabla \times \underline{u} = 0$ then H is not only constant along each streamline, but must be the same constant everywhere.

<u>Examples:</u>

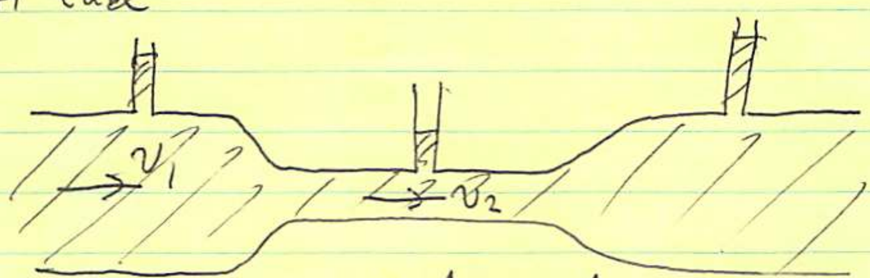
1. water flowing out of a hole at the bottom of a tank



$$v_{out}^2 = 2gH$$

(Q: can we use Bernoulli here? Isn't this a time-dep. problem?)

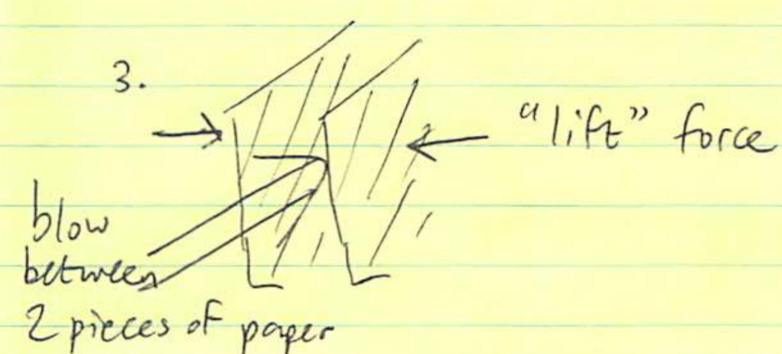
2. Venturi tube



used to measure flow rates

$$\left.\begin{aligned} v_1 A_1 &= v_2 A_2 \\ \frac{\Delta P}{\rho} &= \Delta\left(\tfrac{1}{2}v^2\right) \end{aligned}\right\} \text{can solve for } v_1$$

3.



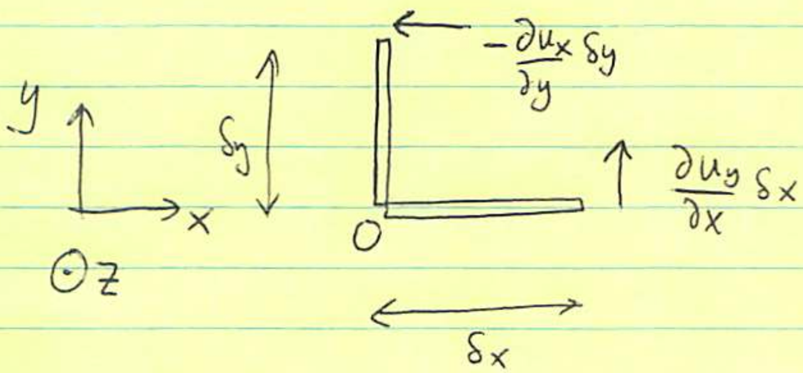
This is perhaps a non-intuitive result: that a faster moving flow has a lower pressure.

Vorticity

The quantity $\underline{\nabla} \times \underline{u}$ is extremely important and is known as the

$$\text{VORTICITY} \quad \underline{\omega} = \underline{\nabla} \times \underline{u}.$$

It measures the local rotation of the fluid at a given point. A way to see this is to consider a "vorticity meter", two infinitesimal rigid rods connected at right angles



mean angular velocity about point O is

$$\frac{1}{2}\left(\frac{\frac{\partial u_y}{\partial x}\delta x}{\delta x} + \frac{-\frac{\partial u_x}{\partial y}\delta y}{\delta y} \right)$$

$$= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right)$$

$$= \frac{1}{2}\omega_z$$

So the instantaneous rotation of the rod is $\frac{1}{2}$ of the magnitude of the vorticity.

A way to emphasize this difference between local and global rotation is to consider the following two flows which involve a rotating fluid:

1) rigid body rotation (uniform rotation) with angular velocity Ω

in cylindrical coordinates, $\underline{\Omega} = \Omega \hat{\underline{z}}$

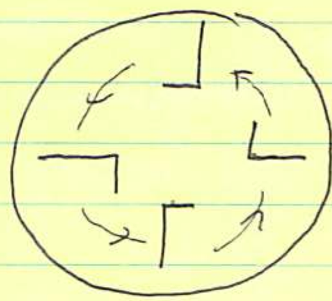
$$\underline{u} = \hat{\underline{\phi}}\, r\Omega$$

The vorticity is

$$\underline{\omega} = \nabla \times \underline{u} = \hat{\underline{z}}\, \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) = \hat{\underline{z}}\, \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \Omega\right)$$
$$= 2\Omega\, \hat{\underline{z}}$$

or $\underline{\omega} = 2\underline{\Omega}$

$\therefore$ we infer that there is a local rotation with angular velocity $\frac{\omega}{2} = \Omega$ at any point. In fact this makes sense because the vorticity meter must rotates as it moves around the rotation axis so that the system is stationary in the rotating frame.

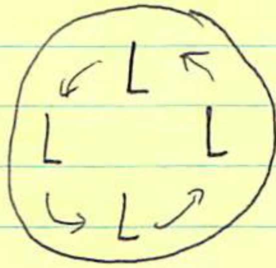


(This is just like how the moon rotates with the same angular velocity as its orbit, meaning that from Earth we always see the same face of the moon).

2) Contrast this with the flow $\underline{u} = \hat{\phi}\, \frac{k}{r}$ for constant k

This has $\underline{\nabla} \times \underline{u} = 0$ everywhere except at the origin.

The vorticity meter keeps the same orientation as it moves around the axis

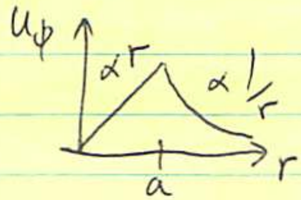


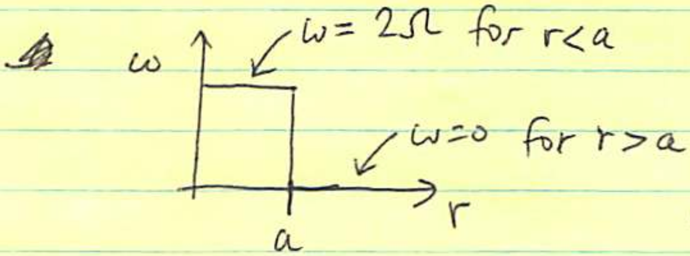
$$\underline{\nabla} \times \underline{u} = \hat{z}\,\frac{1}{r}\frac{\partial}{\partial r}\big(\underbrace{r u_\phi}_{\text{constant}}\big) = 0$$

This is called a <u>line vortex flow</u>.

[A combination of the two flows is a simple idealized model for a vortex known as a <u>Rankine vortex</u>

$$u_\phi = \begin{cases} \Omega r & r < a \\ \dfrac{\Omega a^2}{r} & r > a \end{cases} \qquad u_r = u_z = 0$$





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The <u>vorticity equation</u> describes the time-evolution of $\underline{\omega}$.

We take the curl of $\dfrac{\partial u}{\partial t} - \underline{u} \times (\underline{\nabla} \times u) = -\underline{\nabla} H$ (See (*) on page 14)

$$\Rightarrow \boxed{\frac{\partial \underline{\omega}}{\partial t} - \underline{\nabla} \times (\underline{u} \times \underline{\omega}) = 0}$$

But $\underline{\nabla} \times (\underline{u} \times \underline{\omega}) = (\underline{\omega} \cdot \underline{\nabla})\underline{u} - (\underline{u} \cdot \underline{\nabla})\underline{\omega} + \underline{u}\overbrace{(\underline{\nabla} \cdot \underline{\omega})}^{0} - \underline{\omega}\overbrace{(\underline{\nabla} \cdot \underline{u})}^{0 \text{ for } \rho = \text{constant}}$

$\Rightarrow$ we can also write $\boxed{\dfrac{D\underline{\omega}}{Dt} = (\underline{\omega} \cdot \underline{\nabla})\underline{u}}$ — (*)

Physical interpretation of vorticity equation

There are two different ways we can think about equation (*)

1) The LHS describes the advection of $\underline{\omega}$ by the flow. The terms on the RHS therefore describe how the local angular velocity can change.

To see the physics underlying this term, we can align the z-axis with the local direction of $\underline{\omega}$, ie. $\underline{\omega} = \omega \hat{\underline{z}}$. Then

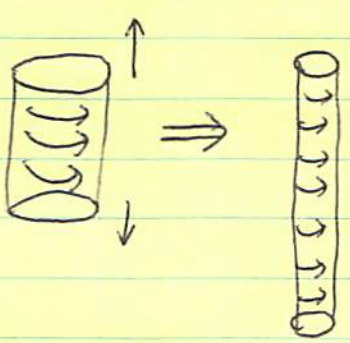
$$\frac{D}{Dt}\left(\omega \hat{\underline{z}}\right) = \hat{z}\, \omega \frac{\partial}{\partial z} \underline{u}$$

Write the fluid velocity as $\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}} + w\hat{\underline{z}}$

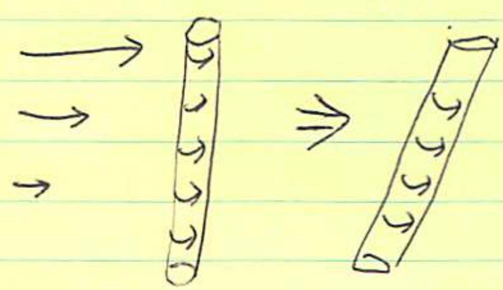
$$\Rightarrow \quad \frac{D}{Dt}\left(\omega \hat{\underline{z}}\right) = \hat{\underline{z}}\, \omega \frac{\partial w}{\partial z} + \hat{\underline{x}}\, \omega \frac{\partial u}{\partial z} + \hat{\underline{y}}\, \omega \frac{\partial v}{\partial z}$$

this term describes "vortex stretching"

"vortex tilting"



angular momentum conservation
$\Rightarrow$ increase in local rotation rate
when the vortex is stretched or squeezed



background shear "tilts" the vortex

2) Another way to interpret (*) is as follows.
Consider the separation between two fluid elements at positions $\underline{r}_1$ and $\underline{r}_2 = \underline{r}_1 + d\underline{l}$

A time δt later they are located at $\underline{r}_1' = \underline{r}_1' + \underline{u}_1 \delta t$
and $\underline{r}_2' = \underline{r}_2 + \underline{u}_2 \delta t$

$$\Rightarrow \quad d\underline{l}' = d\underline{l} + (\underline{u}_2 - \underline{u}_1)\,\delta t$$

But by a Taylor expansion, $\underline{u}_2 = \underline{u}_1 + (d\underline{l}\cdot\underline{\nabla})\,\underline{u}_1$

$$\Rightarrow \quad \frac{\delta(d\underline{l})}{\delta t} = (d\underline{l}\cdot\underline{\nabla})\,\underline{u}$$

this is a Lagrangian time derivative since it applies to particular fluid elements

$$\Rightarrow \quad \boxed{\frac{D}{Dt}\, d\underline{l} = (d\underline{l}\cdot\underline{\nabla})\,\underline{u}}$$

This has the same form as equation (*) with the replacement $\underline{\omega} \rightarrow d\underline{l}$.

This implies that if locally $\underline{\omega}$ is parallel to the separation between two fluid elements, it will always be so since $\underline{\omega}$ and $d\underline{l}$ evolve in the same way.
The <u>vortex lines</u> (the lines that follow the direction of $\underline{\omega}$ at each point) move with the fluid –
we say that they are "frozen" into the fluid.

[In a magnetized fluid, the same equation holds for $\underline{B}$: $\frac{D}{Dt}\underline{B} = (\underline{B}\cdot\underline{\nabla})\,\underline{u}$ and indeed in "magnetohydrodynamics" a basic principle is that magnetic field lines are frozen into the fluid.]

Circulation

The integral quantity is known as the CIRCULATION

$$\Gamma = \oint_{\substack{\text{material}\\\text{curve}}} \underline{u}\cdot d\underline{l} = \int_{\substack{\text{surface}\\\text{bounded by the}\\\text{loop}}} \underline{\omega}\cdot d\underline{S}$$

It is a conserved quantity under certain conditions. To see this, we can evaluate

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_{\substack{\text{material}\\\text{curve}}} \underline{u}\cdot d\underline{l} = \oint \frac{D\underline{u}}{Dt}\cdot d\underline{l} + \oint \underline{u}\cdot\frac{Dd\underline{l}}{Dt}$$

the first term will vanish if $\frac{D\underline{u}}{Dt} = \underline{\nabla}(\text{scalar})$

eg. Constant density fluid with gravity

$$\frac{D\underline{u}}{Dt} = -\underline{\nabla}\left(\frac{P}{\rho} + \chi\right)$$

From the discussion on the previous page, this is

$$\frac{Dd\underline{l}}{Dt} = \delta\underline{u}$$

change in velocity along the curve

So we can change variables to a velocity integration

$$\oint \underline{u}\cdot\frac{Dd\underline{l}}{Dt} = \oint \underline{u}\cdot d\underline{u} = \oint \tfrac{1}{2} du^2$$
$$= 0$$

$$\Rightarrow \boxed{\frac{D\Gamma}{Dt} = 0}$$

KELVIN'S THEOREM

Circulation is conserved around a material curve if the forces are conservative ($\frac{D\underline{u}}{Dt} = \underline{\nabla}(\text{scalar})$).

Vorticity generation and destruction

Kelvin's theorem holds only if the forces are conservative. Similarly, when we derived the vorticity equation, we assumed that the right hand side of the momentum equation was curl-free, ie. we wrote $\underline{F} = -\underline{\nabla} H + \underline{\nabla}(\frac{1}{2}u^2)$.
where $\underline{F}$ is the force per unit mass $\quad \frac{D\underline{u}}{Dt} = \underline{F}$

eg. for pressure + gravity forces

$$\underline{F} = -\frac{\underline{\nabla} P}{\rho} + \underline{g} .$$

More generally, $\underline{\nabla} \times \underline{F}$ may not vanish, and then

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega} . \underline{\nabla})\underline{u} + \underbrace{\underline{\nabla} \times \underline{F}}.$$

a force with non-zero curl can induce fluid rotation and therefore generate (or destroy) vorticity.

Examples:

1) <u>viscous force</u>. We'll look at this in detail later. It leads to diffusion of vorticity, and can be a source or a sink.

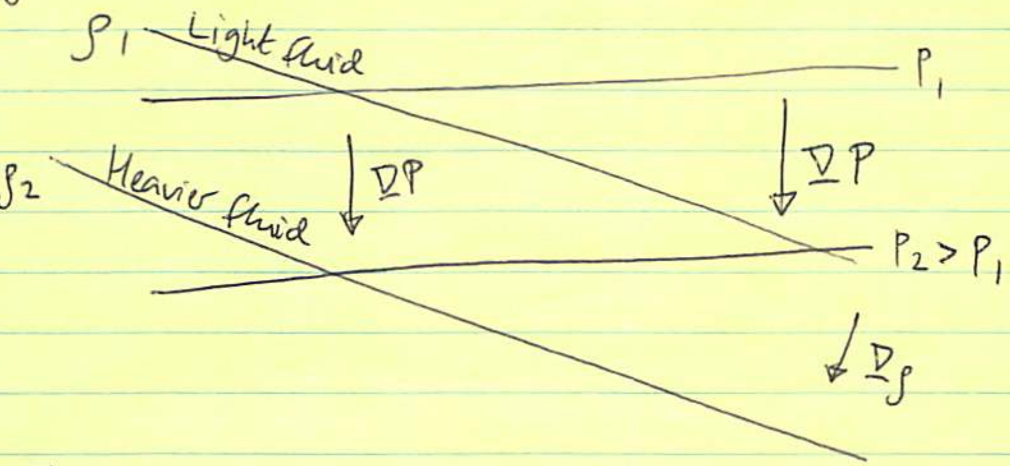
2) <u>baroclinicity</u> If the density is not constant, then

$$\underline{\nabla} \times \underline{F} = -\underline{\nabla} \times \left(\frac{\underline{\nabla} P}{\rho}\right) = \underbrace{-\frac{\underline{\nabla} P \times \underline{\nabla} \rho}{\rho^2}}$$

the BAROCLINIC VECTOR

$\Rightarrow$ vorticity changes when the surfaces of constant pressure and density are misaligned.

eg. consider two isobars



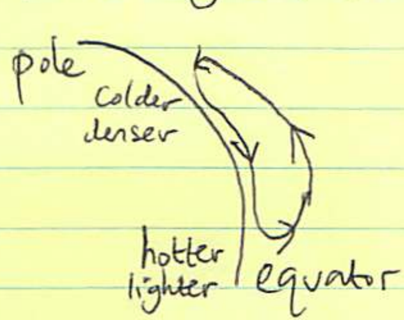
The pressure gradient is the same on both sides, but acts on denser fluid on the left
The acceleration is

$$\uparrow \underline{a} = -\frac{\nabla P}{\rho_2} \qquad\qquad \uparrow \underline{a} = -\frac{\nabla P}{\rho_1}$$

generates a circulation

in the direction $\nabla \rho \times \nabla P$ ✓

Important in geophysical fluid dynamics, eg. Hadley cells maintained by differential heating of Earth's surface.



$\leftarrow$ Solar irradiation
$\leftarrow$ (stronger at equator)

Irrotational flow around a cylinder

We're going to talk about lift on an airplane wing which is an interesting problem in which circulation plays a key role despite the flow being irrotational ($\underline{\nabla} \times \underline{u} = 0$) everywhere!
To set the scene, consider flow around a cylinder. We'll calculate the flow and then the lift force on the cylinder.

In a 2D steady flow, the vorticity equation is $(\underline{u} \cdot \underline{\nabla})\underline{\omega} = 0$ (both the $\frac{\partial}{\partial t}$ and $(\underline{\omega} \cdot \underline{\nabla})\underline{u}$ terms vanish).

$\Rightarrow$ if $\omega = 0$ far from the cylinder (eg. uniform flow) then it will be zero throughout the flow, $\boxed{\underline{\nabla} \times \underline{u} = 0.}$ —(1)

Furthermore if the fluid is incompressible then $\boxed{\underline{\nabla} \cdot \underline{u} = 0.}$ —(2)

It is useful to define scalar fields ϕ and ψ as follows:

1. (1) $\underline{\nabla} \times \underline{u} = 0 \Rightarrow$ we can write the velocity as $\underline{u} = \underline{\nabla}\phi$

"velocity potential"

and then (2) $\Rightarrow$ $\underline{\nabla} \cdot \underline{u} = \boxed{\nabla^2 \phi = 0}$
We can obtain the velocity field by solving Laplace's equation with appropriate boundary conditions.

2. An alternate approach is to start with (2) $\underline{\nabla} \cdot \underline{u} = 0$ which is automatically satisfied if we write

$$\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}}$$

$$\text{with} \quad u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

(then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$).

The scalar ψ is constant along streamlines:

$$\underline{u}\cdot\underline{\nabla}\psi = u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0$$

and is known as the "<u>Stream function</u>".

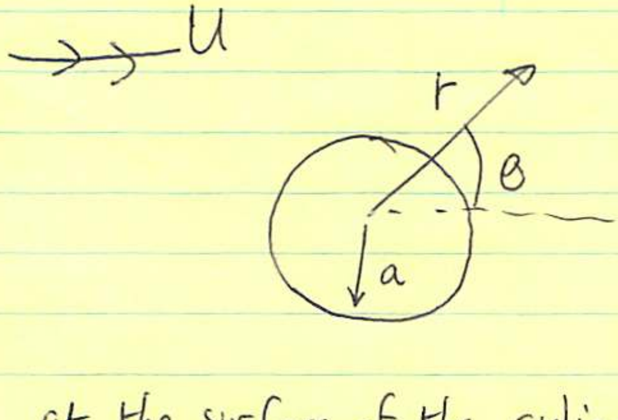
It too obeys Laplace's equation since

$$\underline{\nabla}\times\underline{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} = 0$$

or $\boxed{\nabla^2\psi = 0}$

Note that lines of constant ϕ are perpendicular to lines of constant ψ (streamlines).

Now set up the cylinder problem:



uniform flow $\underline{u} = U\hat{\underline{x}}$ at $r\to\infty$

$\Rightarrow \quad \phi = Ux = Ur\cos\theta \quad (r\to\infty)$

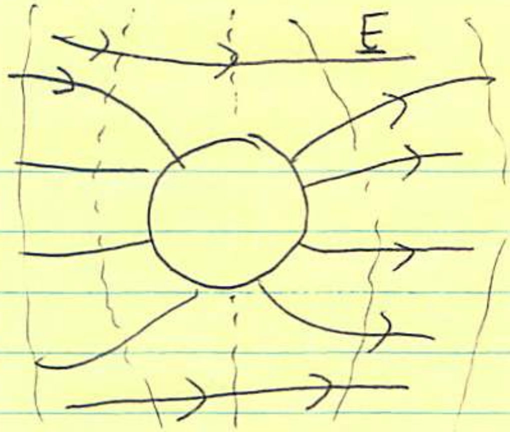
or $\quad \psi = Ur\sin\theta . \quad (r\to\infty)$

at the surface of the cylinder, $r=a$, we impose a "free slip" boundary condition

$$u_r = 0 \text{ at } r=a \quad \text{or} \quad \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0$$

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=a} = 0$$

[The situation is analogous to a conducting cylinder placed in a uniform electric field



E-field lines $\longleftrightarrow$ lines of constant velocity potential ϕ

lines of constant electric potential (equipotentials) $\longleftrightarrow$ lines of constant ψ (streamlines)

The solution to Laplace's equation is

$$\phi = Ur\cos\theta + \frac{C\cos\theta}{r}$$

which has the right form at $r \to \infty$ and the constant C is determined by setting $\frac{\partial \phi}{\partial r} = 0$ at $r = a$

$$\Rightarrow \quad U\cos\theta - \frac{C}{a^2}\cos\theta = 0$$

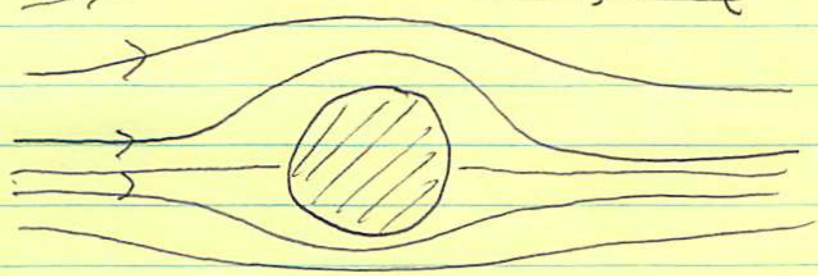
$$\Rightarrow \quad C = Ua^2$$

$$\Rightarrow \quad \boxed{\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)} \quad \text{—}(*)$$

(Similarly you can show that $\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$)

The velocities are $\quad u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$

$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta.$$



Adding circulation and non-uniqueness

Note that the solution (*) has zero circulation around the cylinder:

$$\int_{\theta=0}^{2\pi} r\, d\theta\, u_\theta \propto \left[\cos\theta\right]_0^{2\pi} = 0 .$$

(we can see that from the symmetry of the solution).

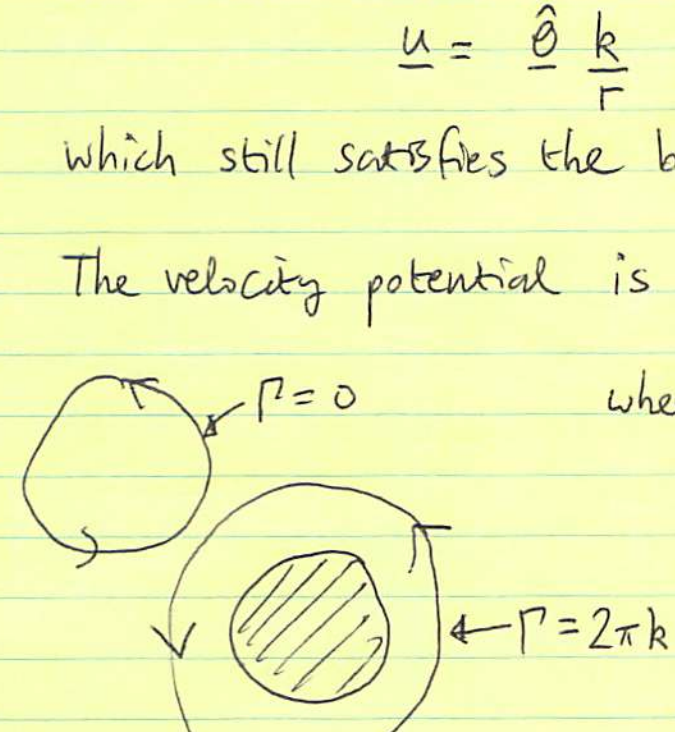
But we can add circulation by adding a line vortex flow

$$\underline{u} = \underline{\hat{\theta}}\, \frac{k}{r}$$

which still satisfies the boundary conditions at $r = a$ and $r = \infty$.

The velocity potential is $\phi = k\theta = \dfrac{\Gamma \theta}{2\pi}$

where Γ is the circulation of the vortex flow.



This implies that the general solution is

$$\boxed{\phi = U \cos\theta \left(r + \frac{a^2}{r} \right) + \frac{\Gamma \theta}{2\pi}}$$

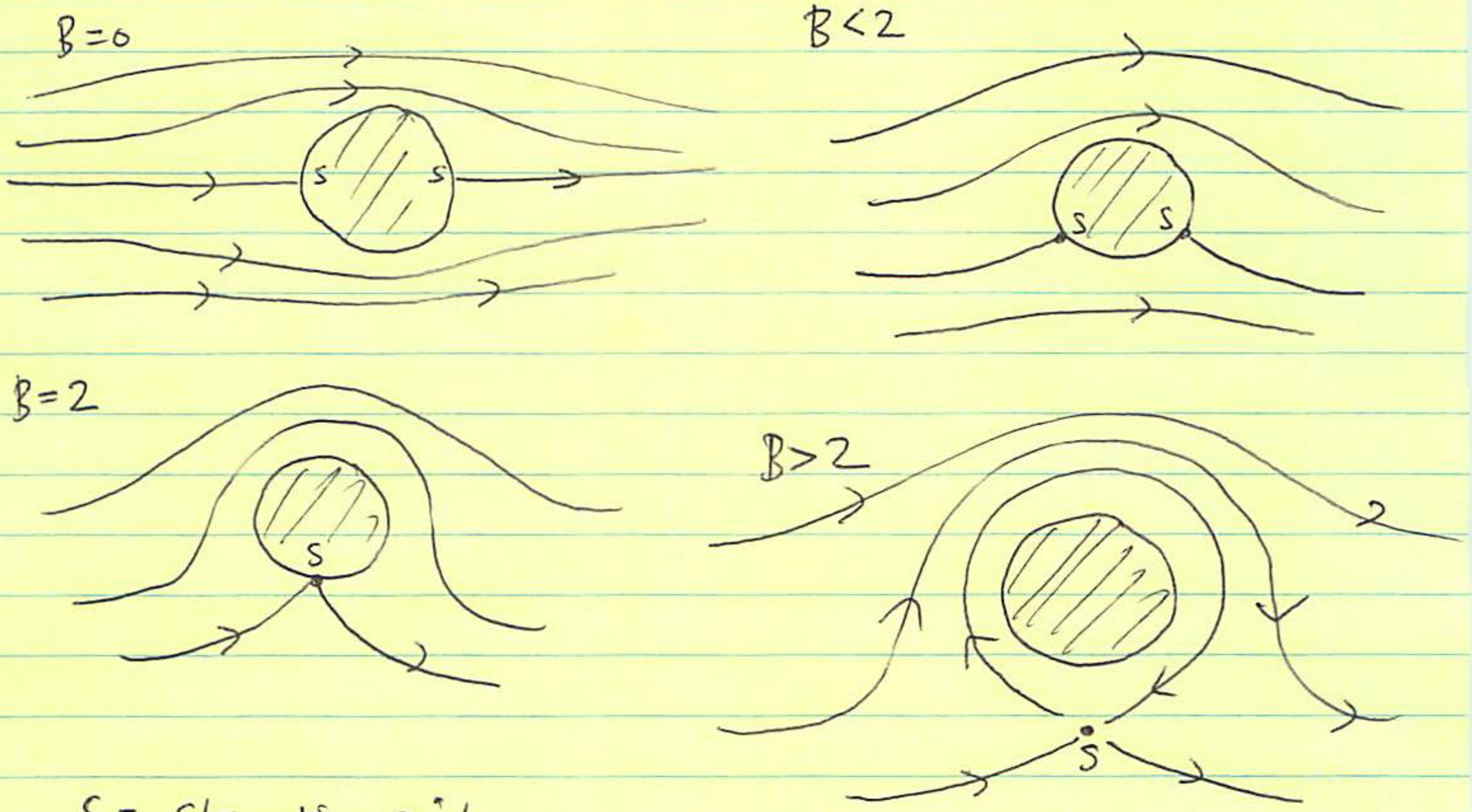
a family of solutions that are parameterised by Γ.

This may seem strange – our intuition from electromagnetism tells us that there is a uniqueness theorem for Laplace's equation. The key difference here is that we have excised the region $r < a$

from the problem. Integrals around closed loops that encircle the cylinder no longer have to vanish, since $\underline{u}$ (and therefore $\underline{\nabla} \times \underline{u}$) is not defined in the region $r < a$.

If we write the dimensionless parameter $B = -\dfrac{\Gamma}{2\pi U a}$

the solutions look like



S = stagnation point

Force on the cylinder

$$\text{Bernoulli's thm} \Rightarrow \quad p + \tfrac{1}{2}\rho u^2 = \text{constant}$$

on the surface $r = a$ (which is a streamline) for $r = a$ $\quad u_r = 0 \quad u_\theta = -2U\sin\theta + \dfrac{\Gamma}{2\pi a}$

$$\Rightarrow \quad -\frac{p}{\rho} = (\text{constant}) + 2U^2\sin^2\theta - \frac{\Gamma U}{\pi a}\sin\theta$$

The net force in the y-direction is (per unit length along the cylinder)

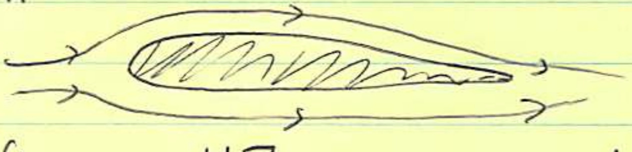
$$-\int p \, a\, d\theta \sin\theta$$

$$= \rho\int_0^{2\pi}\Big(\underbrace{2U^2\sin^2\theta}_{\text{this term is symmetric, doesn't contribute}} - \frac{U\Gamma}{\pi a}\sin\theta\Big)\, a\sin\theta\, d\theta$$

$$= \underline{\underline{-\rho U \Gamma}}$$ the upwards lift force on the cylinder.

The <u>Kutta-Joukowski Lift theorem</u> states that this result applies to any 2D cross-section

eg. a wing



lift $= -\rho U\Gamma$ per unit length along the wing.

The proof relies on conformal mapping to morph the cylindrical boundary into the appropriate shape.

Lift on a wing

We see from the formula $-\rho U \Gamma$ that circulation is crucial for the lift on a wing. But where does it come from and why is it there (after all the wing is not rotating!). The irrotational flow around a wing looks something like

this solution has a problem at the trailing edge where the fluid has to "turn the corner"

The Kutta-Joukowski hypothesis is that the flow develops a circulation that is large enough to move the stagnation point to the trailing edge

The circulation comes from viscous forces that act as the plane initially accelerates (or whenever the speed or angle of attack changes). Since circulation must be conserved, a vortex of opposite sign is shed from the wing

"starting vortex"

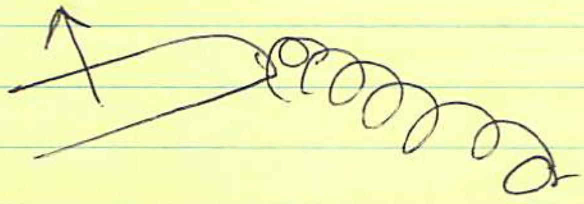
The required circulation is $\propto U$ so that the lift force is $\propto U^2$. (Try plugging numbers for a plane, does it work?)

Two other interesting points are:

1) at large angle of attack the flow becomes turbulent and there is a "stall"

lift drops dramatically
(stall)

2) at the end of the wing the flow is no longer 2D, A trailing vortex is shed from the wingtip.



Viscosity and Viscous Flows

Basic idea and estimates of viscosity

In a viscous fluid, the random motion of molecules transport momentum between adjacent layers that are moving with different bulk velocities.

eg. plane parallel shear flow $\quad \underline{U} = U(z)\,\hat{\underline{x}}$

λ = mean free path

the net flux of momentum across this boundary is

$$-\frac{1}{3}\, n\, m\, v_{th} \left(\lambda \frac{dU}{dz}\right)$$

thermal speed of the molecules

$$v_{th} \simeq \left(\frac{k_B T}{m}\right)^{1/2} \simeq c_s \quad \left(\begin{array}{l}\text{sound}\\ \text{speed}\end{array}\right)$$

but a momentum flux $\left(\begin{array}{l}\text{momentum per unit}\\ \text{area per second}\end{array}\right)$ is a force per unit area or a stress.

Note that this is the x-momentum flux in the z-direction or in other words the stress in the x-direction on a surface whose normal points in the z-direction. A tangential stress.

$\Rightarrow$ off-diagonal term T_{xz} in the stress tensor.

Viscosity is the constant of proportionality between the stress and $\frac{dU}{dz}$:

$$\boxed{\text{stress} = -\mu \frac{dU}{dz}}$$

μ = <u>viscosity</u> $= \frac{1}{3} n m v_{th} \lambda$ units: g/cm s "Poise"

$= \rho \frac{1}{3} v_{th} \lambda$ or $\frac{kg}{m\,s}$ = Pa.s

$= \rho \nu$

ν is the <u>kinematic viscosity</u> units: cm^2/s or m^2/s.

A fluid that has viscous stress $\propto$ velocity gradient is known as a <u>Newtonian fluid</u>. Not all fluids are Newtonian (eg. Corn starch + water — search YouTube for "non-Newtonian fluid"!)

Some values of viscosity:

(these are at 20°C and in cgs units!)

	μ	ν
water	0.01	0.01
air	1.8×10^{-4}	0.15
alcohol	0.018	0.022
glycerine	8.5	6.8
mercury	0.0156	0.0012
molasses		$\simeq 50-100$

<u>Exercise</u>: use the formula above to check the value for air!

Momentum Equation with Viscous term

We already have the machinery to deal with these tangential kind of surface forces. Recall that we wrote the momentum equation as

$$\rho \frac{Du}{Dt} = \underline{\nabla} \cdot \underline{\underline{T}}$$

where T_{ij} is the stress tensor. We add an additional term to this to account for viscous stresses:

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij}$$

where the <u>viscous stress tensor</u> is

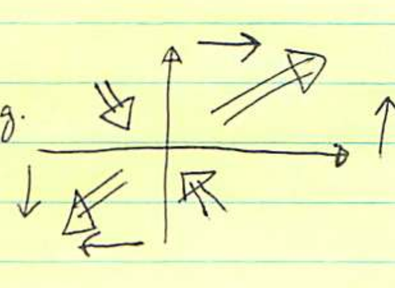
$$\sigma_{ij} = \mu \underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{this is also written as} \\ 2e_{ij}}} - \frac{2}{3}\mu\,\delta_{ij}\,\underline{\nabla}\cdot\underline{u} + \xi\,\delta_{ij}\,\underline{\nabla}\cdot\underline{u} \qquad (*)$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the <u>strain rate tensor</u>

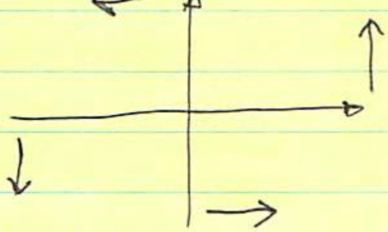
What we've done here is to take the symmetric part of the velocity gradient

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{this term describes deformation of} \\ \text{the fluid element and generates} \\ \text{viscous stress}}} + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

eg.



the second term is rotation of the fluid ($\underline{\nabla} \times \underline{u}$) – this doesn't generate any viscous stress



no deformation

What about the $\underline{\nabla} \cdot \underline{u}$ terms in equation (*)? They are proportional to δ_{ij}, so they look like the pressure term $-P\delta_{ij}$. In fact the reason to write two $\underline{\nabla} \cdot \underline{u}$ terms is that the first two terms in equation (*) are traceless, ie.

$$\sigma_{ii} = 3\zeta \, \underline{\nabla} \cdot \underline{u}$$

so we can think of ζ as the dynamical correction to the equilibrium pressure P.

ζ = coefficient of <u>Bulk viscosity</u>

ie. we can define a mean pressure

$$\bar{P} = -\frac{1}{3} T_{ii}$$

$$= P + \zeta \, \underline{\nabla} \cdot \underline{u}$$

(thermodynamic pressure)

The "Stokes assumption" is that $\zeta = 0$ (σ_{ij} is trace-free) so that volume changes do not lead to dissipation. This is true for a monatomic ideal gas for example.

Let's stick to the case $\underline{\nabla} \cdot \underline{u} = 0$. Then

$$\rho \frac{Du_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\sigma_{ij} = -\frac{\partial P}{\partial x_i} + \underbrace{\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right)}$$

this has a simple physical interpretation: the net force arises from the difference in viscous stress from one side of the fluid element to the other

For μ = constant

$$\boxed{\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} P + \mu \nabla^2 \underline{u}}$$

Note the similarity to the diffusion equation

$$\frac{\partial u}{\partial t} = \mu \nabla^2 u$$

— viscosity causes diffusion of the velocity field

Reynolds number

To determine the importance of viscous effects, compare the relative sizes of the inertia and viscous terms in the momentum equation

$$\underline{u} \cdot \underline{\nabla} \underline{u} \quad \text{vs} \quad \nu \cancel{\mu} \nabla^2 \underline{u}$$

$$\sim \frac{U^2}{L} \qquad\qquad \sim \frac{\nu U}{L^2}$$

The ratio gives the dimensionless Reynolds number

$$\boxed{Re = \frac{UL}{\nu}}$$

(Acheson writes it as "R")

$Re \ll 1$ viscous term dominates
$Re \gg 1$ inertia term dominates

There are many such dimensionless numbers in fluid mechanics. They are important because of the idea of dynamical similarity — two flows can have dramatically different velocity, length or time scales but they will evolve similarly if the underlying dimensionless numbers are the same.

Two interesting features of high and low Re number flows:

1) low $Re \ll 1$ the flow is reversible
eg. dyed blob between two cylinders.

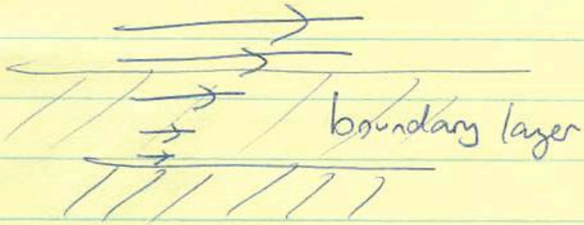
2) high $Re \gg 1$ for moderate values of Re the flow is laminar and viscous effects occur in thin boundary layers. For high $Re > Re_{crit}$ the flow becomes turbulent.

Boundary condition for viscous flow

At the boundary with a solid surface, the fluid obeys the NO SLIP CONDITION $u_{\parallel} = 0$

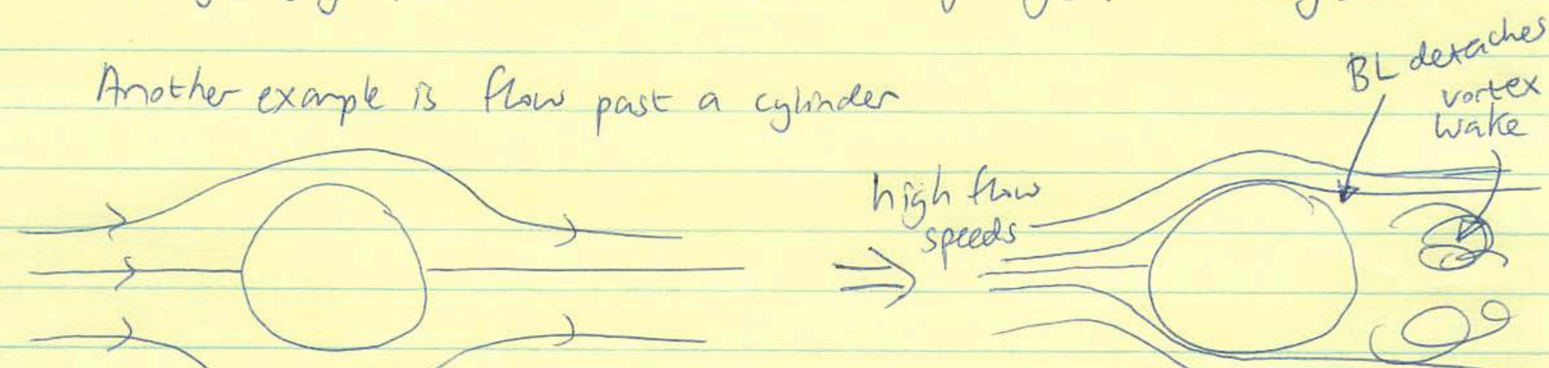
Since $u_{\perp}$ must also vanish, then $\boxed{\text{the total velocity } \underline{u} = 0 \text{ at a solid boundary}}$.

This may seem counter-intuitive. For example in the flow past a wing, the fluid was allowed to have any value of $u_{\parallel}$ at the boundary (we didn't explicitly say this but the only boundary condition at the surface was $u_{\perp} = 0$). We say that the boundary condition is a FREE SLIP CONDITION in that case. But in that case, a thin boundary layer exists in which the fluid velocity falls from the free-slip value to zero at the solid surface. Viscous effects are confined to this thin layer (and indeed are crucial for generating the starting vortex as we discussed).



The assumption in the irrotational flow calculation is that there is a boundary layer at every surface that adjusts the velocity to zero by viscous stresses. But in fact <u>the boundary layer can separate</u> which leads to the catastrophic failure of irrotational flow theory. (eg. loss of lift at the stalling angle for a wing)

Another example is flow past a cylinder



Example: steady flow down an inclined plane (Acheson p 38)

First, use the symmetry to simplify the problem:

no special location along the surface $\Rightarrow$ no x-dependence

The velocity is $\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}}$

but $\underline{\nabla}.\underline{u} = 0 = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \Rightarrow \dfrac{\partial v}{\partial y} = 0$

but $v = 0$ at $y = 0$ (no perpendicular flow at the boundary)

$\Rightarrow v = 0$ everywhere.

$\Rightarrow$ we need to solve for $u(y)$.

The y-cpt of momentum is $\rho \dfrac{\partial v}{\partial t} = 0 = -\rho g \cos\alpha - \dfrac{\partial p}{\partial y}$

$$\Rightarrow \boxed{p = p_0 + \rho g (H - y)\cos\alpha}$$

(hydrostatic balance in y-direction)

(p_0 = atmospheric pressure at $y = H$)

Note that pressure depends on y only so there is no pressure gradient $\dfrac{\partial p}{\partial x}$

$\Rightarrow$ x-momentum eqn is $\dfrac{\partial u}{\partial t} = 0 = \mu \dfrac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha$.

$\Rightarrow$ we need to solve $\mu \frac{d^2u}{dy^2} = -\rho g \sin\alpha$

or $$\boxed{\frac{d^2u}{dy^2} = -\frac{g}{\nu}\sin\alpha}$$

Boundary conditions: $u = 0$ at $y = 0$ (no slip)

$\frac{\partial u}{\partial y} = 0$ at $y = H$

(stress free surface)

$$\Rightarrow \quad \boxed{u(y) = \frac{g}{\nu}\left(H - \frac{y}{2}\right) y \sin\alpha}$$

quadratic dependence of u on height

Notes:

— Viscous stress is linear in height:

$$\mu \frac{\partial u}{\partial y} = \rho g (H - y) \sin\alpha$$

$\Rightarrow$ the stress at the lower boundary is $\rho g H \sin\alpha$
just the component of the weight directed along the surface of the plane!

— The velocity at the top of the fluid is $u = \frac{1}{2} g \frac{H^2}{\nu} \sin\alpha$

can write this as $u = \frac{1}{2} \underbrace{(g \sin\alpha)}_{\text{(acceleration)}} \underbrace{(t_{visc})}_{\text{(time)}}$

where $t_{visc} = \frac{H^2}{\nu}$ is the viscous time across the layer.

The fluid can accelerate for about one viscous time before the effects of viscous drag become significant and it reaches a terminal velocity.

[$t_{visc} = \frac{H^2}{\nu}$ comes from the fluid equation. Without gravity, the time for the fluid to stop moving can be estimated from

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{u}{t} \sim \frac{\nu u}{H^2}$$]

— An obvious application is to lava flow

eg. basalt flow 10 m thick ~ 1 m/s $\rho \simeq 3000$ kg/m^3

$$\Rightarrow \mu \approx \frac{\rho g H^2}{u} \approx \frac{3000 \times 10 \times 10^2}{1}$$

$$\approx 3 \times 10^6 \text{ Pa s}$$

$$\nu = \mu/\rho = 10^3 \frac{m^2}{s}!$$

(water is $\nu = 10^{-6}$ m^2/s)

But this is a complicated problem — viscosity depends on temperature and structure of the lava. There can be non-Newtonian effects such as a lowered viscosity when shearing "shear thinning"

See Griffiths "Dynamics of Lava Flow"
ARFM <u>32</u> 477 (2000)

Example: An impulsively-moved plane boundary (Acheson p 35)

fluid initially at rest

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial p}{\partial x} = 0$$

at $t=0$ boundary starts moving to the right with speed U

$$\begin{aligned} u(y,0) &= 0 \quad y>0 \\ u(0,t) &= U \quad t>0 \\ u(\infty,t) &= 0 \quad t>0 \end{aligned}$$

This is a neat problem because it illustrates the idea of a SIMILARITY SOLUTION. There is no lengthscale in the problem except the distance we can diffuse in time t

$\Rightarrow$ the solution must be a function of $\dfrac{y}{\sqrt{\nu t}} \equiv \eta$.

ie. $u = f(\eta)$

Change variables:

$$\frac{\partial u}{\partial t} = f'(\eta)\frac{\partial \eta}{\partial t} = -f'(\eta)\frac{y}{2\nu^{1/2}t^{3/2}}$$

$$\frac{\partial u}{\partial y} = f'(\eta)\frac{\partial \eta}{\partial y} = f'(\eta)\frac{1}{\nu^{1/2}t^{1/2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f''(\eta)}{\nu t}$$

$$\Rightarrow \quad \frac{f''}{\nu t} = -\frac{1}{\nu} f' \frac{y}{2\sqrt{\nu}\, t^{3/2}}$$

$$\Rightarrow \quad f'' = -\frac{f' y}{2\sqrt{\nu t}} = -\frac{f'}{2}\eta$$

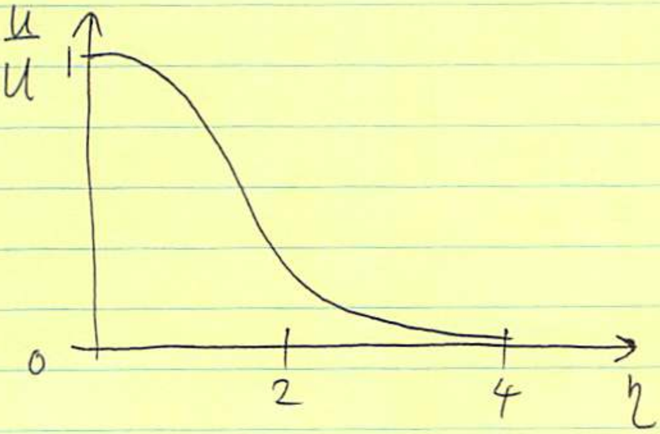
$$\Rightarrow \quad \boxed{f'' + \frac{1}{2} f' \eta = 0}$$

Solution is $f' = Be^{-\eta^2/4} \Rightarrow f = A + B\int_0^\eta e^{-s^2/4}\, ds$

b.c.'s determine A and B:

$$f(\infty) = 0 \qquad \text{(velocity vanishes at large distance or early time)}$$
$$f(0) = U$$

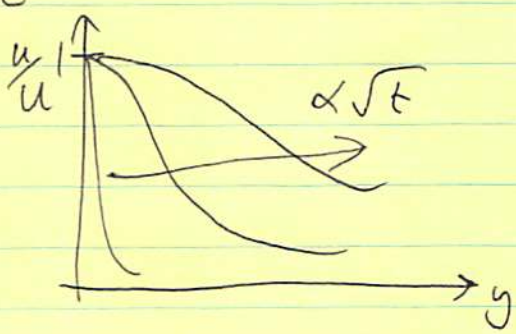
$$\Rightarrow \quad u = U\left[1 - \frac{1}{\sqrt{\pi}}\int_0^{\eta} e^{-s^2/4}\, ds\right]$$
$$= U\left[1 - \mathrm{erf}\left(\eta/2\right)\right] .$$



the function $\frac{u}{U}(\eta)$ is fixed

where $\eta = y/\sqrt{\nu t}$

$\Rightarrow$ in terms of y the profile stretches out $\propto \sqrt{t}$



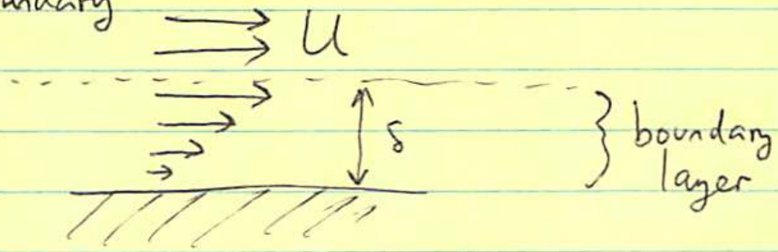
The vorticity is $\quad \omega = -\frac{\partial u}{\partial y} = \frac{U}{(\pi \nu t)^{1/2}} e^{-y^2/4\nu t}$

so we see that vorticity diffuses away from the wall.

[Note that $\int dy\, \omega = U \frac{1}{(\pi\nu t)^{1/2}} \int_0^{\infty} e^{-y^2/4\nu t}\, dy$
(circulation)
$= U$ = constant]

Estimate of boundary layer width

Imagine that we have a flow with characteristic lengthscale L velocity U and $Re \gg 1$. The viscous term is small except in a thin boundary layer in which the flow velocity drops from $\sim U$ to zero at the boundary



The boundary layer thickness δ is such that the viscous time across the BL is the flow time

ie. $$\frac{\delta^2}{\nu} \sim \frac{L}{U}$$

$$\Rightarrow \qquad \delta \sim \left(\frac{L\nu}{U}\right)^{1/2}$$

or $$\frac{\delta}{L} \sim \left(\frac{\nu}{UL}\right)^{1/2} = \frac{1}{Re^{1/2}}$$

$\Rightarrow$ the typical size of a BL is $Re^{1/2}$ times smaller than the scale of the flow.

The Energy Equation

Let's write an equation for the energy of the fluid without viscosity first, and then we'll come back and add viscosity later.

The bulk kinetic energy density is $\frac{1}{2}\rho u^2$. To derive an equation for the kinetic energy, take $\underline{u} \cdot \begin{pmatrix}\text{momentum}\\ \text{equation}\end{pmatrix}$

$$\underline{u} \cdot \left[\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla} P \right]$$

this is $\rho \frac{\partial}{\partial t}(\frac{1}{2}u^2)$

this is
$$\rho\underline{u}\cdot\left(-\underline{u}\times(\underline{\nabla}\times\underline{u}) + \underline{\nabla}\tfrac{1}{2}u^2\right)$$
$$= \rho\,\underline{u}\cdot\underline{\nabla}\tfrac{1}{2}u^2$$

$$-\underline{u}\cdot\underline{\nabla}P$$
$$= -\underline{\nabla}\cdot(\underline{u}P) + P(\underline{\nabla}\cdot\underline{u})$$

add $(\frac{1}{2}u^2) \times$ (continuity equation)

$$\frac{1}{2}u^2\frac{\partial \rho}{\partial t} + \frac{1}{2}u^2\,\underline{\nabla}\cdot(\rho\underline{u}) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right) + \underline{\nabla}\cdot\left(\rho\underline{u}\tfrac{1}{2}u^2\right) + \underline{\nabla}\cdot(\underline{u}P) = P(\underline{\nabla}\cdot\underline{u})$$

or
$$\boxed{\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right) + \underline{\nabla}\cdot\left(\underline{u}\left[\tfrac{1}{2}\rho u^2 + P\right]\right) = P\,\underline{\nabla}\cdot\underline{u}} \quad \text{---} (*)$$

Kinetic energy equation

For internal energy, we use the 1st law of thermodynamics

$$T\,dS = dE - \frac{P}{\rho^2}\,d\rho$$

where E = internal energy per unit mass
S = entropy per unit mass

For an adiabatic flow $\quad T\dfrac{DS}{Dt} = 0$

$$\Rightarrow \quad \frac{DE}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} = -\frac{P}{\rho}\underline{\nabla}.\underline{u}$$

$$\Rightarrow \quad \frac{D}{Dt}(\rho E) = \rho\frac{DE}{Dt} + E\frac{D\rho}{Dt}$$

$$= -P\underline{\nabla}.\underline{u} \cancel{+} - \rho E\,\underline{\nabla}.\underline{u}$$

$$\frac{\partial}{\partial t}(\rho E) + \underline{u}.\underline{\nabla}(\rho E) + \rho E\,\underline{\nabla}.\underline{u} = -P\underline{\nabla}.\underline{u}$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}(\rho E) + \underline{\nabla}.\left(\underline{u}(\rho E)\right) = -P\underline{\nabla}.\underline{u}} \quad (\dagger)$$

The term $P\underline{\nabla}.\underline{u} = -\dfrac{P}{\rho}\dfrac{D\rho}{Dt}$ represents PdV work. It appears in both (*) and (†) but with opposite sign – PdV work transfers energy from bulk k.e. to internal energy, and vice versa.

The total energy equation (*) + (†) is

$$\boxed{\frac{\partial}{\partial t}\left(\rho E + \tfrac{1}{2}\rho u^2\right) + \underline{\nabla}.\left(\underline{u}\left[\rho E + P + \tfrac{1}{2}\rho u^2\right]\right) = 0} \quad (**)$$

A non-adiabatic flow has a term $\rho T\dfrac{Ds}{Dt}$ on the RHS of (†) and (**). For example, if there is a heat flux

$$\underline{F} = -K\underline{\nabla}T$$

thermal conductivity K

then $\rho T \frac{DS}{Dt} = -\underline{\nabla}.\underline{F} + \varepsilon$

local sources or sinks of energy (J/kg/s)

If the flow is adiabatic, we can write

$$\frac{DS}{Dt} = 0 = \frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right)$$

γ = ratio of specific heats $\frac{C_p}{C_v}$.

$$\Rightarrow \frac{1}{p}\frac{Dp}{Dt} - \frac{\gamma}{\rho}\frac{D\rho}{Dt} = 0$$

A good approximation if (time for heat transport or generation) $\gg$ flow time

Viscous dissipation

With viscosity included, there is an extra term in the kinetic energy equation

$$u_i \frac{\partial}{\partial x_j} \sigma_{ij} = u_i \frac{\partial}{\partial x_j} 2\mu e_{ij}$$

viscous stress tensor

$$= \frac{\partial}{\partial x_j}\left(2\mu u_i e_{ij}\right) - 2\mu e_{ij} \frac{\partial u_i}{\partial x_j}$$

(integrate by parts) (surface term)

Use a trick to simplify

$$-2\mu\, e_{ij} \frac{\partial u_i}{\partial x_j} = -2\mu\left[e_{ij} \frac{\partial u_i}{\partial x_j} + e_{ji} \frac{\partial u_j}{\partial x_i}\right]\frac{1}{2}$$

$$= -2\mu\, e_{ij}\, \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= -2\mu\, e_{ij} e_{ij}$$

$$= -2\mu \left(e_{ij}\right)^2$$

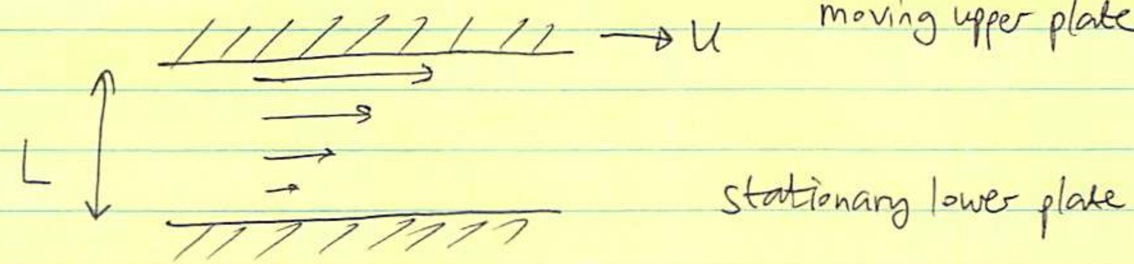
Notice that this term is < 0, ie. kinetic energy decreases due to the action of viscosity. The energy goes into internal energy.

The <u>viscous dissipation rate</u> is

$$\Phi_V = \sigma_{ij} \frac{\partial u_i}{\partial x_j} = \underline{\underline{2\mu \left(e_{ij}\right)^2}}$$

This term should be added to the RHS of (†) and subtracted from the RHS of (*).

Example: fluid between two plates



steady-state has $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} = 0 \Rightarrow \boxed{u = \frac{U}{L} z}$

linear velocity profile

The viscous stress is $\mu \frac{\partial u}{\partial z} = \frac{\mu U}{L}$.

The viscous dissipation rate is $2\mu \left(e_{xz}^2 + e_{zx}^2 \right)$

$$= 2\mu \left(\left(\frac{1}{2} \frac{\partial u}{\partial z} \right)^2 + \left(\frac{1}{2} \frac{\partial u}{\partial z} \right)^2 \right)$$

$$= \mu \left(\frac{\partial u}{\partial z} \right)^2 = \frac{\mu U^2}{L^2}.$$ (energy per unit volume per second)

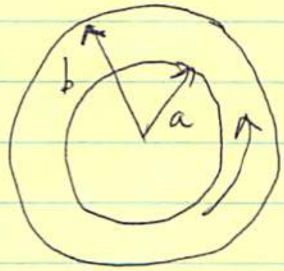
[The k.e. in the flow is $\int_0^L \frac{1}{2} \rho \frac{U^2}{L^2} z^2 \, dz = \frac{1}{2} \rho U^2 \frac{L}{3}$ per unit area]

The viscous dissipation matches the rate of work needed to move the upper boundary at constant speed U against the stress $\tau = \frac{\mu U}{L}$

ie. $$\tau U = \underbrace{\frac{\mu U^2}{L}}_{\text{rate of work by upper plate}} = \underbrace{\frac{\mu U^2}{L^2} \cdot L}_{\text{viscous dissipation per unit area}}.$$

Taylor-Couette Flow

An important example in fluid mechanics, in which a viscous fluid flows between two concentric cylinders.



$$\underline{u} = \hat{\underline{\theta}}\, u_\theta(r)$$

The momentum equations in cylindrical coordinates are in the Appendix of Acheson (p.353)

For steady flow

$$\frac{\partial u_\theta}{\partial t} = 0 = \nu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right]$$

$$= \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right]$$

$$= \nu\left[u_\theta'' + \frac{u_\theta'}{r} - \frac{u_\theta}{r^2}\right]$$

Power law solution $u_\theta \propto r^n \Rightarrow n(n-1) + n - 1 = 0$

$$\Rightarrow n^2 - 1 = 0 \Rightarrow n = \pm 1$$

$$\therefore \boxed{u_\theta = Ar + \frac{B}{r}} \quad \text{or} \quad \boxed{\Omega = \frac{u_\theta}{r} = A + \frac{B}{r^2}}$$

no slip boundary conditions $\Rightarrow$ $u_\theta = \Omega_a a$ at $r = a$

$\Omega_b b$ at $r = b$

$$\Rightarrow A = \frac{\Omega_b b^2 - \Omega_a a^2}{b^2 - a^2} \qquad B = \frac{(\Omega_a - \Omega_b)a^2 b^2}{b^2 - a^2} = \Delta\Omega \Big/ \left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

What is the viscous stress? $\tau = 2\mu e_{ij}$

$$= 2\mu e_{r\theta}$$ (only non-zero piece: see eq. (A.36) of Acheson)

$$= 2\mu \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)$$

$$= \underline{\underline{\mu r \frac{\partial \Omega}{\partial r}}}$$

<u>differential</u> rotation $\Rightarrow$ viscous stress

In our case, $\tau = -\mu r \frac{2B}{r^3} = -\frac{2\mu B}{r^2}$.

The torque is $(2\pi r \tau) \times r = -4\pi\mu B$ = constant
as it should be for a steady state (analogous to τ = constant on page 18)

the torque is > 0 if $\Omega_a < \Omega_b$
< 0 $\Omega_a > \Omega_b$

[Q: does that make sense? What is the physical picture?]

Viscous heating rate

$$2\mu\left(e_{r\theta}^2 + e_{\theta r}^2\right)$$

$$= 4\mu \frac{1}{4} r^2 \left(\frac{\partial \Omega}{\partial r}\right)^2 = \mu r^2 \left(\frac{d\Omega}{dr}\right)^2$$

$$= \frac{4B^2\mu}{r^4} = \Phi_v$$

Total dissipation per unit length: $\int 2\pi r \, dr \, \Phi_v = 4\pi\mu B^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

Rate of work by cylinders: $2\pi b^2 \tau_b \Omega_b - 2\pi a^2 \tau_a \Omega_a = 4\pi B \mu \Delta\Omega$
$= 4\pi\mu B^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ ✓

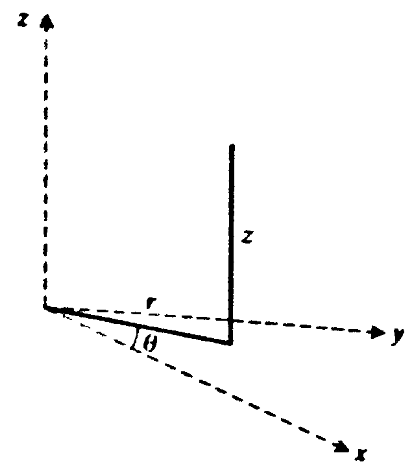


Fig. A.2 Cylindrical polar coordinates.

Also,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z, \tag{A.30}$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}, \tag{A.31}$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}, \tag{A.32}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}, \tag{A.33}$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}. \tag{A.34}$$

The Navier–Stokes equations in cylindrical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right),$$

(A.35)

$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0.$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}, \qquad 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{A.36}$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

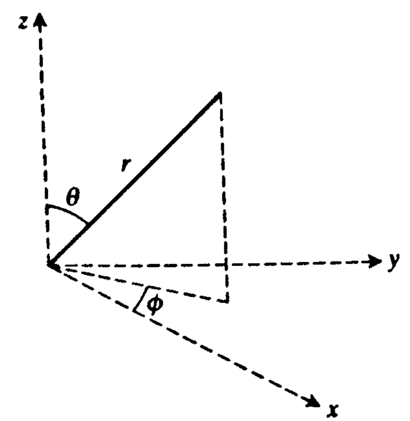


Fig. A.3. Spherical polar coordinates.

Example:

Accretion disk

In astrophysics, accretion describes the process whereby matter falls onto a central star at some rate $\dot{M}$ $(g s^{-1})$. This can release a lot of energy. The luminosity is $\sim \left(\frac{GM}{R}\right)\dot{M}$

grav. binding energy of central star (per gram)

Compared to the rest mass coming in $\dot{M}c^2$

The ratio is $\left(\frac{GM}{Rc^2}\right)$ For a black hole this ratio is substantial – several % of the rest mass energy is released.

One problem is that the incoming matter has angular momentum. It forms a disk in which viscous forces transport angular momentum outward allowing the matter to move inwards.

Let's solve a "thin accretion disk"

Geometrically thin disk, fluid rotates on Kepler orbits $\Omega^2 = \frac{GM}{r^3}$ (gravity of central star dominates)

ie. the flow is $\underline{u} = \hat{e}_\theta\, r\Omega(r) + \hat{e}_r\, u_r(r)$

inwards flow

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right) = 0$$

Integrate this over the vertical height $\Sigma = \int \rho\, dz$

$$\boxed{\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma u_r\right) = 0} \quad (*)$$

$g\,cm^{-2}$ through the disk

The momentum equation is (see Acheson appendix)

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r}\right) = \rho\nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right)$$

Or height integrated $\rho \rightarrow \Sigma$.

Compared to our earlier equation on page 36,
We now include the advection terms on the LHS – the inwards radial flow advects angular momentum inwards. To see this, combine the continuity and momentum equations:

$$\frac{\partial}{\partial t}\left(r u_\theta \Sigma \right) = -u_\theta \frac{\partial}{\partial r}\left(r \Sigma u_r \right) \cancel{+} - \Sigma r u_\theta \frac{\partial u_\theta}{\partial r} - \frac{\Sigma r u_r u_\theta}{r}$$

$$+ \left(\text{viscous term} \right)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma r^2 u_\theta u_r \right) + \left(\text{viscous term} \right)$$

But $r u_\theta \Sigma$ is the angular momentum per unit area $J \equiv r u_\theta \Sigma$

$$\Rightarrow \boxed{\frac{\partial J}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r J \right) = \left(\text{viscous term} \right).}$$

Writing $\Omega = \frac{u_\theta}{r}$, the viscous term $\Sigma \nu r \left[u_\theta'' + \frac{u_\theta'}{r} - \frac{u_\theta}{r^2} \right]$

$$= \Sigma \nu \frac{1}{r^{\cancel{2}}} \frac{\partial}{\partial r}\left(r^3 \Omega' \right)$$

This form for the viscous term assumes $\mu = \rho \nu =$ constant. In fact, in an accretion disk μ changes with position. The correct equation has the $\nu\Sigma$ inside the derivative $\frac{1}{r^{\cancel{2}}}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \Omega' \right)$

giving the result

$$(\dagger) \quad \boxed{\frac{\partial}{\partial t}\left(r^2 \Omega \Sigma \right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r r^2 \Omega \Sigma \right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \Omega' \right)}$$

The viscous term makes sense because the torque is

$$\underbrace{\left(\rho \nu r \frac{d\Omega}{dr} \right)}_{\text{stress}} \times \underbrace{\left(2\pi r H \right)}_{\text{area}} \times \underbrace{r}_{\text{lever arm}} = 2\pi \nu \Sigma r^3 \frac{d\Omega}{dr}$$

So the viscous term says that the angular momentum of a ring of fluid will change if there is a difference in torque across it.
Equations (*) and (†) can be solved, eg. see Pringle 1981 (ARAA 19 137) for the Green's function.

Let's look at the steady-state solution:

(*) $\Rightarrow$ $r\Sigma u_r = \text{constant} = -\dfrac{\dot{M}}{2\pi}$

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

(†) $\Rightarrow$ $$\frac{\partial}{\partial r}\left(-\frac{\dot{M}}{2\pi} r^2\Omega\right) = \frac{\partial}{\partial r}\left(\nu\Sigma r^3\Omega'\right)$$

integrate from the inner edge at $r = R_*$ to a point r in the disk:

$$-\frac{\dot{M}}{2\pi}\left[r^{1/2} - R_*^{1/2}\right] = \nu\Sigma r^3\left(-\frac{3}{2}\frac{1}{r^{5/2}}\right) + (0)$$

assume no viscous stress at inner edge

$$\Rightarrow \boxed{\frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{r}\right)^{1/2}\right] = \nu\Sigma}$$

What is the viscous dissipation rate?

(energy per second per unit area)

$$\Phi_\nu = \cancel{\text{per}}\ \Sigma\nu r^2\left(\frac{d\Omega}{dr}\right)^2 = \Sigma\nu r^2\left(\frac{9}{4}\frac{\Omega^2}{r^2}\right)$$

$$= \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{r}\right)^{1/2}\right]\frac{9}{4}\frac{GM}{r^3}$$

$$= \frac{3GM\dot{M}}{r}\frac{1}{4\pi r^2}\left[1 - \left(\frac{R_*}{r}\right)^{1/2}\right]$$

What might we have guessed?

as matter moves from $r+dr$ to r, the gravitational energy relased is $\left(\dfrac{GM\dot{m}}{2r^2}dr\right)\dfrac{1}{(2\pi r\, dr)}$ per unit area

The energy of a circular orbit is $-\dfrac{1}{2}\dfrac{GM}{r}$ per unit mass

$$= \frac{GM\dot{M}}{4\pi r^3}.$$

We have an extra factor of $3\left[1 - \left(\frac{R_*}{r}\right)^{1/2}\right]$

so at large distances $r \gg R_*$ 3 times more energy is released than we would have expected! The extra energy comes from viscous dissipation associated with viscous transport of angular momentum. Note that the overall energy release is correct

$$\int \Phi_\nu \, 2\pi r \, dr = \frac{3GM\dot{M}}{2} \int_{R_*}^{\infty} dr \, \frac{1}{r^2}\left(1 - \left(\frac{R_*}{r}\right)^{1/2}\right)$$

$$= \frac{3GM\dot{M}}{2R_*} \int_1^{\infty} \frac{dx}{x^2}\left(1 - x^{-1/2}\right) = \underline{\underline{\frac{GM\dot{M}}{2R_*}}} \checkmark$$

III: Numerical Techniques

We've looked at a number of cases of linear and non-linear flows that can be solved analytically, but in general this is not the case, and we must proceed by either trying to make simplified models or by solving the fluid equations numerically. This is a vast subject and here we have time only to give a brief introduction and highlight some of the main ideas.

Acheson does not cover numerical techniques. Two good introductory references are Chapter 19 of "Numerical Recipes" by Press et al. and the book by Michael Thompson "An Introduction to Astrophysical Fluid Dynamics" Chapter 6.

We start by looking at how to solve the 1D advection-diffusion equation by finite differencing. The equation is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$

(constant) diffusivity

velocity (assumed constant)

which serves as a basic model for the kind of terms that appear in the fluid equations.

The finite differencing technique is to represent f on a grid:

f_j $j = 1$ to N

is the value of f at $x = x_j$

For simplicity, assume the grid spacing $x_{j+1} - x_j = \Delta x = \text{constant}$.

To derive expressions for the derivatives of f with respect to x, Taylor expand:

$$f_{j+1} = f_j + \Delta x \, f_j' + \frac{\Delta x^2}{2} f_j'' + O(\Delta x^3) \quad \text{—(1)}$$

$$f_{j-1} = f_j - \Delta x \, f_j' + \frac{\Delta x}{2} f_j'' + O(\Delta x^3) \quad \text{—(2)}$$

Subtracting these gives an expression for $f_j' = \frac{\partial f}{\partial x}$ at $x = x_j$

$$f_j' = \frac{f_{j+1} - f_{j-1}}{2\Delta x} + O(\Delta x^2)$$

the error is of order Δx^2
– we say that this is a
"2nd order accurate" expression

An alternative would be to use (1) or (2) seperately to write

$$f_j' = \frac{f_{j+1} - f_j}{\Delta x} + O(\Delta x)$$
$$= \frac{f_j - f_{j-1}}{\Delta x} + O(\Delta x)$$

these expressions are only first order accurate.

Adding (1) and (2) gives $\partial^2 f / \partial x^2$:

$$f_j'' = \frac{f_{j+1} - 2f_j + f_{j-1}}{(\Delta x)^2} + O(\Delta x^2).$$

Now we can use these expressions to write a finite difference representation of the differential equation we are trying to solve.

Let's focus first on advection only $\quad \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0.$

The first thing we might try is the FTCS scheme:

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$$

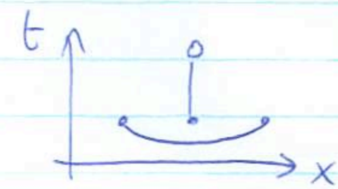
where we introduce a new label n that gives the time

$$\Rightarrow \quad f_j^{n+1} = f_j^n - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right). \quad -(*)$$

which gives a rule for updating the values of f to move from time n to $n+1$.

Because the value of f_j^{n+1} is written only in terms of values of f^n (ie. the values at the current timestep) we say that this method is "explicit".

We can represent it on a diagram:



The problem is that as we will now show this scheme is numerically unstable!

Von Neumann stability analysis

We look for solutions

$$f_j^n = \xi^n e^{ik(j\Delta x)} \qquad \text{—}(\dagger)$$

and the idea is that if $|\xi| > 1$ for any wavevector k then the scheme is unstable because f_j will grow exponentially with time.

For the FTCS scheme, substituting $(\dagger)$ into $(*)$ gives

$$e^{ikj\Delta x}\,\xi^{n+1} = e^{ikj\Delta x}\,\xi^n - \frac{v\Delta t}{2\Delta x}\left(\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}\right)$$

$$\Rightarrow \quad \xi = 1 - \frac{v\Delta t}{2\Delta x}\, 2i\sin(k\Delta x)$$

$$\Rightarrow \quad |\xi| = 1 + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)$$

> 1 for all k! FTCS is unstable.

As I showed in class, you soon see the instability if you implement the FTCS scheme – large fluctuations develop from grid point to grid point (large k grows fastest).

The Lax method

This method is stable. We modify the first term on the RHS of $(*)$

$$f_j^{n+1} = \frac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right) \qquad -(1)$$

This scheme has

$$\xi = \cos k\Delta x - \frac{iv\Delta t}{\Delta x}\sin k\Delta x \qquad -(2)$$

or $|\xi|^2 = \cos^2 k\Delta x + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x$

$$= 1 + \sin^2(k\Delta x)\left[\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1\right]$$

$\Rightarrow$ this scheme is stable if $\boxed{\frac{v\Delta t}{\Delta x} \leq 1}$ "Courant condition"

(or Courant-Friedrichs-Levy criterion)

You can understand this in terms of causality — we're only using the two nearest neighbors $j-1$ and $j+1$ to update j. This means that we don't have enough information to step further ahead than $\Delta x/v$ in time.

Why is this method stable? One way to see it is to write (1) as

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v\left(\frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}\right) + \frac{1}{2}\left(\frac{f_{j+1}^n - 2f_j^n + f_{j-1}^n}{\Delta t}\right)$$

which is the FTCS representation of

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \underbrace{\frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 f}{\partial x^2}}_{\text{diffusion term}}$$

This scheme has "numerical dissipation". For $|v|\Delta t < \Delta x$ then $|\xi| < 1$ and the amplitude decreases. The damping is small for long wavelength features $k\Delta x \ll 1$. Short scales with $k\Delta x \sim 1$ damp away quickly.

This is just one kind of error that a numerical scheme can make. $|\xi| < 1$ is known as an amplitude error.

There are also phase errors. Eq. (2) is

$$\xi = e^{-ik\Delta x} + \underbrace{i\left(1 - \frac{v\Delta t}{\Delta x}\right)\sin k\Delta x}$$

this term introduces dispersion when $v\Delta t \neq \Delta x$

Another kind of error is a transport error. For example, in the Lax method, information from both $j-1$ and $j+1$ travels to j at the next time step. But since the fluid velocity has a definite direction, this is unphysical.

To avoid this, we can use upwind differencing

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v_j^n \begin{cases} \dfrac{f_j^n - f_{j-1}^n}{\Delta x} & v_j^n > 0 \\[2ex] \dfrac{f_j^{n+1} - f_j^n}{\Delta x} & v_j^n < 0 \end{cases}$$

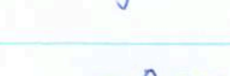
the stability condition is again the Courant condition.

Diffusion

Now let's look at the diffusion term $\quad \dfrac{\partial f}{\partial t} = D \dfrac{\partial^2 f}{\partial x^2}$ assume constant diffusivity here

Straightforward differencing gives

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right)$$



Is this stable? $f_j^n = \xi^n e^{ikj\Delta x}$

$$\Rightarrow \quad \xi^{n+1} - \xi^n = \xi^n \frac{\Delta t D}{(\Delta x)^2} \underbrace{\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)}_{2(\cos k\Delta x - 1)}$$

$$\Rightarrow \quad |\xi| = 1 - \frac{4D\Delta t}{(\Delta x)^2} \sin^2\left(\frac{k\Delta x}{2}\right)$$

$$\Rightarrow \quad \text{stable if} \quad \boxed{\frac{2\Delta t D}{(\Delta x)^2} \leq 1}$$

roughly this is
$\Delta t <$ diffusion time across gridcell

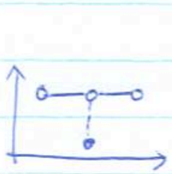
We have a stable scheme, but the problem is that it is slow. If we are interested in following diffusion across a macroscopic lengthscale $L \gg \Delta x$, the number of timesteps needed is

$$\simeq \frac{L^2}{D}\frac{1}{\Delta t} \simeq \frac{L^2}{(\Delta x)^2} \simeq N_{grid}^2 .$$

We need a scheme that allows larger timesteps (at the expense of accuracy on the smallest scales). Two possibilities are:

i) <u>implicit scheme</u>

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left(f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}\right)$$



or if we write $\alpha = \dfrac{D\Delta t}{(\Delta x)^2}$

this is

$$-\alpha f_{j+1}^{n+1} + (1+2\alpha) f_j^{n+1} - \alpha f_{j-1}^{n+1} = f_j^n$$

which we can write as a matrix equation $\underline{\underline{A}}\,\underline{f}^{n+1} = \underline{f}^n$

where $\underline{\underline{A}}$ is a <u>tridiagonal matrix</u>*

$$\begin{pmatrix} \ddots & \ddots & & & 0 \\ & -\alpha & 1+2\alpha & -\alpha & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

Inverting the matrix gives $\boxed{\underline{f}^{n+1} = \underline{\underline{A}}^{-1}\,\underline{f}^n}$

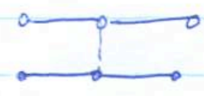
*(We use the fact that the matrix is tridiagonal when inverting it numerically)

The stability analysis for this scheme gives

$$\xi = \frac{1}{1 + 4\alpha \sin^2\left(\frac{k\Delta x}{2}\right)} < 1 \quad \text{for all } \Delta t.$$

Of course, a large timestep comes at the expense of numerical accuracy. In this scheme the solution goes to the steady-state solution (which obeys $f'' = 0$) for large Δt. The short wavelengths are not followed accurately, but adopt their steady-state solution (which makes physical sense — on timescales long compared to the local diffusion time we expect the steady-state or equilibrium solution).

2) <u>Crank–Nicholson</u> (semi-implicit)



Whereas the fully-implicit method is first order in time, this method is second order in time and space.

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left[\frac{1}{2}\left(f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}\right) + \frac{1}{2}\left(f_{j+1}^{n} - 2f_j^{n} + f_{j-1}^{n}\right)\right]$$

Also stable for all choices of Δt.

Boundary conditions

Boundary conditions are usually implemented using a dummy or ghost cell. For example, suppose our scheme uses f_{j-1} and f_j to update f_j. Then to update f_1, we need to know the value of f_0 which lies off the grid. The idea is to let the boundary condition inform us about f_0.

eg. the boundary condition $\frac{df}{dx} = C$ = constant implies

$$\frac{f_2 - f_0}{2\Delta x} = C \Rightarrow \underline{f_0 = f_2 - 2C\Delta x}.$$ This value for f_0 can be inserted into the equation used to update f_1.

Operator splitting

We started off with the advection - diffusion equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$

but considered advection and diffusion separately. How do we put them together? One way is to come up with a scheme that does both at once, but we can also apply them separately using operator splitting.

Two methods

1) suppose we have an equation $\frac{\partial f}{\partial t} = Lf = (L_1 + L_2 \ldots)f$

eg. L_1 = advection
L_2 = diffusion

then

$$\left.\begin{aligned} f^{n+1/m} &= U_1(f^n, \Delta t) \\ f^{n+2/m} &= U_2(f^{n+1/m}, \Delta t) \\ &\vdots \\ f^{n+1} &= U_m(f^{n+\frac{m-1}{m}}, \Delta t) \end{aligned}\right\}$$ apply each operator sequentially for the full timestep.

eg. advection diffusion

one possibility is $$f_j^{n+1/2} = \frac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right)$$

$$f_j^{n+1} = f_j^{n+1/2} + \frac{D\Delta t}{(\Delta x)^2}\left(f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}\right)$$

2) we can use an update scheme for the entire operator L at each step, where the update at each step need only be stable for each piece L_1, L_2, etc.

$$f^{n+1/m} = U_1\left(f^n, \frac{\Delta t}{m}\right)$$

$$\vdots$$

$$f^{n+1} = U_m\left(f^{n+\frac{m-1}{m}}, \frac{\Delta t}{m}\right)$$

eg. 2D diffusion

$$\frac{\partial f}{\partial t} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

(here we put $\frac{\alpha}{2}$ because we're taking $\frac{1}{2}$ a time step)

Alternating-direction implicit scheme (ADI)

$$f_{j,l}^{n+1/2} = f_{j,l}^n + \frac{\alpha}{2}\left(f_{j+1,l}^{n+1/2} - 2f_{j,l}^{n+1/2} + f_{j-1,l}^{n+1/2} + f_{j,l+1}^n - 2f_{j,l}^n + f_{j,l-1}^n\right)$$

$$f_{j,l}^{n+1} = f_{j,l}^{n+1/2} + \frac{\alpha}{2}\left(f_{j+1,l}^{n+1/2} - 2f_{j,l}^{n+1/2} + f_{j-1,l}^{n+1/2} + f_{j,l+1}^{n+1} - 2f_{j,l}^{n+1} + f_{j,l-1}^{n+1}\right)$$

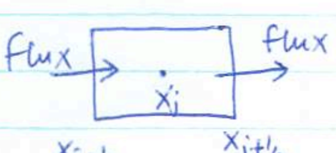
Flux-conserving formulation – Finite volume methods

The equations of hydrodynamics are of the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\underbrace{(uf)}_{\text{flux } J(f)} = (\text{sources and sinks})$$

ie. conservation equations. If possible we should use a formulation that conserves the quantity f (in the absence of sources and sinks).

In finite volume methods, we divide the volume into cells. The positions x_j now label the cell centers (rather than being grid points). The cell boundaries $(N+1)$ are at $x_{j\pm 1/2} = \frac{1}{2}(x_{j\pm 1} + x_j)$



Then $$\underbrace{\frac{d}{dt}(f_j \Delta x)}_{\substack{\text{rate of change of}\\ \text{amount of } f \text{ in cell } j}} = J_{j-1/2} - J_{j+1/2}$$

rate of change of amount of f in cell j = flux in − flux out

or $$\boxed{\frac{f_j^{n+1} - f_j^{n}}{\Delta t} = \frac{J_{j-1/2}^{n+1/2} - J_{j+1/2}^{n+1/2}}{\Delta x}}$$

where the fluxes are averaged over the timestep

$$J_{j+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} dt \; J_{j+1/2}(t) .$$

Summing over all cells

$$\sum_j \frac{d}{dt}(f_j \Delta x) = J_{-1/2} - J_{N+1/2}$$

we see that f is automatically conserved (except for flow through the boundaries). The idea is to choose the fluxes J to accurately represent the flow between cells.

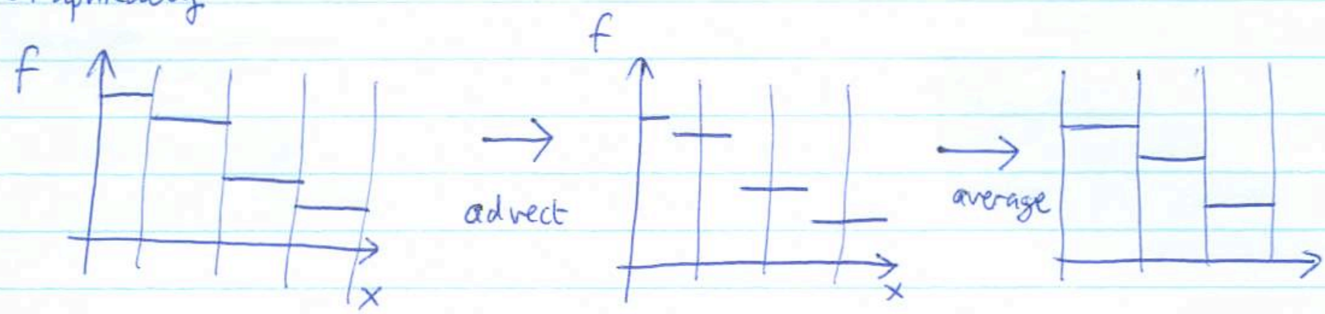
Donor cell advection (1st order accurate)

The simplest way to choose the fluxes. We assume that the quantity f is constant throughout the cell. → this is our 'subgrid model'

$$J_{j+\frac{1}{2}} = \begin{cases} v_{j+\frac{1}{2}} f_j^n & v_{j+\frac{1}{2}} > 0 \\ v_{j+\frac{1}{2}} f_{j+1}^n & v_{j+\frac{1}{2}} < 0 \end{cases}$$

$$J_{j-\frac{1}{2}} = \begin{cases} v_{j-\frac{1}{2}} f_{j-1} & v_{j-\frac{1}{2}} > 0 \\ v_{j-\frac{1}{2}} f_j & v_{j-\frac{1}{2}} < 0 \end{cases}$$

Graphically

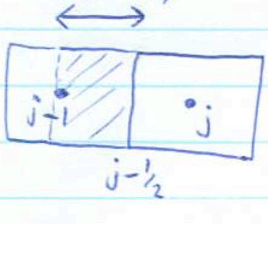


Piecewise linear schemes (second order)

Here we try to do better by assuming f varies linearly across the cell.

$$f(x, t=t^n) = \underbrace{f_j^n}_{\text{average value in the cell}} + \underbrace{\sigma_j^n}_{\text{slope}}(x - x_j) \qquad x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$$

For example, consider $v > 0$ (flow to the right) and constant v. Then



$$J_{j-\frac{1}{2}}(t) = v f(x = x_{j-\frac{1}{2}}, t)$$

$$= v f_{j-1}^n + v\sigma_{j-1}^n \left(x_{j-\frac{1}{2}} - v(t-t_n) - x_{j-1}\right)$$

$$= v f_{j-1}^n + v\sigma_{j-1}^n \left(\tfrac{1}{2}\Delta x - v(t-t_n)\right)$$

Now average over the timestep:

$$J_{j-1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}=t_n+\Delta t} dt\; J_{j-1/2}(t)$$
$$= v f_{j-1}^n + \frac{1}{2} v \sigma_{j-1}^n (\Delta x - v\Delta t).$$

The update is therefore

$$f_j^{n+1} - f_j^n = -\frac{v\Delta t}{\Delta x}\left(f_j^n - f_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(\sigma_j^n - \sigma_{j-1}^n\right) * (\Delta x - v\Delta t)$$

Three different choices for the slope:

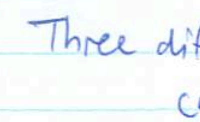
centered $\quad \sigma_j^n = \dfrac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$ $\quad$ Fromm's

upwind $\quad \sigma_j^n = \dfrac{f_j^n - f_{j-1}^n}{\Delta x}$ $\quad$ beam warming

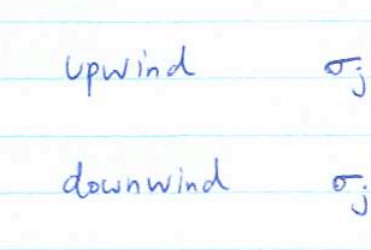
downwind $\quad \sigma_j^n = \dfrac{f_{j+1}^n - f_j^n}{\Delta x}$ $\quad$ Lax-Wendroff.

eg. if we choose the centered expression, we get Fromm's method

$$f_j^{n+1} - f_j^n = -\frac{v\Delta t}{4\Delta x}\left(f_{j+1}^n + 3f_j^n - 5f_{j-1}^n + f_{j-2}^n\right)$$
$$-\left(\frac{v\Delta t}{2\Delta x}\right)^2\left(f_{j+1}^n - f_j^n - f_{j-1}^n + f_{j-2}^n\right)$$



Graphically:



then advect by $v\Delta t$ and average as before

recall that we assume here v to the right, so we need the slope in cells $j-1$ and j $\Rightarrow$ we need the values of f in cells $j-2, j-1, j, j+1$.

A 1D hydro code example — see Heidelberg lecture course on Numerical methods in hydrodynamics (Dullemond and Johansen 2007 Chp5)

write $f_1 = \rho$ and take the equation of state to be
$f_2 = \rho u$ $\qquad P = \rho c_s^2 \quad c_s^2 = \text{constant}$.

then
$$\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial x}(u f_1) = 0$$
$$\frac{\partial f_2}{\partial t} + \frac{\partial}{\partial x}(u f_2) = -\frac{\partial P}{\partial x}.$$

The algorithm is

1) $$f_{1,j}^{n+1/2} = f_{1,j}^{n} - \frac{\Delta t}{\Delta x}\left(J_{1,j+1/2} - J_{1,j-1/2}\right)$$

with donor cell flux
$$J_{1,j+1/2} = \begin{cases} f_{1,j}^{n}\, u_{j+1/2}^{n} & \text{if } u_{j+1/2}^{n} > 0 \\ f_{1,j+1}^{n}\, u_{j+1/2}^{n} & \text{"} \quad < 0 \end{cases}$$

where the velocity is
$$u_{j+1/2}^{n} = \frac{1}{2}\left(\frac{f_{2,j}}{f_{1,j}} + \frac{f_{2,j+1}}{f_{1,j+1}}\right)$$

and similarly for $J_{1,j-1/2}$.

The same rule also updates $f_{2,j}^{n} \to f_{2,j}^{n+1/2}$

2) Now add the source term
$$f_{1,j}^{n+1} = f_{1,j}^{n+1/2}$$
$$f_{2,j}^{n+1} = f_{2,j}^{n+1/2} - \frac{c_s^2}{\Delta x}\left(f_{1,j+1}^{n+1/2} - f_{1,j-1}^{n+1/2}\right).$$

Part 4 Waves

Let's try to understand some of the behavior we saw in the numerical simulations — waves and steepening.

Sound waves

Consider a constant density gas at rest.
Perturb the gas density $\rho + \delta\rho$
velocity $\underline{u} + \underline{\delta u}$

To first order in the perturbations,

continuity $$\frac{\partial \delta\rho}{\partial t} + \rho\, \underline{\nabla}\cdot \delta\underline{u} = 0 \qquad \text{— (1)}$$

momentum $$\frac{\partial \delta\underline{u}}{\partial t} = -\frac{\underline{\nabla}\delta P}{\rho}$$

Notice in particular that the non-linear term $(\underline{u}\cdot\underline{\nabla})\underline{u}$ has gone away — it is second order in $\delta\underline{u}$. We are left with a linear system of equations.
To close the equations we need a relation between δP and $\delta\rho$.
If the perturbations are adiabatic then

$$\frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho}$$

$$\Rightarrow \frac{\partial}{\partial t}\delta\underline{u} = -\frac{\gamma P}{\rho}\frac{\underline{\nabla}\delta\rho}{\rho} \qquad \text{— (2)}$$

Combining (1) and (2) gives a wave equation

$$\boxed{\frac{\partial^2 \delta\rho}{\partial t^2} = \frac{\gamma P}{\rho}\nabla^2\delta\rho} \quad \text{— } (*) \qquad \left(\begin{matrix}\text{same for}\\ \delta\underline{u}\end{matrix}\right)$$

A wave equation with wave speed $c_s^2 = \dfrac{\gamma P}{\rho}$

c_s is the <u>adiabatic sound speed</u>.

Since we have a linear equation, we can decompose into modes

$$\delta\rho,\ \delta\underline{u} \ \propto\ e^{i\underline{k}.\underline{r} - i\omega t}$$

$$(*) \Rightarrow \qquad -\omega^2 \delta\rho = c_s^2 (-k^2)\,\delta\rho$$

$$\Rightarrow \qquad \boxed{\omega^2 = c_s^2 k^2}$$

The <u>dispersion relation</u> for the modes

The phase and group velocities are

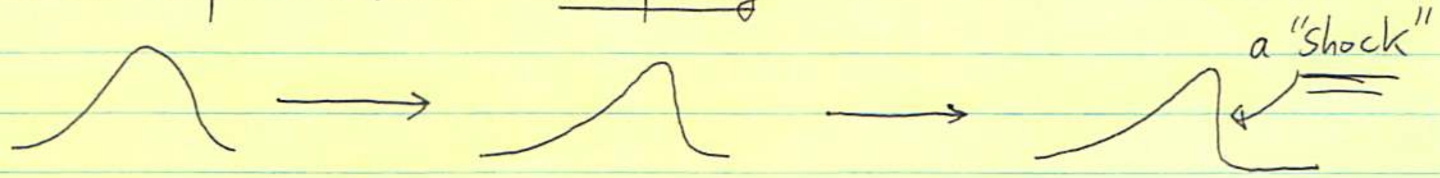
$$v_p = \frac{\omega}{k} = c_s \qquad \leftarrow \text{independent of frequency, no dispersion}$$

$$v_g = \frac{\partial\omega}{\partial k} = c_s$$

Air at room temperature has a sound speed $\simeq 330$ m/s.

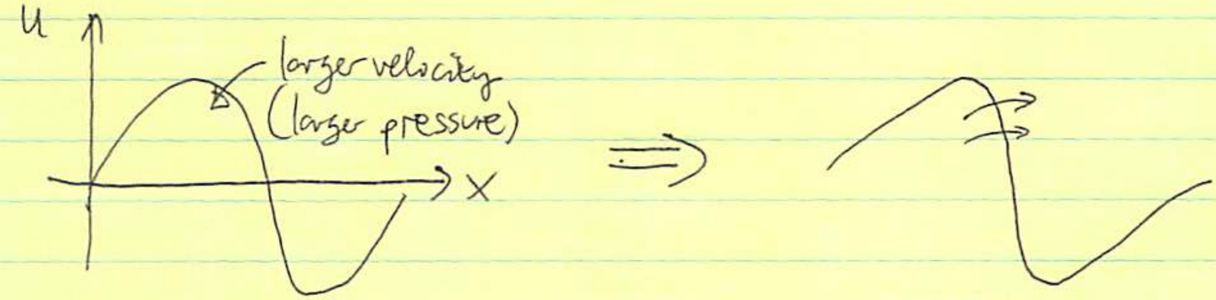
<u>Non-linear steepening</u>

We saw numerically that only small amplitude perturbations propagated without changing shape. Larger amplitude perturbations show the phenomenon of "<u>steepening</u>"



Steepening occurs because of the $(\underline{u}.\underline{\nabla})\underline{u}$ term in the momentum equation.
Different ways to think about this:

1. in a disturbance, the peaks have a larger ρ and therefore a larger velocity (c_s is larger)



2. generation of harmonics: a wave $\propto e^{ikx}$ has $\underline{u}.\underline{\nabla}\underline{u} \propto e^{i2kx}$

3. we can look at solutions of the advection terms in 1D, $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ (Burgers equation)

$\Rightarrow \quad u = f(x - ut)$ is the solution for some function f

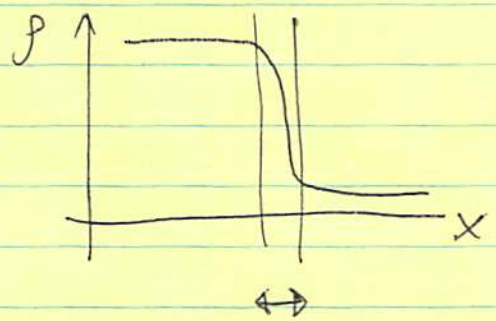
[Exercise: show that this form for u satisfies the equation]

The velocity gradient is $\dfrac{\partial u}{\partial x} = \dfrac{f'}{1+f't}$

where $f' \equiv \left.\dfrac{\partial u}{\partial x}\right|_{t=0}$. This shows that if $\dfrac{\partial u}{\partial x} < 0$ at $t=0$ then $\dfrac{\partial u}{\partial x}$ increases with time, becoming infinite when

$$t = \left(- \frac{1}{\partial u/\partial x|_{t=0}} \right)$$ ie. the initial "turnover time"

Of course the profile never gets to $\frac{\partial u}{\partial x} = \infty$ — diffusion acts to prevent the wave steepening too much



Shock thickness δ set by $\underline{u} \cdot \underline{\nabla}\, \underline{u} \sim \nu \nabla^2 \underline{u}$

$$\Rightarrow \frac{u^2}{\delta} \sim \frac{\nu u}{\delta^2}$$

$$\Rightarrow \delta \sim \nu/u$$

We can see that this is very thin because recall that $\nu \sim c_s \lambda$ where λ = mean free path

$$\Rightarrow \delta \sim \lambda \left(\frac{c_s}{u} \right)$$

$\Rightarrow$ shock thickness is comparable to mean free path.

General solution to the linearized wave equation

(eq. * on p 1)

The general solution to the wave equation is

$$\frac{\delta\rho}{\rho} = f(x - c_s t) + g(x + c_s t) \qquad (\dagger)$$

propagates to the right propagates to the left

[To see this change variables to $\xi = x - ct$ and $\eta = x + ct$ $\Rightarrow$ $\frac{\partial^2 \delta\rho}{\partial\xi\partial\eta} = 0$

$\Rightarrow$ $\frac{\delta\rho}{\rho} = f(\xi) + g(\eta)$]

Note that if $\delta u(x \pm c_s t)$, then

$$\frac{\partial}{\partial x}\delta u = \pm\frac{1}{c_s}\frac{\partial}{\partial t}\delta u$$

so the continuity equation is

$$\frac{1}{\rho}\frac{\partial}{\partial t}\delta\rho = -\frac{\partial}{\partial x}\delta u = \mp\frac{1}{c_s}\frac{\partial}{\partial t}\delta u$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho} \pm \frac{\delta u}{c_s}\right) = 0$$

$$\Rightarrow \quad \boxed{\frac{\delta\rho}{\rho} = \mp\frac{\delta u}{c_s}}$$

This means we can write

$$\frac{\delta u}{c_s} = f(x - c_s t) - g(x + c_s t)$$

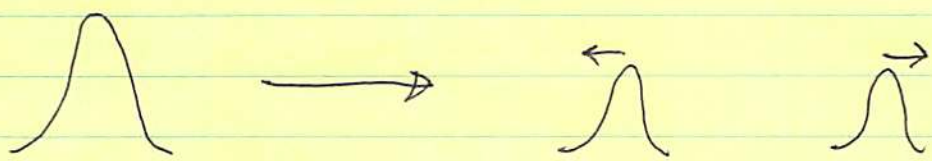
where f and g are the same functions as in $(\dagger)$.

These functions are determined by the initial conditions

$$f = \frac{1}{2}\left[\frac{\delta\rho}{\rho}(x, t=0) + \frac{\delta u}{c_s}(x, 0) \right]$$

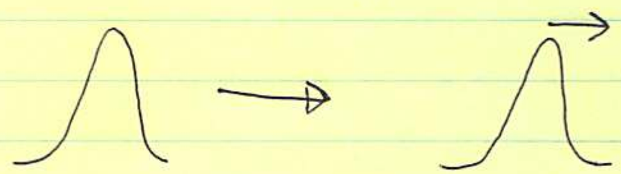
$$g = \frac{1}{2}\left[\frac{\delta\rho}{\rho}(x, 0) - \frac{\delta u}{c_s}(x, 0) \right]$$

So we see that for example an initial disturbance with $\delta u = 0$ has equal left and right going pieces

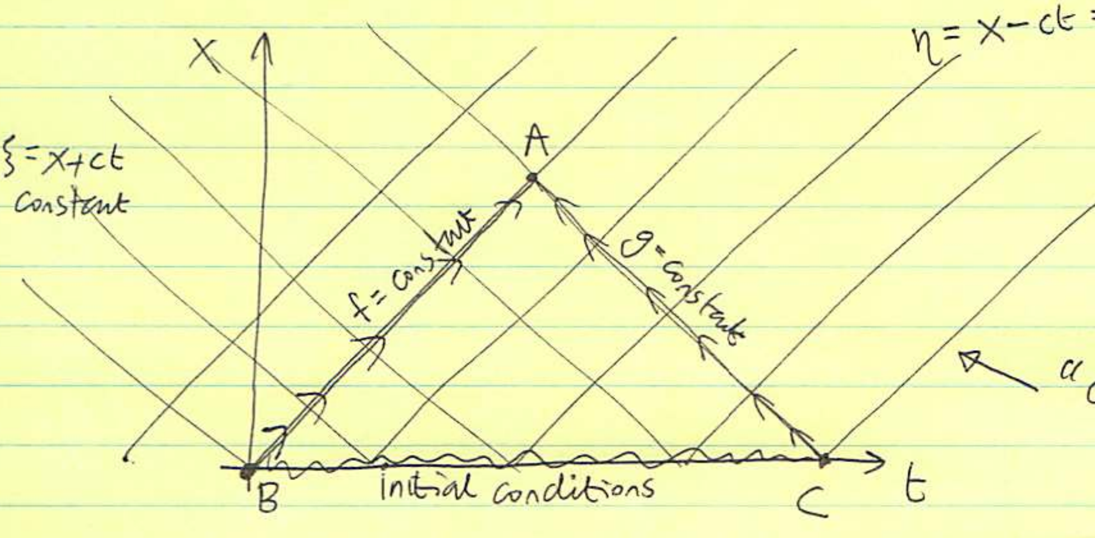


If we choose $\frac{\delta u}{c_s} = \frac{\delta\rho}{\rho}$ initially, then $g = 0$

$\Rightarrow$ right going pulse only.



In the (x,t) plane:



The solution at A is determined by f at B and g at C.

"characteristic curves"

Non-linear disturbances: the method of characteristics

Last time we showed that the linear wave equation for sound waves has a solution

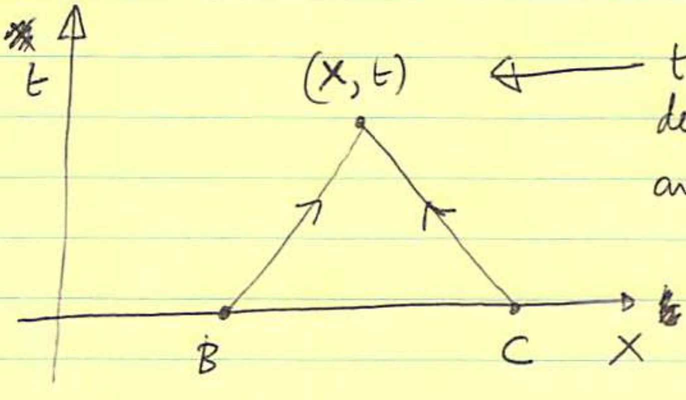
$$\frac{\delta \rho}{\rho} = f(x - c_s t) + g(x + c_s t)$$

$$\frac{\delta u}{c_s} = f(x - c_s t) - g(x + c_s t)$$

where the functions f and g are set by the initial conditions

at $t=0$
$$f = \frac{1}{2}\left[\frac{\delta \rho}{\rho} + \frac{\delta u}{c_s}\right]$$
$$g = \frac{1}{2}\left[\frac{\delta \rho}{\rho} - \frac{\delta u}{c_s}\right]$$

Graphically,



the solution here depends on f at point B and g at point C.

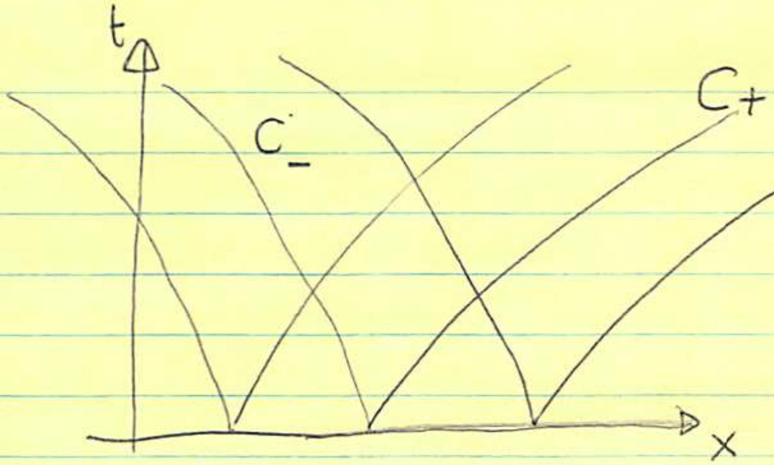
In this case, the characteristic curves along which f and g are constant are straight lines because information propagates to the left and right at the sound speed.

If the fluid is moving, however, the waves propagate at velocities

$u+c$ "to the right"
and $u-c$ "to the left"

[Q: why did I put quotes here?]

The characteristic curves are then more complicated:



The C_+ characteristics are described by

$$\frac{dx_+}{dt} = u + c$$

and C_- by $$\frac{dx_-}{dt} = u - c$$

but the same idea applies — the initial conditions propagate along the characteristic curves.

Consider an <u>isentropic flow</u> (everywhere $P = K\rho^\gamma$)

continuity $\Rightarrow$ $$\frac{D\rho}{Dt} = -\rho \frac{\partial u}{\partial x}$$

or $$\frac{1}{c_s^2}\frac{DP}{Dt} + \rho\frac{\partial u}{\partial x} = 0$$

since $c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_s = \frac{\gamma P}{\rho}$.

momentum $\Rightarrow$ $$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

Therefore $$\frac{1}{\rho c_s}\frac{\partial P}{\partial t} + \frac{u}{\rho c_s}\frac{\partial P}{\partial x} + c_s\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

Add and subtract: $$\left[\frac{\partial u}{\partial t} + (u \pm c_s)\frac{\partial u}{\partial x}\right] \pm \frac{1}{\rho c_s}\left[\frac{\partial P}{\partial t} + (u \pm c_s)\frac{\partial P}{\partial x}\right] = 0$$

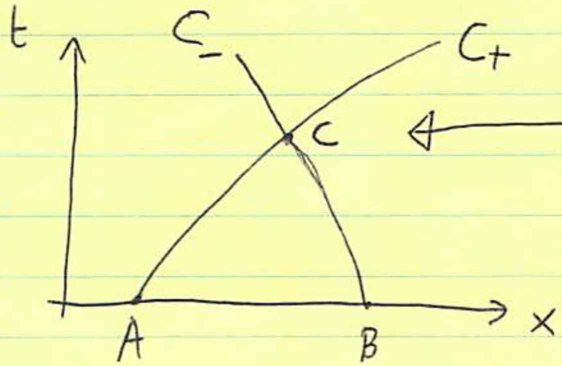
We then define the <u>Riemann invariants</u> $\quad J_\pm = u \pm \int \frac{dP}{\rho c_s}$

$$= u \pm \frac{2c_s}{\gamma - 1}$$

$$\Rightarrow \quad \boxed{\frac{\partial J_\pm}{\partial t} + (u \pm c_s)\frac{\partial J_\pm}{\partial x} = 0}$$

which shows that $J_\pm$ is constant along the curves $x_\pm(t)$

where $\frac{dx_\pm}{dt} = u \pm c_s$. We label these curves $C_\pm$.



(u, c_s) at point C depend on J_+ from A and J_- from B

We reconstruct them using

$$u = \frac{1}{2}(J_+ + J_-)$$

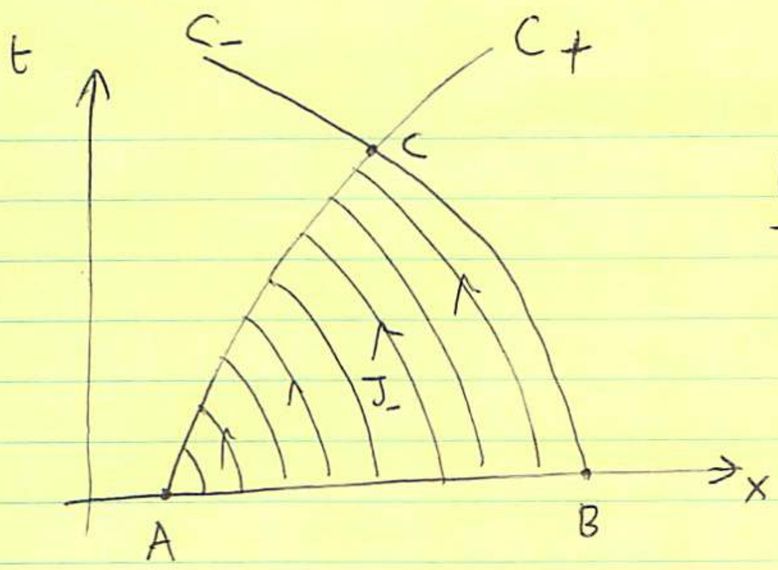
$$c_s = \left(\frac{\gamma - 1}{4}\right)(J_+ - J_-)$$

It is instructive to rewrite the equations for $x_\pm$ in terms of $J_\pm$:

$$\frac{dx_+}{dt} = u + c_s = \left(\frac{\gamma+1}{4}\right)J_+ + \left(\frac{3-\gamma}{4}\right)J_-$$

$$\frac{dx_-}{dt} = u - c_s = \left(\frac{3-\gamma}{4}\right)J_+ + \left(\frac{\gamma+1}{4}\right)J_-$$

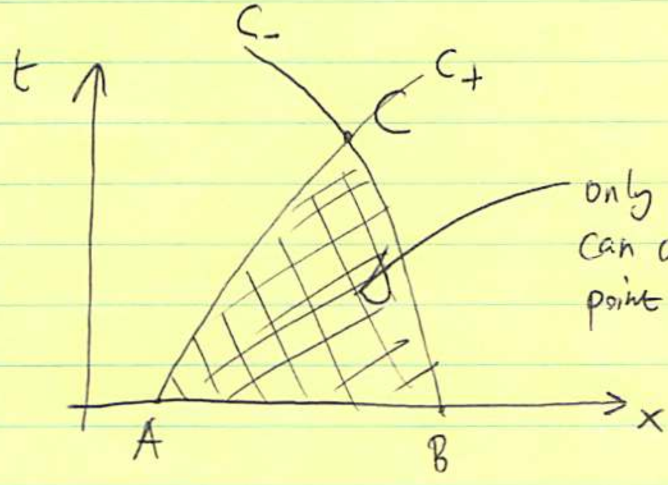
We see that the shape of $x_+(t)$ for example depends on how J_- varies along the curve (J_+ is constant).



The shape of C_+ is determined by the values of J_- along it. These values of J_- are set by the initial conditions between A and B.

Vice-versa for the curve C_- from A to C.

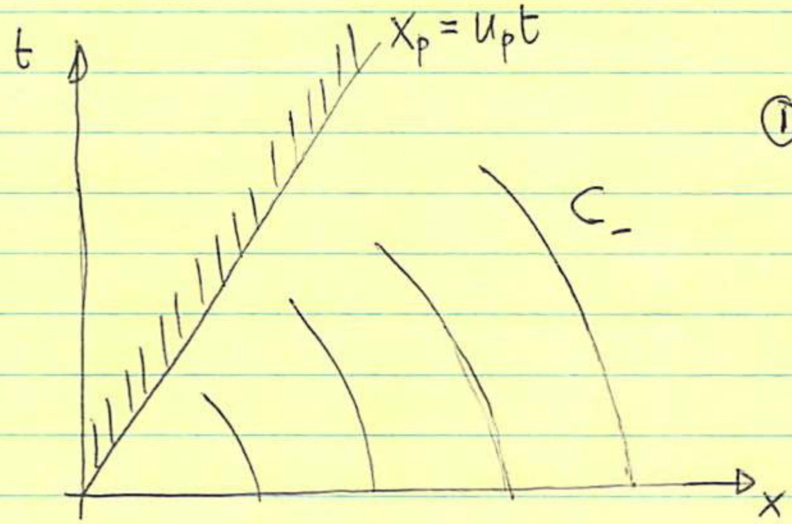
We see that only the initial conditions between A and B can affect the solution at point C. The idea of causality therefore arises naturally in this picture:



only points in this region can communicate information to point C.

Example: Piston propagating into shock tube

A piston is pushed into a semi-infinite tube of gas with constant velocity, so that the position of the piston is $x_p = u_p t$. What happens?



① At $t=0$ $J_- = -\dfrac{2c_0}{\gamma-1}$ = constant since the fluid is initially at rest and has the same sound speed c_0 everywhere.

② the value of J^- is carried into the rest of the fluid by the C_- characteristics $\Rightarrow$

$$\boxed{J_- \text{ is constant everywhere}}$$

But $J_- = u - \dfrac{2c}{\gamma-1} = -\dfrac{2c_0}{\gamma-1}$

$$\Rightarrow \quad \boxed{c = c_0 + \left(\frac{\gamma-1}{2}\right)u}$$

The sound speed is a function of the local fluid velocity only, $c(u)$.

③ New we can solve for the shape of the C_+ curves, because both J_+ and J_- are constant along each one

$$\frac{dx_+}{dt} = \left(\frac{\gamma+1}{4}\right)J_+ + \left(\frac{3-\gamma}{4}\right)J_- = \text{constant}$$

$\Rightarrow$ $\boxed{\text{the } C_+ \text{ characteristics are straight lines}}$ with slope

given by $$\frac{dx_+}{dt} = c + u = c_0 + \left(\frac{\gamma+1}{2}\right) u$$

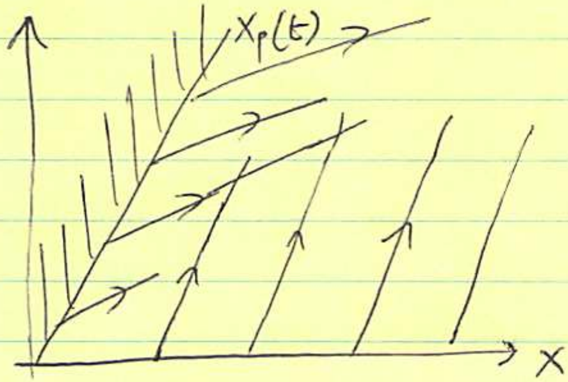
$\therefore$ constant slope $\Rightarrow$ $\boxed{u \text{ is constant along the } C_+ \text{ characteristics}}$

There are two sets of C_+ curves:

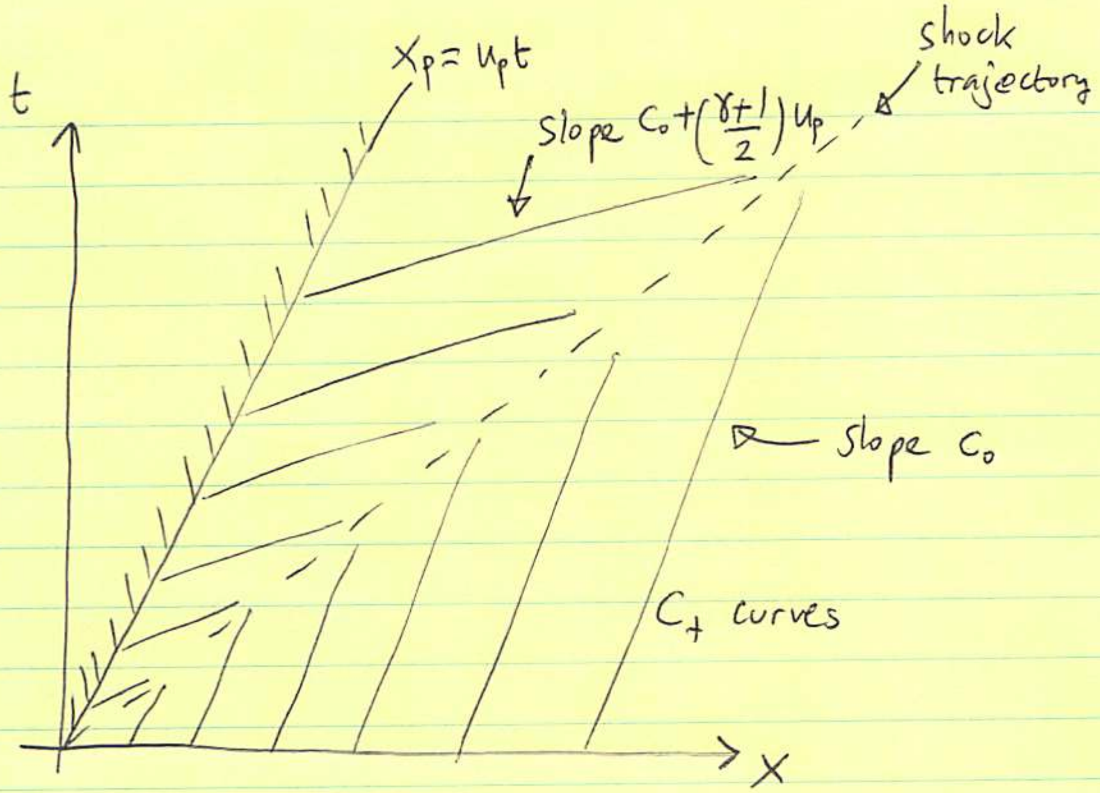
1) emerging from the undisturbed gas at $t = 0$
they have $\frac{dx_+}{dt} = c_0$, $u = 0$

2) emerging from the piston (with $u = u_p$) $\quad \frac{dx_+}{dt} = c_0 + \left(\frac{\gamma+1}{2}\right) u_p$

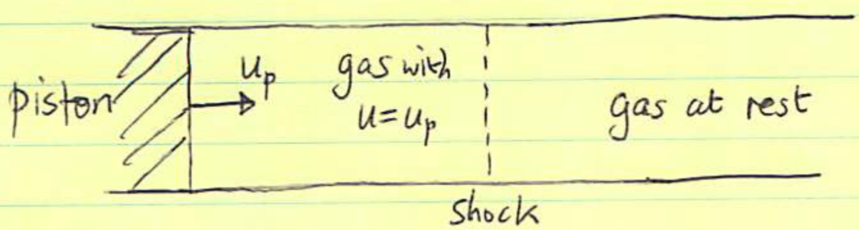
These two curves intersect because of their different slopes:



There appears to be a problem because only one value of J_+ is needed at each point, so the solution appears overdetermined. In fact what happens is that a <u>shock</u> forms in which the fluid properties change discontinuously from one value of J_+ to the other:



Here's a picture of what is happening:

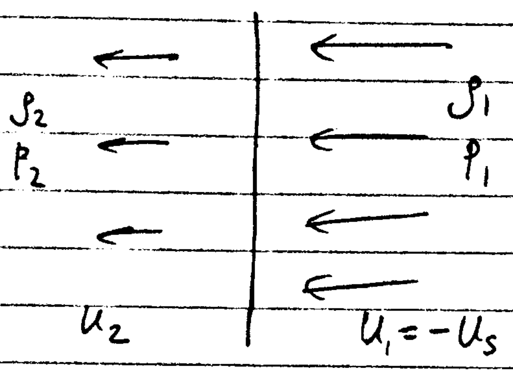


Question for next time: how can we get the shock speed?

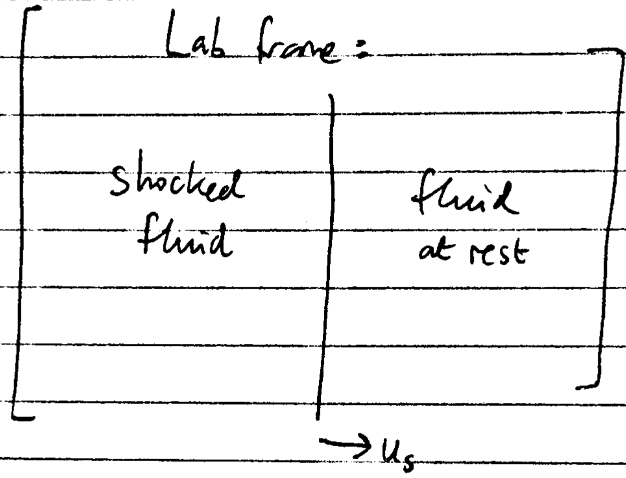
Shock jump conditions (Rankine-Hugoniot relations)

The fluid velocity and thermodynamic variables (P, ρ, T, c_s) change over a very short lengthscale at a shock. Rather than calculating the details of the shock structure, we can treat it as a discontinuity and use conservation laws to relate quantities on each side.

In the frame of the shock:



[Lab frame:



]

Mass conservation (continuity)

steady flow in 1D $\Rightarrow$ $\frac{\partial}{\partial x}(\rho u) =$ constant

integrate across the shock: $\boxed{\rho_1 u_1 = \rho_2 u_2}$ — (1)

Momentum:

$$\rho u \frac{du}{dx} = \frac{d}{dx}\left(\rho u^2\right) = -\frac{dP}{dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\rho u^2 + P\right) = 0 \quad \Rightarrow \quad \boxed{\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2} \text{ — (2)}$$

[Q: viscous terms do not contribute here — why? Hint: $du/dx = 0$ away from the shock]

Energy: $\frac{d}{dx}\left(u\left[\frac{1}{2}\rho u^2 + \rho E + P\right]\right) = 0 \quad \Rightarrow \quad \frac{1}{2}u^2 + E + \frac{P}{\rho}$ is the same on both sides

To simplify this, write $\rho E = \frac{P}{(\gamma-1)}$ (eg. monatomic ideal gas $\gamma = \frac{5}{3}$ $E = \frac{3}{2} P = \frac{3}{2} \frac{\rho k T}{\mu m_p}$)

$$\Rightarrow E + \frac{P}{\rho} = \frac{P}{\rho} \frac{\gamma}{\gamma - 1}$$

$$\Rightarrow \boxed{\frac{1}{2} u_1^2 + \left(\frac{\gamma}{\gamma-1}\right) \frac{P_1}{\rho_1} = \frac{1}{2} u_2^2 + \left(\frac{\gamma}{\gamma-1}\right) \frac{P_2}{\rho_2}} \quad (3)$$

Equations (1), (2), (3) relate the "upstream" conditions (ρ_1, P_1, u_1) to the "downstream" ones (ρ_2, P_2, u_2)

Some interesting results follow:

① Use (1) and (2) to eliminate P_2 and u_2 from (3)

$$\Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2}}$$

where $M_1 = u_1 / c_1$ (upstream Mach number) $= \frac{\text{shock velocity}}{\text{undisturbed sound speed}}$

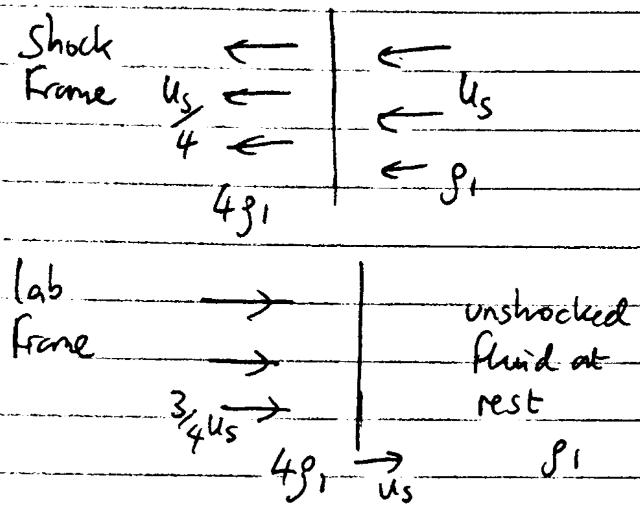
For $M_1 \gg 1$ $\quad \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$ Maximum compression eg. $\gamma = 5/3 \Rightarrow \frac{\rho_2}{\rho_1} = 4$.

② The pressure jump is $\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{(\gamma+1)}$

The $P_2 - \rho_2$ relation is known as the <u>shock adiabat</u> or <u>Hugoniot curve</u>. But note that the flow across the shock is definitely not adiabatic! There is a large jump in entropy as the ordered bulk motion of the incoming fluid is converted into heat in the compressed gas.

e.g. strong shock $M_1 \to \infty$
ideal gas $\gamma = 5/3$

$$\frac{\rho_2}{\rho_1} = 4 = \frac{u_1}{u_2}$$



$$\frac{P_2}{P_1} = \frac{10}{7} M_1^2$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{5}{14} M_1^2 = \frac{5}{14} \frac{u_s^2 \mu m_p}{\frac{5}{3} k_B T_1}$$

$$\Rightarrow \boxed{\frac{k_B T_2}{\mu m_p} = \frac{3}{14} u_s^2}$$

In HW4, you will use the shock jump conditions to calculate the speed of the shock in the 1D piston experiment we talked about last time

Lagrangian vs. Eulerian perturbations

We can take a Lagrangian or Eulerian view when thinking about perturbations.

Let $\underline{x}_0$ label each fluid element (eg. $\underline{x}_0$ = initial position)

Then define the displacement $\underline{\xi} = \underline{r}(\underline{x}_0, t) - \underline{r}_0(\underline{x}_0, t)$

position of the fluid element in the perturbed flow

position of the fluid element in the unperturbed flow

The Eulerian perturbation in the quantity f is

$$\delta f(\underline{r}, t) = f(\underline{r}, t) - f_0(\underline{r}, t)$$

value in perturbed flow at $\underline{r}$

value in unperturbed flow at $\underline{r}$

The Lagrangian perturbation is

$$\Delta f(\underline{x}_0, t) = f(\underline{x}_0, t) - f_0(\underline{x}_0, t)$$

or $\Delta f(\underline{r}, t) = f(\underline{r}, t) - f_0(\underline{r}_0, t)$

where $\underline{r} = \underline{r}_0 + \underline{\xi}$

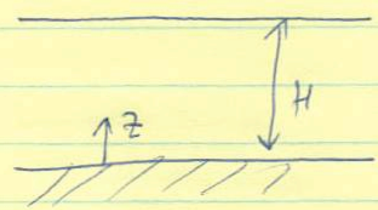
The relation between Δf and δf is

$$\Delta f = \delta f + \left[f_0(\underline{r}, t) - f_0(\underline{r}_0, t) \right]$$

To first order in $\underline{\xi}$, $\boxed{\Delta f = \delta f + \underline{\xi} \cdot \underline{\nabla} f_0}$

Surface gravity waves

A plane-parallel layer of incompressible fluid, initially at rest, in hydrostatic balance with a vertical gravitational field — eg. the ocean.



The continuity equation is $\underline{\nabla} . \delta \underline{u} = 0$

Momentum $\rho \frac{\partial}{\partial t} \delta \underline{u} = - \underline{\nabla} \delta P$ [no gravity term because $\delta \rho = 0$]

take the divergence $\Rightarrow \nabla^2 \delta P = 0$

Look for solution $\delta P = f(z) e^{i(k_\perp x - \omega t)}$

$$\Rightarrow \quad f'' - k_\perp^2 f = 0$$

$$\Rightarrow \quad \boxed{f = A e^{-k_\perp z} + B e^{+k_\perp z}}$$

What are the boundary conditions? At the floor of the ocean there is a hard surface $\Rightarrow \delta u_z = 0$ at $z = 0$

the z-cpt of momentum is $\rho \frac{\partial \delta u_z}{\partial t} = - \frac{\partial \delta P}{\partial z}$

$$\Rightarrow \quad -i\omega \rho \, \delta u_z = - f'$$

$$\Rightarrow \quad f' = 0 \quad \text{at} \quad z = 0$$

$$\Rightarrow \quad \boxed{A = B}$$

At the top of the ocean, the pressure must match the external atmospheric pressure. Therefore $\underline{\Delta P = 0}$ at $z = H$

or $\quad \delta P + \xi_z \frac{dP}{dz} = 0 = \delta P + \rho g \xi_z$

$$\Rightarrow \quad \delta p = -\rho g \xi_z \qquad \text{at } z = H$$

$$\underbrace{A e^{-k_\perp H} + B e^{k_\perp H}}_{\delta P} = -\rho g \cdot \underbrace{\frac{k_\perp}{\rho \omega^2}\left[-A e^{-k_\perp H} + B e^{k_\perp H}\right]}_{\xi_z}$$

[where we use the fact that $\delta u_z = -i\omega \xi_z \Rightarrow \xi_z = \frac{f'}{\rho\omega^2}$]

$$\Rightarrow \quad \boxed{\tanh(k_\perp H) = \frac{\omega^2}{g k_\perp}}$$ This is the dispersion relation for the waves

Two limits: (1) deep $k_\perp H \gg 1 \quad \tanh(k_\perp H) \to 1$

$$\Rightarrow \quad \boxed{\omega^2 \simeq g k_\perp}$$

[the waves don't know about the finite thickness]

(2) shallow $k_\perp H \ll 1 \quad \tanh(k_\perp H) \simeq k_\perp H$

$$\Rightarrow \quad \boxed{\omega^2 \simeq g k_\perp^2 H}$$

Eigenfunctions

$$\delta p \propto \cosh(k_\perp z)$$
$$\xi_z \propto \sinh(k_\perp z)$$

in the deep limit $\delta p \propto \xi_z \propto e^{k_\perp z}$

(most of the wave displacement + energy is within λ below the surface)

" shallow " $\delta p \simeq$ constant, $\xi_z \propto z$

{ whole depth of ocean participates.

What about the horizontal displacement? The momentum equations are $-\rho\omega^2 \xi_z = -\frac{d\,\delta P}{dz} \qquad -\rho\omega^2 \xi_\perp = -ik_\perp \delta P$

$$\Rightarrow \text{ if } \delta P = \cosh(k_\perp z)$$
$$\xi_z = \frac{k_\perp}{\rho\omega^2} \sinh(k_\perp z)$$
$$\xi_\perp = \frac{i k_\perp}{\rho\omega^2} \cosh(k_\perp z)$$
$$\Rightarrow \frac{\xi_z}{-i\xi_\perp} = \tanh(k_\perp z)$$

('i' factor ⇒) 90° out of phase

For deep waves, $\xi_z \simeq -i\xi_\perp$ circular motions

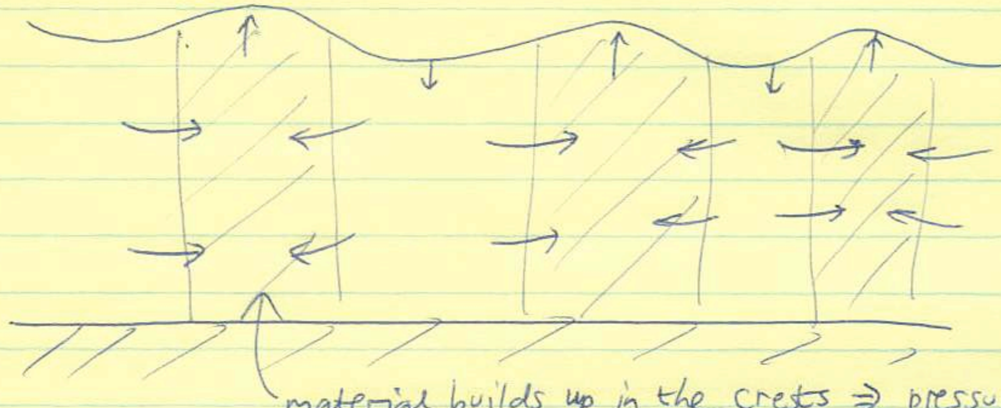
shallow waves, $\frac{\xi_z}{-i\xi_\perp} \simeq k_\perp H \ll 1$ eg. Tsunami

Estimate typical periods

deep limit period $\simeq \frac{2\pi\lambda}{g}$ $\simeq$ 1 second for $\lambda \simeq 1\text{m}$.

shallow period $\simeq \frac{\lambda}{\sqrt{gH}}$ $\simeq$ 1 hour for $\lambda = 1000\text{km}$, $H = 10\text{km}$ (Tsunami case)

Physics of the wave:



material builds up in the crests ⇒ pressure gradient horizontally ⇒ restoring force.

The shallow water wave is <u>non-dispersive</u> $\omega = k_\perp \sqrt{gH}$

$$\Rightarrow \quad v_g = v_p = \sqrt{gH}$$ * see note below

$\left(\approx 300\,\mathrm{m/s} \text{ for } H = 10\,\mathrm{km},\ g = 10 \right)$

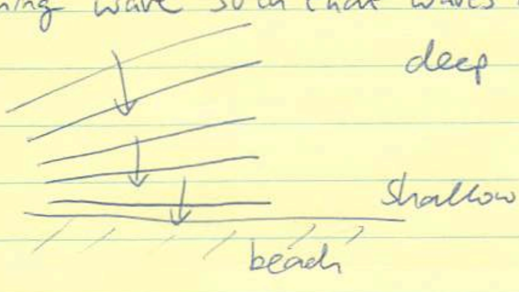
The deep waves are <u>dispersive</u> however:

$$\omega^2 = g k_\perp$$

$$v_p = \frac{\omega}{k_\perp} = \sqrt{\frac{g}{k_\perp}} \qquad v_g = \frac{\partial \omega}{\partial k_\perp} = \frac{1}{2}\sqrt{\frac{g}{k_\perp}} \quad \propto \sqrt{\lambda}$$

longer wavelengths travel faster

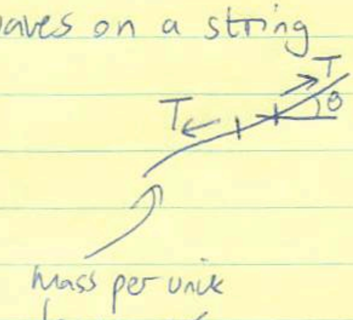
* The $v \propto \sqrt{H}$ dependence in the shallow case leads to refraction of an approaching wave such that waves always come in parallel to the beach!



Capillary waves

These are waves restored by surface tension.

As a reminder, recall how to calculate the wave equation for waves on a string



tension T

vertical component is $T\sin\theta = T\dfrac{dy}{dx}$

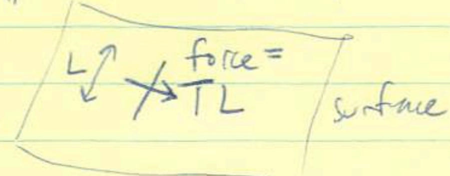
$$\text{net force} = T\frac{dy}{dx}\Big|_{x+dx} - T\frac{dy}{dx}\Big|_{x}$$

$$= T\frac{d^2y}{dx^2}\,dx$$

$$\Rightarrow \quad \Sigma \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \text{wave speed } c^2 = \frac{T}{\Sigma}$$

Now include surface tension in our discussion of surface waves: On the surface of the ocean there is a similar tension force T (per unit length perpendicular to the force)



A similar argument to the string gives

$$\text{force per unit area} = T\frac{\partial^2 \xi_z}{\partial x^2}$$

(upwards force)

$$\frac{\partial^2 \xi_x}{\partial x^2} < 0$$

$$\frac{\partial^2 \xi_x}{\partial x^2} > 0$$

Previously, the boundary condition at the surface was $\Delta P = 0$ so that the surface fluid element remains at atmospheric pressure. Now we must have

$$\Delta P = -T \frac{\partial^2 \xi_z}{\partial x^2}$$

(so that a downwards tension force gives an excess pressure $\Delta P > 0$ to balance it).

ie. $$\delta P + \underbrace{\xi_z \frac{dP_0}{dz}}_{-\rho g \xi_z} = -T \frac{\partial^2 \xi_z}{\partial x^2}$$

$$\Rightarrow \quad \delta P = \rho \xi_z \left(g + \frac{k_\perp^2 T}{\rho} \right)$$

Since this was where g entered the calculation, we can use the previous result with $g \Rightarrow g + \frac{k_\perp^2 T}{\rho}$.

$\Rightarrow$ The <u>dispersion relation</u> is

$$\tanh(k_\perp H) = \frac{\omega^2}{k_\perp} \frac{1}{g + k_\perp^2 T/\rho}$$

Limits: 1) $g = 0 \quad k_\perp H \gg 1$ (deep) $\quad \boxed{\omega^2 = k_\perp^3 \frac{T}{\rho}}$

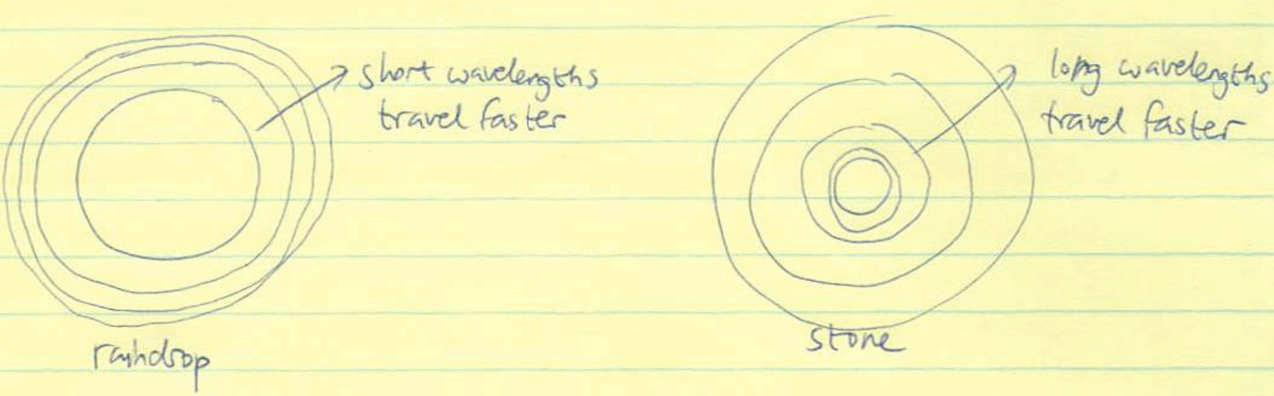
capillary waves

$$\left.\begin{aligned} v_p &= \left(\frac{T}{\rho}\right)^{1/2} k_\perp^{1/2} \\ v_g &= \frac{3}{2} v_p \end{aligned}\right\} \text{short wavelengths travel faster}$$

$\Rightarrow$ opposite to gravity waves in deep ocean $\omega^2 = g k_\perp \Rightarrow v_p = \sqrt{g/k_\perp}$

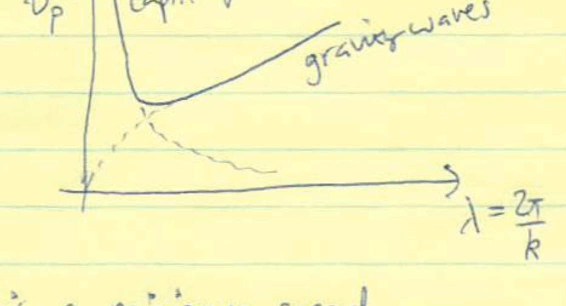
$v_g = \frac{1}{2} v_p$

Acheson points out that this opposite behavior gives different wave patterns for a raindrop hitting water vs. a large stone



2) $k_\perp H \gg 1$ but keep both gravity and surface tension

Then



(capillary have $v_p \propto \frac{1}{\lambda^{1/2}}$

gravity waves " $v_p \propto \lambda^{1/2}$)

There is a minimum speed

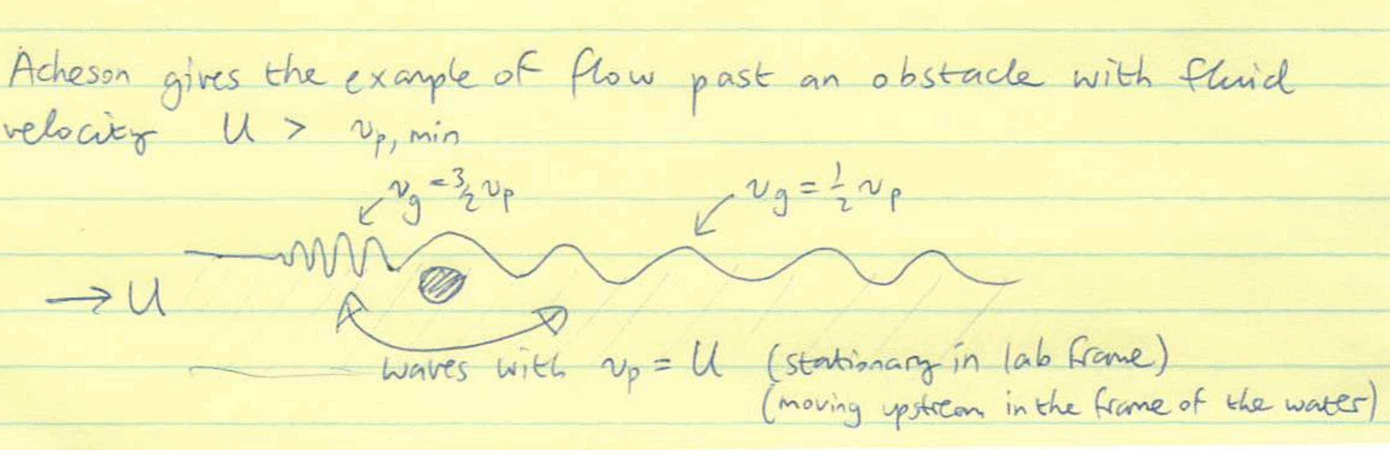
$$v_{p,min} = \left(\frac{4gT}{\rho}\right)^{1/4} \quad \text{for } k_\perp = \left(\frac{\rho g}{T}\right)^{1/2}$$

For air-water interface at room temperature, $T = 0.074$ N/m

$\Rightarrow$ $v_{p,min} \simeq 20$ cm/s

$\lambda \simeq 2$cm

Acheson gives the example of flow past an obstacle with fluid velocity $U > v_{p,min}$



A moving obstacle generates a standing wave pattern which involves waves whose speed matches that of the underlying flow.

The way to understand this mathematically is to go back to our perturbation equations

eg. for gravity waves $\frac{\partial}{\partial t} \delta \underline{u} = -\frac{\underline{\nabla} \delta P}{\rho}$

now we must add a term
$i U_0 k_x \delta \underline{u}$

(which comes from $\underline{u} \cdot \underline{\nabla} \underline{u}$
the background flow is $U_0 \hat{\underline{x}}$)

$\Rightarrow$ the perturbation equation is $-i(\omega - k_x U_0)\, \delta \underline{u} = -\frac{\underline{\nabla} \delta P}{\rho}$

We get the same dispersion relation as before but with
$\omega \rightarrow \omega - k_x U_0$.

eg. ~~shallow~~ deep water waves $\omega - k_x U_0 = \pm \sqrt{g} \sqrt{k_x}$

assume $k_\perp = k_x$ for simplicity.

$\Rightarrow$ if we choose $-k_x U_0 = \pm \sqrt{g} \sqrt{k_x}$

$k_x^2 U_0^2 = g k_x$

In other words the wave with $v_p = -U_0$ $\longrightarrow$ $k_x = g/U_0^2$

then $\omega = 0$ time independent solution //

(zero frequency mode)

Better way to write it — previously we obtained a dispersion relation which we could write as

$$\omega = v_{p,x} k_x$$

phase velocity in the x-direction

Now we get

$$\omega - k_x U_0 = v_{px} k_x$$

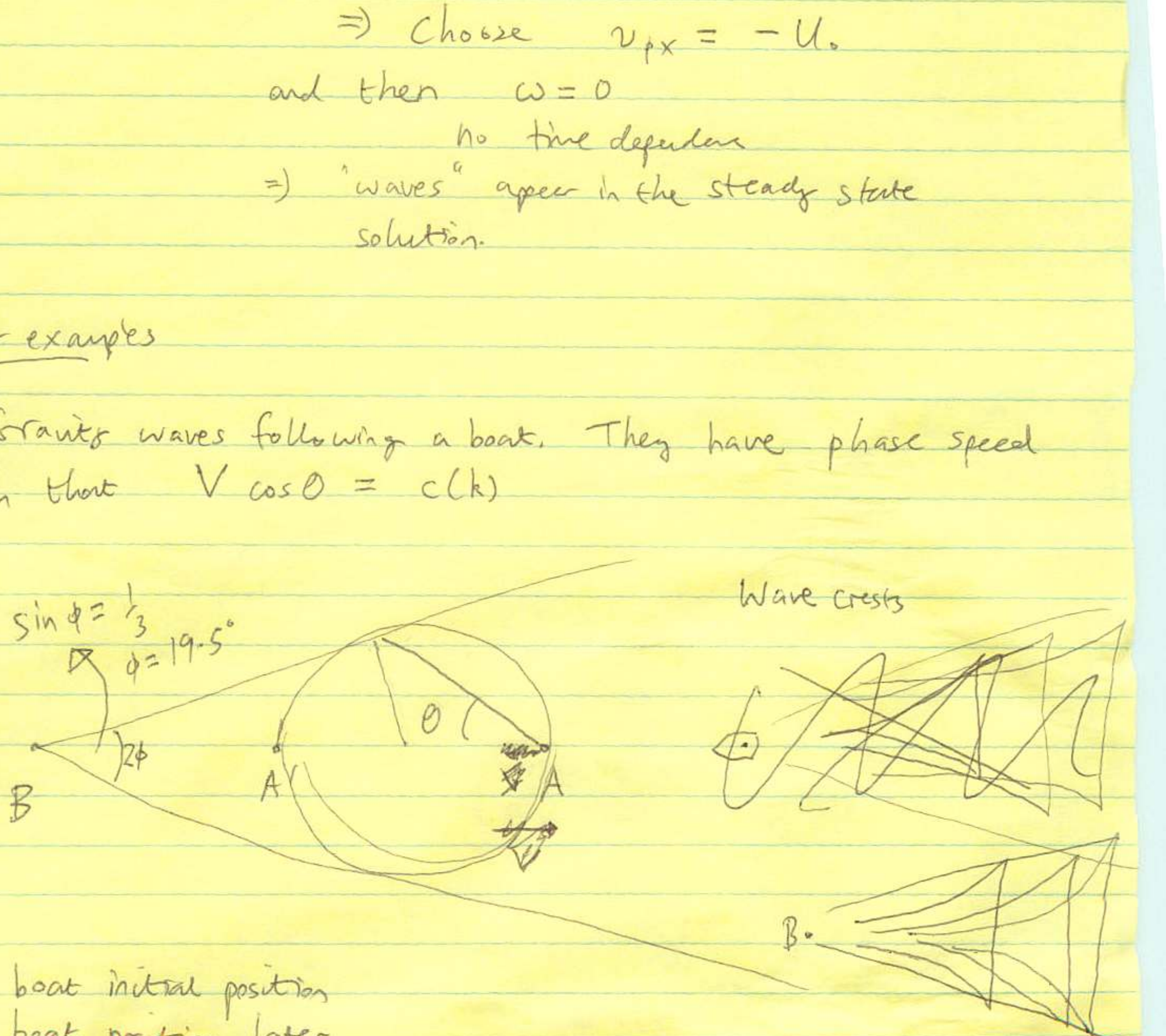
$$\Rightarrow \text{Choose} \quad v_{px} = -U_0$$

and then $\omega = 0$
no time dependence

$\Rightarrow$ "waves" appear in the steady state solution.

Other examples

- Gravity waves following a boat. They have phase speed such that $V \cos\theta = c(k)$

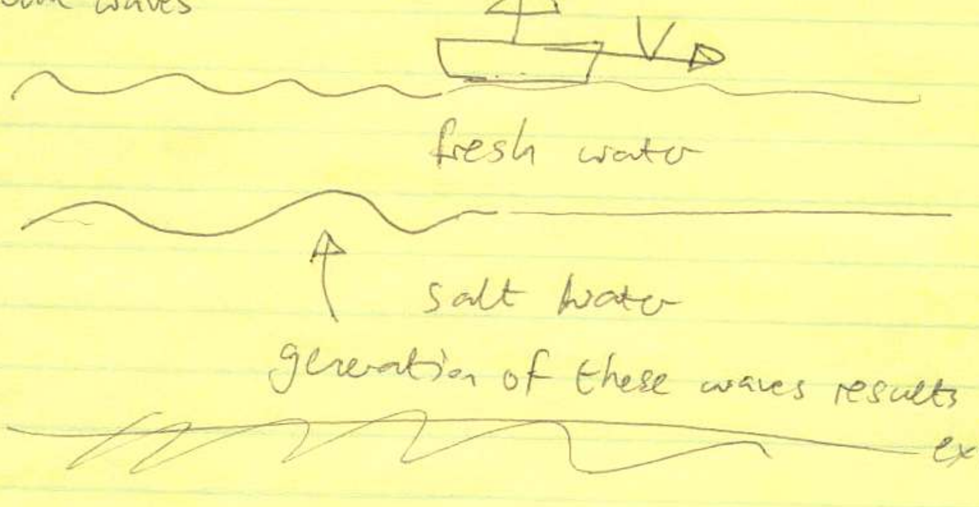


A = boat initial position
B = boat position later

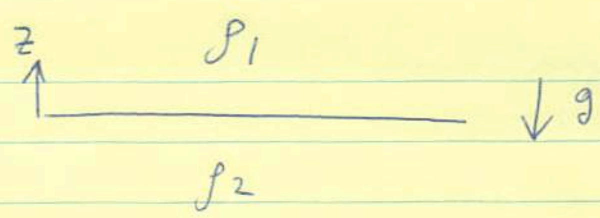
$'B = AA' = \frac{AB}{2}$ since $v_g = \frac{v_p}{2}$ for gravity waves.

- "Dead water"

Sudden slowing of a boat – extra drag from generation of interfacial waves



Gravity Waves at an interface



We can use our earlier solution and write

$$\delta p = f(z)\, e^{ik_\perp x - i\omega t}$$

where
$$f_1(z) = A e^{-\cancel{i}k_\perp z} \qquad z > 0$$
$$f_2(z) = B e^{k_\perp z} \qquad z < 0$$

(Choosing the solutions that remain finite as $|z| \to \infty$)

The boundary conditions at the interface are that

1) $\xi_{z,1} = \xi_{z,2} = \xi_z \;\Rightarrow\; -\dfrac{A k_\perp}{\rho_1 \omega^2} = +\dfrac{B k_\perp}{\rho_2 \omega^2}$

2) $\Delta P_1 = \Delta P_2 \;\cancel{\Rightarrow \delta p_1 = \delta p_2 \Rightarrow A = B}$

$$\Rightarrow \delta p_1 - \rho_1 g \xi_z = \delta p_2 - \rho_2 g \xi_z$$

$$\Rightarrow A - \rho_1 g\left(\frac{-A k_\perp}{\rho_1 \omega^2}\right) = \cancel{8}\, B - \rho_2 g \left(\frac{B k_\perp}{\rho_2 \omega^2}\right)$$

$$A\left(1 + \frac{g k_\perp}{\omega^2}\right) = B\left(1 - \frac{g k_\perp}{\omega^2}\right)$$

$$\Rightarrow \left(1 + \frac{g k_\perp}{\omega^2}\right) = \left(1 - \frac{g k_\perp}{\omega^2}\right)\left(-\frac{\rho_2}{\rho_1}\right)$$

$$\Rightarrow \boxed{\omega^2 = g k_\perp \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)}$$

As a check for $\rho_1 \ll \rho_2$ (eg. water-air interface) we recover our previous result $\omega^2 = g k_\perp$.

The wave "lives" on the interface – the eigenfunctions decay exponentially in either direction away from it.

Note that if $\rho_1 > \rho_2$ (heavy fluid overlying light fluid)
then $\omega^2 < 0$

$\Rightarrow$ there is a growing mode $(\delta p, \xi_z) \propto e^{\sigma t}$

$$\sigma = \sqrt{g k_\perp} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^{1/2}$$

The arrangement is unstable!

This instability is the RAYLEIGH-TAYLOR INSTABILITY.

More on the boundary conditions:
if the boundary conditions 1) and 2) are not obvious to you, remember that you can derive them by integrating the equations of motion across the boundary.

continuity $\quad \nabla . \underline{\xi} = 0 \quad \Rightarrow \quad \dfrac{d\xi_z}{dz} = -ik_\perp \xi_\perp$

$$\int_{-\varepsilon}^{\varepsilon} \frac{d\xi_z}{dz} \, dz = -\int_{-\varepsilon}^{\varepsilon} i k_\perp \xi_\perp \, dz$$

as $\varepsilon \to 0$ RHS vanishes

$$\Rightarrow \quad [\xi_z]_{-\varepsilon}^{\varepsilon} = 0 \quad \Rightarrow \quad \underline{\xi_z} \text{ is continuous}$$

momentum $\quad -\rho\omega^2 \xi_z = -\dfrac{d\delta P}{dz} - \delta\rho \, g$

but $\delta\rho + \xi_z \dfrac{d\rho}{dz} = \Delta\rho = 0$ for incompressible fluid

$$\Rightarrow \quad \rho\omega^2 \xi_z = \frac{d\delta P}{dz} - \xi_z \frac{d\rho}{dz} g$$

integrating $\displaystyle\int_{-\varepsilon}^{\varepsilon} dz$ and taking $\varepsilon \to 0$

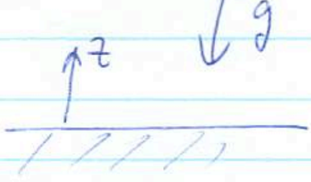
gives $\quad \left[\delta P - \xi_z \rho g \right]_{-\varepsilon}^{\varepsilon} = 0 \quad \Rightarrow \quad \underline{\Delta P} \text{ is continuous}$

Example: internal gravity waves

Incompressible perturbations with a background density gradient set by hydrostatic balance (eg. waves in an atmosphere).

The perturbation equations are

$$\underline{\nabla}\cdot\underline{\xi} = 0 \qquad \frac{\Delta\rho}{\rho} = 0$$



$$-\rho\omega^2\xi_z = -\frac{d}{dz}\delta P \;\cancel{+}\!- g\,\delta\rho$$

$$-\rho\omega^2\xi_\perp = -ik_\perp\delta P$$

We define the "convective discriminant" $A = \dfrac{d\ln\rho}{dz}$

then $\Delta\rho = 0 \Rightarrow \delta\rho = -\rho\xi_z A$

$$= \frac{\rho N^2\xi_z}{g}$$

where $N^2 = -gA = -g\dfrac{d\ln\rho}{dz}$

is the buoyancy frequency.

$$\Rightarrow \left.\begin{aligned} \rho(N^2-\omega^2)\xi_z &= -\frac{d\delta P}{dz} \\ -\rho\omega^2\xi_\perp &= -ik_\perp\delta P \\ \frac{d\xi_z}{dz} + ik_\perp\xi_\perp &= 0 \end{aligned}\right\} \qquad \boxed{\begin{aligned} \frac{d}{dz}\delta P &= \rho(\omega^2-N^2)\xi_z \\ \frac{d\xi_z}{dz} &= \frac{k_\perp^2\delta P}{\rho\omega^2} \end{aligned}}$$

(*)

Equations (*) are two coupled equations for ξ_z and δP. With appropriate boundary conditions they form an eigenvalue problem for the frequency ω. To get a sense of the solutions, make a WKB approximation

$$\xi_z,\ \delta P \propto e^{ik_z z}$$

$$\Rightarrow \quad ik_z\,\delta P = \rho(\omega^2 - N^2)\,\xi_z$$

$$ik_z\,\xi_z = \frac{k_\perp^2\,\delta P}{\rho\omega^2}$$

$$\Rightarrow \quad -k_z^2 = \frac{k_\perp^2}{\omega^2}(\omega^2 - N^2)$$

$$\Rightarrow \quad \boxed{\omega^2 = \frac{N^2 k_\perp^2}{k_\perp^2 + k_z^2}}$$

Dispersion relation for internal gravity waves

Notes

1) for $\omega^2 > 0$ we require $N^2 > 0$ or $\frac{d\ln\rho}{dz} < 0$

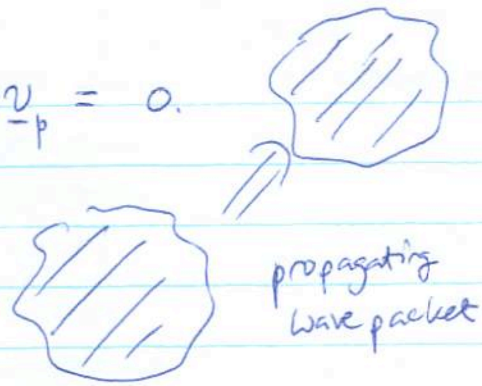
	STABLE (waves)		UNSTABLE (convection)
$g \downarrow$	light	$\uparrow z$	heavy
	heavy		light
	$\frac{d\rho}{dz} < 0$		$\frac{d\rho}{dz} > 0$

2) The phase and group velocities are perpendicular for these waves.

$$\underline{v}_p = \frac{\omega}{k}\hat{\underline{k}} = \frac{\omega}{k}\left(\hat{\underline{z}}\,k_z + \hat{\underline{x}}\,k_\perp\right)$$

$$\underline{v}_g = \hat{\underline{z}}\frac{\partial\omega}{\partial k_z} + \hat{\underline{x}}\frac{\partial\omega}{\partial k_\perp} = -\hat{\underline{z}}\frac{Nk_\perp k_z}{k^3} + \hat{\underline{x}}\frac{Nk_z^2}{k^3}$$

$\Rightarrow \quad \underline{v}_g \cdot \underline{v}_p = 0.$



3) More generally, for a gas $\Delta\rho = 0$ may not be appropriate. Instead, for example the condition may be adiabatic perturbations

$$\frac{\Delta P}{P} = \gamma \frac{\Delta\rho}{\rho}$$ (mode period << time for heat flow)

$$\Rightarrow \quad \frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\delta p}{p} + \frac{\xi_z}{p}\frac{dP}{dz}\frac{1}{\gamma} - \frac{\xi_z}{\rho}\frac{d\rho}{dz}$$

$$= \frac{\delta p}{p} + \frac{N^2 \xi_z}{g}$$

where we define $N^2 = g\left(\frac{1}{\gamma}\frac{d\ln P}{dz} - \frac{d\ln\rho}{dz}\right)$

As before $N^2 < 0 \Rightarrow$ instability (convection)

Since entropy $S = k_B \ln\left(\frac{P}{\rho^\gamma}\right)$ + constant

we can see that $N^2 \propto \frac{dS}{dz}$

$g \downarrow$ low entropy $\uparrow z$ high entropy

high entropy low entropy

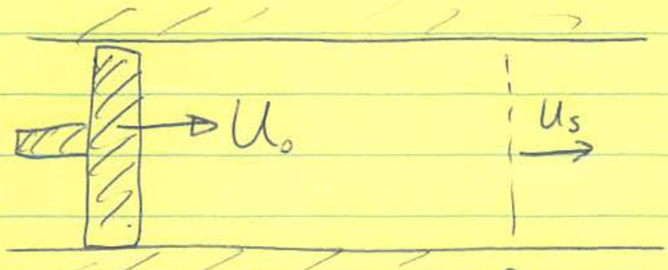
$\frac{dS}{dz} < 0$ $N^2 < 0$ unstable $N^2 > 0$ stable

CONVECTION GRAVITY WAVES

Non-linear waves

We mentioned that a shock has a finite thickness set by viscosity. The smoothing / diffusive effect of viscosity on the velocity profile balances the non-linear steepening effect, so that the shock propagates without change of shape.

eg. piston moving into a shock tube



a shock moves ahead of the piston into the fluid

Using the method of characteristics (unfortunately, we don't have time to go into that here) it can be shown that

$$\frac{\partial}{\partial t}(2u) + \underbrace{\left[\left(\frac{\gamma+1}{2}\right)u + c_0\right]\frac{\partial}{\partial x}(2u)}_{\substack{\text{non-linear advective term} \\ \text{(leads to steepening)}}} = \underbrace{\frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}}_{\text{viscous term}}$$

c_0 is the sound speed in the undisturbed fluid ahead of the shock.

[To see where the viscous term comes from, recall that the viscous stress tensor is $\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\,\delta_{ij}\,\underline{\nabla}\cdot\underline{u} \Rightarrow \sigma_{xx} = \frac{4}{3}\mu\frac{\partial u}{\partial x}$ in this case.]

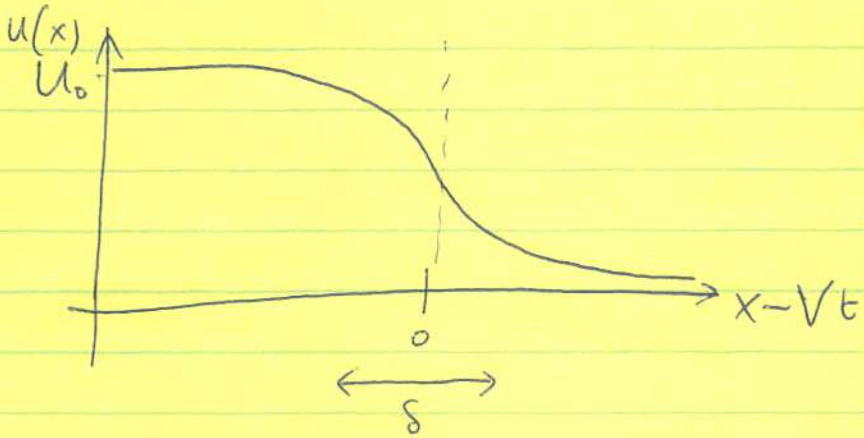
This is Burger's equation, and has a solution

$$u(x,t) = \frac{U_0}{1 + \exp\left(\frac{x - Vt}{\delta}\right)}$$

of the form $u = f(x - Vt)$ a travelling wave

where $V = c_0 + \left(\frac{\gamma+1}{4}\right) U_0$ is the shock speed

and $\delta = \frac{8}{3} \frac{\nu}{(\gamma+1) U_0}$ is a measure of shock thickness



This thickness is typically very small, eg. air $\nu = 0.1$ cgs

$$\Rightarrow \quad \delta \simeq 10^{-3}\,\text{cm} \left(\frac{1\,\text{m/s}}{U_0}\right)$$

Another way to balance non-linear steepening is with dispersion.
eg. shallow water waves have a dispersion relation

$$\frac{\omega^2}{gk} = \tanh(kH) \qquad \text{(see previous notes)}$$

$$\simeq kH - \frac{(kH)^3}{3} \quad \text{for } kH \ll 1 \qquad \text{(shallow limit)}$$

or $$\omega^2 \simeq k^2 gH\left(1 - \frac{(kH)^2}{3}\right)$$

$$\Rightarrow \quad \omega \simeq k\sqrt{gH}\left(1 - \frac{(kH)^2}{6}\right) \qquad \text{—}(*)$$

The phase speed $v_p = \frac{\omega}{k}$ depends on k, so that different components of an initial disturbance travel at different speeds, leading to smoothing of the initial profile.

A famous example in which this smoothing balances the non-linear steepening effect is <u>solitary waves</u> in shallow water.

First note that a wave with dispersion relation $(*)$ satisfies

$$\frac{\partial u}{\partial t} + \sqrt{gH}\,\frac{\partial u}{\partial x} + \sqrt{gH}\,\frac{H^2}{6}\,\frac{\partial^3 u}{\partial x^3} = 0.$$

[Check: $u \sim e^{i\omega t} e^{-ikx}$

$$\Rightarrow \quad i\omega + \sqrt{gH}\,(-ik) + \sqrt{gH}\,\frac{H^2}{6}(-ik)^3 = 0$$

$$\Rightarrow \quad \omega - k\sqrt{gH} + k\sqrt{gH}\,\frac{(kH)^3}{6} = 0 \qquad \checkmark$$
]

Adding the non-linear term, we get the famous Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \left(\sqrt{gH} + \tfrac{3}{2}u\right)\frac{\partial u}{\partial x} + \sqrt{gH}\,\frac{H^2}{6}\,\frac{\partial^3 u}{\partial x^3} = 0.$$

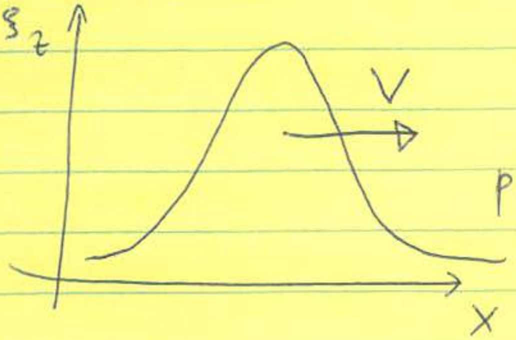
~~This has a solution~~ (the $\frac{3}{2}$ factor comes from method of characteristics again)

There is a solution to this equation of the form $f(x - Vt)$

with $V = \sqrt{gH}\left(1 + \dfrac{a}{2H}\right)$ — larger amplitude waves travel faster

and displacement $\xi_z = a\,\mathrm{sech}^2\left(\left(\dfrac{3a}{4H^3}\right)^{1/2}(x - Vt)\right)$

width decreases with increasing amplitude a



propagates without change in shape.

First observed on a canal by Russell in 1834.

Remarkably two solitary waves pass through each other without change of shape — they retain their identity upon collision! (as if they were linear waves, except there is a change in phase that results from the interaction). Solitary waves that show this behavior are known as <u>solitons</u>. Applications in many areas of physics.

The wave of translation

Work	
Institution memberships	Royal Society of Edinburgh, Royal Society, Institution of Naval Architects

In 1834, while conducting experiments to determine the most efficient design for canal boats, he discovered a phenomenon that he described as the **wave of translation**. In fluid dynamics the wave is now called a Scott Russell **solitary wave** or soliton. The discovery is described here in his own words:[1][2]

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [300–450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [2–3 km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Scott Russell spent some time making practical and theoretical investigations of these waves, he built wave tanks at his home and noticed some key properties:

- The waves are stable, and can travel over very large distances (normal waves would tend to either flatten out, or steepen and topple over)
- The speed depends on the size of the wave, and its width on the depth of water.
- Unlike normal waves they will never merge—so a small wave is overtaken by a large one, rather than the two combining.
- If a wave is too big for the depth of water, it splits into two, one big and one small.

Solitary wave in a laboratory wave channel.

Scott Russell's experimental work seemed at contrast with the Isaac Newton and Daniel Bernoulli's theories of hydrodynamics. George Biddell Airy and George Gabriel Stokes had difficulty to accept Scott Russell's experimental observations because Scott Russell's observations could not be explained by the existing water-wave theories. His contemporaries spent some time attempting to extend the theory but it would take until the 1870s before an explanation was provided.

Lord Rayleigh published a paper in Philosophical Magazine in 1876 to support John Scott Russell's experimental observation with his mathematical theory. In his 1876 paper, Lord Rayleigh mentioned Scott Russell's name and also admitted that the first theoretical treatment was by Joseph Valentin Boussinesq in 1871. Joseph Boussinesq mentioned Scott Russell's name in his 1871 paper. Thus Scott Russell's observations on solitons were accepted as true by some prominent scientists within his own life time.

Korteweg and de Vries did not mention John Scott Russell's name at all in their 1895 paper but they did

Incompressible vs. compressible flow

Go back to sound waves ... first say something about under what conditions we can consider a flow to be incompressible.

Consider a 1D isentropic flow that is steady $\partial/\partial t = 0$.

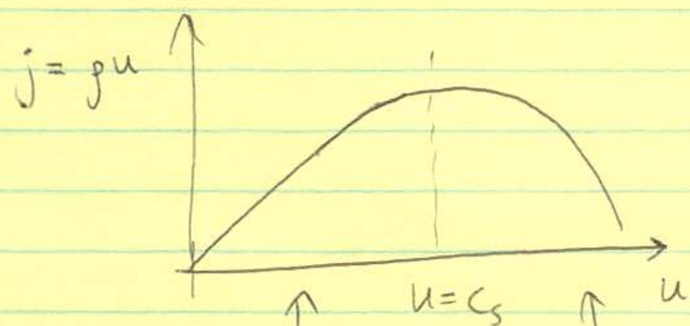
Then $$u \frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x}$$

$$\Rightarrow \quad \frac{u}{\rho}\frac{d\rho}{du} = -\frac{u^2}{c_s^2} \qquad \left(M = \frac{u}{c_s} \ \text{Mach number}\right)$$

$$\text{or} \quad \frac{d}{du}(\rho u) = \rho + u\frac{d\rho}{du} = \rho\left(1 - \frac{u^2}{c_s^2}\right) = \rho(1 - M^2)$$

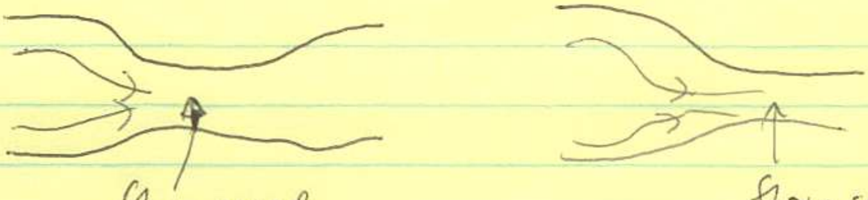
For $u \ll c_s$ the mass flux ρu increases $\propto u$ as in an ~~flu~~ incompressible fluid

for $u \gg c_s$, the mass flux ρu decreases with u.



"incompressible" eg. a river

"compressible" freeway



flow speed increases

flow speed decreases

So even a compressible fluid like air we can think of as being incompressible if $u \ll c_s$.

Example: Spherical blast wave in uniform medium (Taylor-Sedov solution)

Consider input of energy E into a small volume at the origin

eg. Supernova explosion in astrophysics
thermonuclear explosion

A shock wave propagates outwards into the rest of the gas.
Assume that there is no time to cool – ie. the energy E is constant, and that the ram pressure $P_2 \simeq \rho_1 u_s^2 \gg P_1$.

What characteristic lengthscale should we expect in this problem? The only parameters are the energy E and density of the undisturbed gas ρ_1 $\Rightarrow$ at time t $\quad r \propto t^{2/5}\left(\frac{E}{\rho_1}\right)^{1/5}$

This implies

1) we expect that the solution for physical quantities inside the blast wave should depend on r and t only through the combination $\xi = r\left(\frac{\rho_1}{Et^2}\right)^{1/5}$.

(A similarity solution.)

2) The shock front will correspond to some particular value of $\xi = \xi_s$.

$$\Rightarrow \quad \boxed{r_{shock} = \xi_s \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5}}$$

3) The velocity of the shock $\quad u_s = \frac{dr_s}{dt} = \frac{2}{5}\frac{r_s}{t} = \frac{2}{5}\xi_s\left(\frac{E}{\rho_1 t^3}\right)^{1/5}$

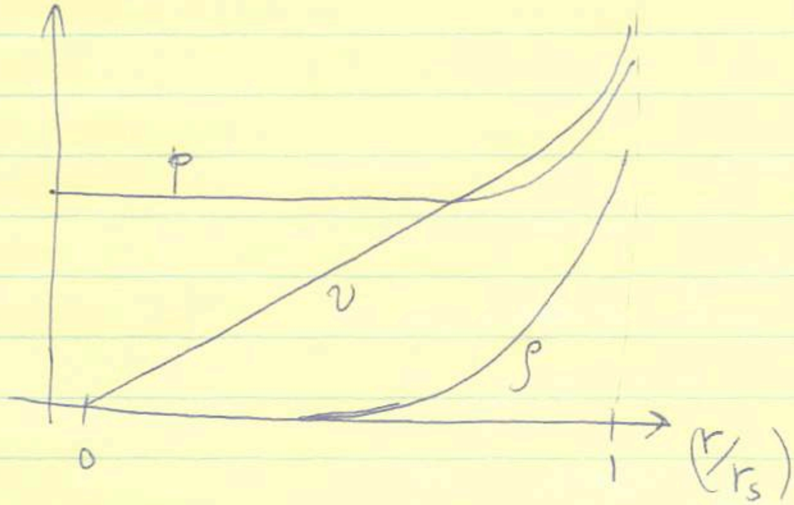
$$\propto \frac{1}{t^{3/5}}$$

the shock weakens over time.

There is indeed a self-similar solution of the fluid equations for this

problem. For $\gamma = 1.4$ the position of the shock is $\xi_s = 1.03$.
(See eg. Taylor 1950 Proc. Roy. Soc. London A 201, 159).

The interior solution looks like



For most of the interior: 1) $v \propto r$

2) $P \simeq$ constant ($\Rightarrow$ uniform distribution of internal energy)

3) the density drops to zero in the interior $\Rightarrow$ very high central temperatures.

4) most of the mass is right behind the shock.

Plug in some numbers:

For a <u>supernova</u>

$$R = 5\,\text{pc} \left(\frac{t}{1000\,\text{yrs}}\right)^{2/5} \left(\frac{E}{10^{51}\,\text{ergs}}\right)^{1/5} \left(\frac{1\,\text{cm}^{-3}}{n}\right)^{1/5}$$

[1pc ≃ 3 light years
Observed supernova remnants are of this scale]

$$\dot{R} = \frac{2}{5}\frac{R}{t} = 1800\,\text{km/s} \left(\frac{t}{1000\,\text{yrs}}\right)^{-3/5} \left(\frac{E_{51}}{n}\right)^{1/5}$$

<u>Atomic explosion</u> Taylor (1950) used photographs of the first atomic explosion in New Mexico to estimate the energy of the explosion.

His Figure 1 shows that $R \propto t^{2/5}$ fits the data extremely well.

From his table, we get $R \simeq 40\text{m}$ at $t = 1\text{msec}$.

Take $\rho_1 = 10^{-3}\ \text{g/cm}^3$

$$\Rightarrow \quad E = \rho_1 r^5/t^2 \simeq (10^{-3})(4\times10^3)^5/(10^{-3})^2$$
$$= \underline{10^{21}\ \text{ergs}}$$

agrees with Taylor's number

[One ton of TNT 4×10^{16} ergs
(10^6g) $\Rightarrow$ 2.5×10^4 tons of TNT equivalent.]

Velocity of the shock

$$\dot{R}_s = \frac{2}{5}\frac{R_s}{t} \approx 10^6\ \text{cm/s} = 10\ \text{km/sec}.$$

(sound speed 0.3 km/sec)

One of the uncertainties is what value of γ to take. At high temperatures, molecular vibrations are excited and γ drops.

For a strong shock
$$u_2 \simeq \tfrac{1}{6}u_s \qquad (\gamma = 7/5)$$
$$\rho_2 \simeq 6\rho_1$$

$$P_2 \simeq \tfrac{1}{2}\rho_1 u_s^2$$
$$\Rightarrow \frac{\rho_2 k_B T_2}{\mu m_p} \simeq \tfrac{1}{2}\rho_1 u_s^2$$
$$T_2 = \frac{\mu m_p u_s^2}{12 k_B} = 7000\ \text{K}$$

(Taylor estimates 2800K)

The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945

By Sir Geoffrey Taylor, F.R.S.

(*Received* 10 *November* 1949)

[Plates 7 to 9]

Photographs by J. E. Mack of the first atomic explosion in New Mexico were measured, and the radius, R, of the luminous globe or 'ball of fire' which spread out from the centre was determined for a large range of values of t, the time measured from the start of the explosion. The relationship predicted in part I, namely, that $R^{\frac{5}{2}}$ would be proportional to t, is surprisingly accurately verified over a range from $R=20$ to 185 m. The value of $R^{\frac{5}{2}}t^{-1}$ so found was used in conjunction with the formulae of part I to estimate the energy E which was generated in the explosion. The amount of this estimate depends on what value is assumed for γ, the ratio of the specific heats of air.

Two estimates are given in terms of the number of tons of the chemical explosive T.N.T. which would release the same energy. The first is probably the more accurate and is 16,800 tons. The second, which is 23,700 tons, probably overestimates the energy, but is included to show the amount of error which might be expected if the effect of radiation were neglected and that of high temperature on the specific heat of air were taken into account. Reasons are given for believing that these two effects neutralize one another.

After the explosion a hemispherical volume of very hot gas is left behind and Mack's photographs were used to measure the velocity of rise of the glowing centre of the heated volume. This velocity was found to be 35 m./sec.

Until the hot air suffers turbulent mixing with the surrounding cold air it may be expected to rise like a large bubble in water. The radius of the 'equivalent bubble' is calculated and found to be 293 m. The vertical velocity of a bubble of this radius is $\frac{2}{3}\sqrt{(g\,29300)}$ or 35·7 m./sec. The agreement with the measured value, 35 m./sec., is better than the nature of the measurements permits one to expect.

Comparison with photographic records of the first atomic explosion

Two years ago some motion picture records by Mack (1947) of the first atomic explosion in New Mexico were declassified. These pictures show not only the shape of the luminous globe which rapidly spread out from the detonation centre, but also gave the time, t, of each exposure after the instant of initiation. On each series of photographs a scale is also marked so that the rate of expansion of the globe, or 'ball of fire', can be found. Two series of declassified photographs are shown in figure 6, plate 7.

These photographs show that the ball of fire assumes at first the form of a rough sphere, but that its surface rapidly becomes smooth. The atomic explosive was fired at a height of 100 ft. above the ground and the bottom of the ball of fire reached the ground in less than 1 msec. The impact on the ground does not appear to have disturbed the conditions in the upper half of the globe which continued to expand as a nearly perfect luminous hemisphere bounded by a sharp edge which must be taken as a shock wave. This stage of the expansion is shown in figure 7, plate 8 which corresponds with $t = 15$ msec. When the radius R of the ball of fire reached about 130 m., the intensity of the light was less at the outer surface than in the interior. At

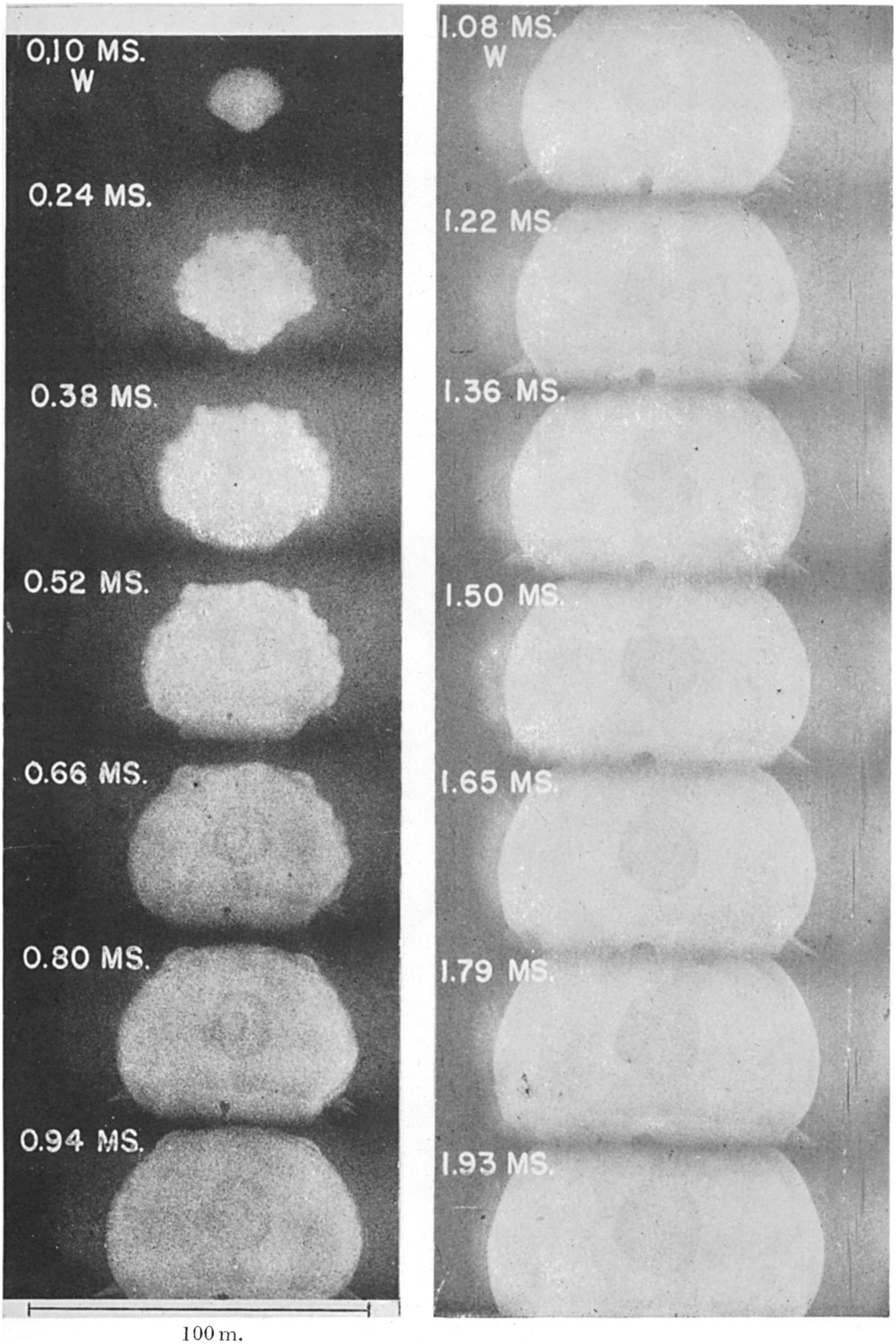


FIGURE 6. Succession of photographs of the 'ball of fire' from $t = 0{\cdot}10$ msec. to $1{\cdot}93$ msec.

(*Facing p.* 182)

later times the luminosity spread more slowly and became less sharply defined, but a sharp-edged dark sphere can be seen moving ahead of the luminosity. This must be regarded as showing the position of the shock wave when it ceases to be luminous. This stage is shown in figure 8, plate 9, taken at $t = 127$ msec. It will be seen that the edge of the luminous area is no longer sharp.

The measurements given in column 3 of table 1 were made partly from photographs in Mack (1947), partly from some clearer glossy prints of the same photographs kindly sent to me by Dr N. E. Bradbury, Director of Los Alamos Laboratory and partly from some declassified photographs lent me by the Ministry of Supply. The times given in column 2 of table 1 are taken directly from the photographs.

TABLE 1. RADIUS R OF BLAST WAVE AT TIME t AFTER THE EXPLOSION

authority	t (msec.)	R (m.)	$\log_{10} t$	$\log_{10} R$	$\frac{5}{2}\log_{10} R$
strip of small images MDDC 221	0·10	11·1	$\bar{4}\cdot0$	3·045	7·613
	0·24	19·9	$\bar{4}\cdot380$	3·298	8·244
	0·38	25·4	$\bar{4}\cdot580$	3·405	8·512
	0·52	28·8	$\bar{4}\cdot716$	3·458	8·646
	0·66	31·9	$\bar{4}\cdot820$	3·504	8·759
	0·80	34·2	$\bar{4}\cdot903$	3·535	8·836
	0·94	36·3	$\bar{4}\cdot973$	3·560	8·901
strip of declassified photographs lent by Ministry of Supply	1·08	38·9	$\bar{3}\cdot033$	3·590	8·976
	1·22	41·0	$\bar{3}\cdot086$	3·613	9·032
	1·36	42·8	$\bar{3}\cdot134$	3·631	9·079
	1·50	44·4	$\bar{3}\cdot176$	3·647	9·119
	1·65	46·0	$\bar{3}\cdot217$	3·663	9·157
	1·79	46·9	$\bar{3}\cdot257$	3·672	9·179
	1·93	48·7	$\bar{3}\cdot286$	3·688	9·220
strip of small images from MDDC 221	3·26	59·0	$\bar{3}\cdot513$	3·771	9·427
	3·53	61·1	$\bar{3}\cdot548$	3·786	9·466
	3·80	62·9	$\bar{3}\cdot580$	3·798	9·496
	4·07	64·3	$\bar{3}\cdot610$	3·809	9·521
	4·34	65·6	$\bar{3}\cdot637$	3·817	9·543
	4·61	67·3	$\bar{3}\cdot688$	3·828	9·570
large single photographs MDDC 221	15·0	106·5	$\bar{2}\cdot176$	4·027	10·068
	25·0	130·0	$\bar{2}\cdot398$	4·114	10·285
	34·0	145·0	$\bar{2}\cdot531$	4·161	10·403
	53·0	175·0	$\bar{2}\cdot724$	4·243	10·607
	62·0	185·0	$\bar{2}\cdot792$	4·267	10·668

To compare these measurements with the analysis given in part I of this paper, equation (38) was used. It will be seen that if the ball of fire grows in the way contemplated in my theoretical analysis, $R^{\frac{5}{2}}$ will be found to be proportional to t. To find out how far this prediction was verified, the logarithmic plot of $\frac{5}{2}\log R$ against $\log t$ shown in figure 1 was made. The values from which the points were plotted are given in table 1. It will be seen that the points lie close to the 45° line which is drawn in figure 1. This line represents the relation

$$\tfrac{5}{2}\log_{10} R - \log_{10} t = 11\cdot915. \tag{1}$$

The ball of fire did therefore expand very closely in accordance with the theoretical prediction made more than four years before the explosion took place. This is surprising, because in those calculations it was assumed that air behaves as though γ, the ratio of the specific heats, is constant at all temperatures, an assumption which is certainly not true.

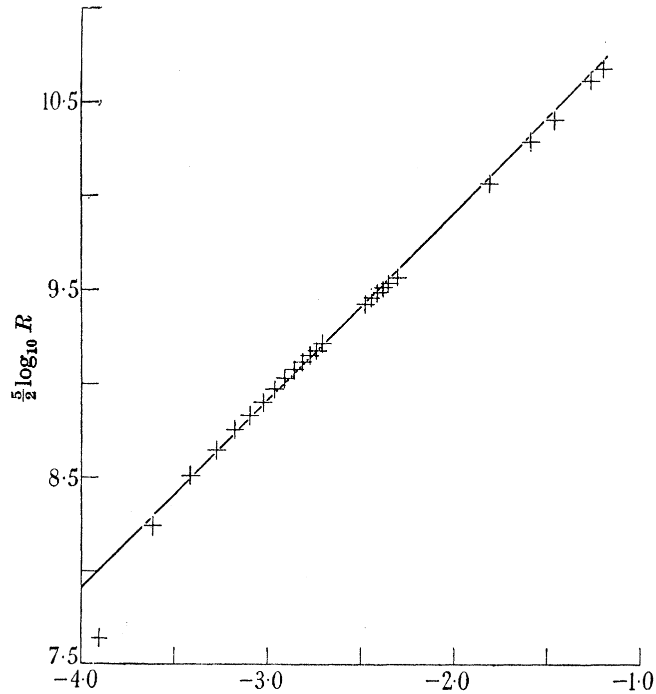


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

At room temperatures $\gamma = 1{\cdot}40$ in air, but at high temperatures γ is reduced owing to the absorption of energy in the form of vibrations which increases C_v. At very high temperatures γ may be increased owing to dissociation. On the other hand, the existence of very intense radiation from the centre and absorption in the outer regions may be expected to raise the apparent value of γ. The fact that the observed value of R^5t^{-2} is so nearly constant through the whole range of radii covered by the photographs of the ball of fire suggests that these effects may neutralize one another, leaving the whole system to behave as though γ has an effective value identical with that which it has when none of them are important, namely, 1·40.

Calculation of the energy released by the explosion

The straight line in figure 1 corresponds with

$$R^5t^{-2} = 6{\cdot}67 \times 10^2\,(\text{cm.})^5\,(\text{sec.})^{-2}. \tag{2}$$

The energy, E, is then from equation (18) of part I

$$E = \rho_0 A^2\left\{2\pi I_1 + \frac{4\pi}{\gamma(\gamma-1)} I_2\right\}, \tag{3}$$

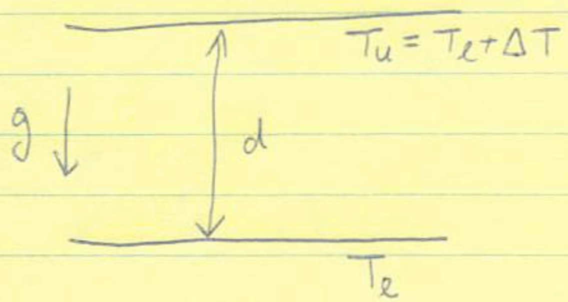
IV : Instability and Turbulence

(Acheson Chapter 9)

Rayleigh-Benard convection

A layer of fluid is heated from below. For large enough temperature contrast between the top and bottom, convection results. For the experiment, see the NCFM film on "Flow Instability" (go to minute 19:50 – 23:00).

Let's first work out the <u>linear theory</u>.



constant temperature boundaries
constant viscosity ν
thermal diffusivity κ

Equation of state: $\rho = \bar{\rho}\left[1 - \alpha(T - \bar{T})\right]$

volume coeff of thermal expansion

The fluid equations are

$$\underline{\nabla} . \underline{u} = 0$$

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho\nu\nabla^2\underline{u} + \rho\underline{g}$$

$$\frac{\partial T}{\partial t} + \underline{u}.\underline{\nabla} T = \kappa\nabla^2 T$$

we'll assume the fluid is incompressible and put in density perturbations only in the buoyancy term ($g\rho'$).
This means that we filter out sound waves from the solution.
<u>Boussinesq approximation</u>.

The background state has $\underline{u}_0 = 0$

$$\kappa \frac{d^2 T_0}{dz^2} = 0 \Rightarrow \boxed{T_0(z) = T_\ell - \frac{z}{d}\Delta T}$$

(constant heat flux $\propto \frac{dT}{dz}$)

and $\frac{dp_0}{dz} = -\rho_0 g$ (H.B.)

Now make small perturbations

$$T = T_0(z) + T_1$$
$$\rho = \rho_0(z) + \rho_1$$
$$\underline{u} = \underline{u}_1$$

(follow Acheson's notation)

$$\Rightarrow \underline{\nabla}.\underline{u}_1 = 0 \qquad \rho_1 = -\alpha\bar{\rho}T_1$$

$$\rho_0 \frac{\partial \underline{u}_1}{\partial t} = -\underline{\nabla}p_1 + \rho_0 \nu \nabla^2 \underline{u}_1 + \rho_1 \underline{g} \quad -(*)$$

$$\frac{\partial T_1}{\partial t} + w_1 \frac{dT_0}{dz} = \kappa \nabla^2 T_1 \quad -(\dagger)$$

PERTURBATION EQUATIONS

Now manipulate these to get a single equation for w_1:

take the curl of $(*)$ $\quad \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) \underline{\nabla} \times \underline{u}_1 = -\alpha(\underline{\nabla}T_1) \times \underline{g}$

and again $\quad \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) \nabla^2 \underline{u}_1 = \alpha\left[(\underline{g}.\underline{\nabla})\underline{\nabla}T_1 - \underline{g}\nabla^2 T_1\right]$

z-component $\quad \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) \nabla^2 w_1 = \alpha g \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T_1 \quad -(**)$

use $(\dagger)$

$$\boxed{\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\left(\frac{\partial}{\partial t} - \nu\nabla^2\right) \nabla^2 w_1 = -\alpha g \frac{dT_0}{dz}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_1}$$

Look for a separable solution $\quad w_1 = W(z)\, f(x,y)\, e^{st}$

f must satisfy $\quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = -a^2 f$

separation constant (~ horizontal wavevector)

$$\Rightarrow \boxed{\left[\nu (D^2-a^2)-s\right]\left[\kappa(D^2-a^2)-s\right](D^2-a^2)W = \alpha g \frac{dT_0}{dz} a^2 W} \qquad -(1)$$

$D \equiv \frac{d}{dz}$ $\qquad$ a 6th order ODE for W

With boundary conditions, this is an ~~6th order~~ eigenvalue problem for the growth rate s.

The boundary conditions are:

at $z=0$ and $z=d$ $\qquad u_1 = v_1 = \boxed{w_1 = 0}$ no slip

$T_1 = 0$ isothermal

$$u_1 = v_1 = 0 \Rightarrow \frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial w_1}{\partial z} = 0}$$

(since $\underline{\nabla}\cdot\underline{u}_1 = 0$)

$T_1 = $ constant $= 0 \Rightarrow$ eq (**) is

$$\boxed{\left[\nu(D^2-a^2)-s\right](D^2-a^2)W = 0}$$

these are 3 b.c.'s at $z=0,d$ $\rightarrow$ 6 in total ✓

Simplifying the last one using the others, we get the b.c.'s

$$\boxed{W=0 \quad DW=0 \quad D^4W - \left(2a^2+\frac{s}{\nu}\right)D^2W = 0 \quad \text{at } z=0,d} \qquad -(2)$$

(1) and (2) are an eigenvalue problem for s.

The solution is much simpler if we choose instead the b.c.'s

$$W = D^2 W = D^4 W = 0$$

which correspond to stress-free boundaries.

The solution is $\quad W = \sin\left(\dfrac{n\pi z}{d}\right) \qquad n = 1, 2, 3 \ldots$

$$\Rightarrow \quad (s + \nu a_*^2)(s + \kappa a_*^2)\, a_*^2 = -\alpha g \frac{dT_0}{dz} a^2$$

where $a_*^2 = a^2 + \dfrac{n^2 \pi^2}{d^2}$ (total wavevector)

$$\Rightarrow \quad s = -\left(\frac{\nu + \kappa}{2}\right) a_*^2 \pm \left[\frac{(\nu+\kappa)^2}{4} a_*^4 + \left\{ \frac{\alpha g \Delta T}{d} \frac{a^2}{a_*^2} - \nu\kappa a_*^4 \right\} \right]^{1/4}$$

(dispersion relation).

For $\Delta T > 0$ you can show that s is real, and

$$s > 0 \quad \text{if} \quad \frac{\alpha g \Delta T}{d \nu \kappa} > \frac{1}{a^2}\left(a^2 + \frac{n^2\pi^2}{d^2}\right)^3$$

Instability will occur if the LHS is larger than the minimum value of the RHS (which depends on the wavelength n, a) – ie. all we need is one unstable mode.

This is when $\quad n = 1 \quad a = a_c = \dfrac{\pi}{\sqrt{2}\, d}$

The instability criterion is

the Rayleigh number $\longrightarrow$ $$Ra \equiv \frac{\alpha g \Delta T d^3}{\nu \kappa} > \frac{2 d^4}{\pi^2} d^2 \left(\frac{\pi^2}{2d^2} + \frac{\pi^2}{d^2}\right)^3$$

$$\boxed{Ra > \frac{27\pi^4}{4} = 658}$$ Rayleigh (1916)

With b.c.'s (2) the critical Rayleigh number is $Ra_c = 1708$
with $a_c = 3.1/d$.

Experimentally, the critical Ra # is in good agreement with linear theory. The theory also nicely explains why the cells are of a size proportional to the fluid depth ($a_c \propto 1/d$) as pointed out in the movie.

The Rayleigh number compares the stabilizing (ν, κ) and destabilizing (ΔT) factors. We can write it as

$$Ra \sim \left(\frac{g}{d}\frac{\Delta\rho}{\rho}\right)\frac{d^2}{\nu}\frac{d^2}{\kappa} \sim N^2\, t_{therm}\, t_{visc}$$

we saw this earlier $N^2 \sim g \frac{d \ln \rho}{dz}$

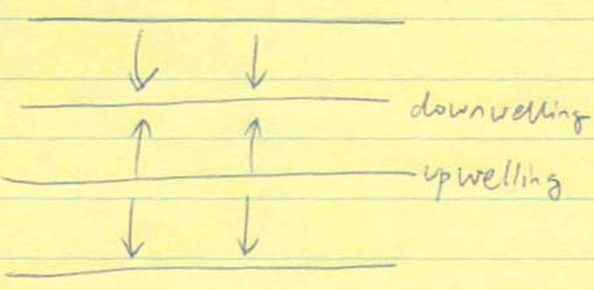
viscosity or heat conduction can quench the instability if they can act quickly enough.

Non-linear development

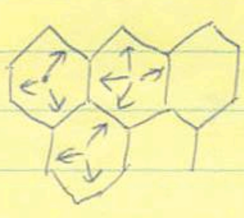
The linear theory does not account for:

- the growth is not exponential, but saturates due to non-linear terms
- the shape of the cells. Linear theory only fixes the lengthscale $\sim \frac{1}{a}$.

2D rolls



hexagonal cells



[The original hexagonal cells observed by Bénard in ~~the~~ 1900 which prompted Rayleigh's work on the instability are in fact driven by the temperature dependence of the surface tension!

surface tension weaker when the surface is heated → fluid pulled out of that region along the surface

Surface tension dominates in shallow layers.

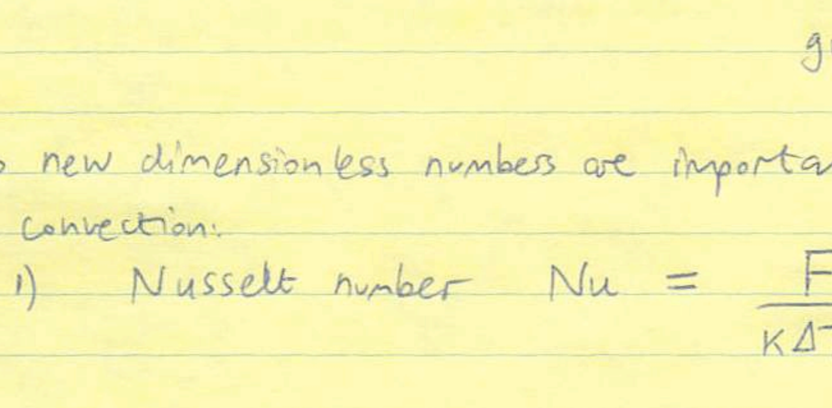
In a liquid, the fluid rises in the center of the cell whereas in a gas, the fluid sinks at the cell center – due to different behavior of ν with T

liquid $\nu \downarrow$ as $T \uparrow$
gas $\nu \uparrow$ as $T \uparrow$

Two new dimensionless numbers are important for characterizing the convection:

1) Nusselt number $Nu = \dfrac{F}{K\Delta T/d} = \dfrac{\text{heat flux}}{\text{conductive heat flux}}$

when the fluid is at rest $Nu = 1$ – all heat transported by thermal conduction.
Convection leads to enhanced transport $Nu > 1$.



see handout for data

$Nu \propto Ra^{1/3}$ arises when the thickness of the layer drops out
$Nu \propto d$
$Ra \propto d^3$

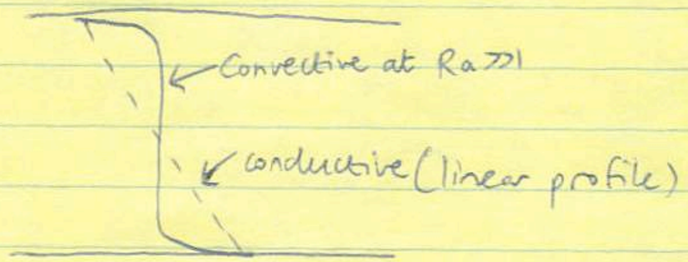
a transition to turbulence is observed at large Ra

2) Prandtl number $Pr = \frac{\nu}{\kappa}$ ~~dtt~~ measures which diffusivity is dominant.

The behavior of the flow as Ra increases depends significantly on whether $Pr < 1$ or $Pr > 1$ — see handout for a schematic diagram.

For $Pr < 1$ the transition to turbulence is very rapid, and cells can only be seen for a small range of Ra #. eg. Mercury which has $Pr = 0.03$.

Temperature profile at large Ra:



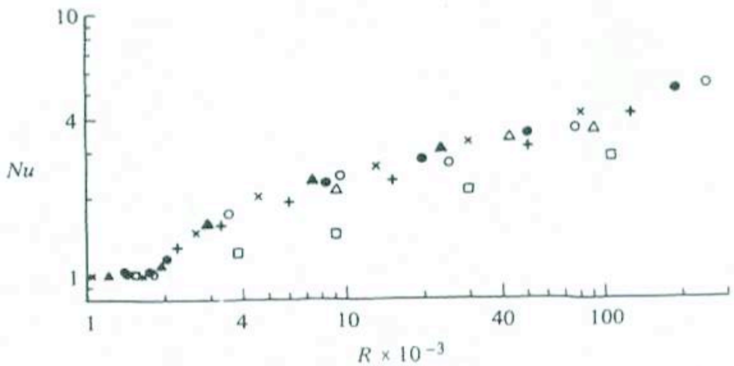


Fig. 2.6. Some experimental results on the heat transfer in various fluids in various containers. The Nusselt number is plotted against the Rayleigh number; ○ water; + heptane; × ethylene glycol; ● silicone oil AK 3; ▲ silicone oil AK 350; △ air; □ mercury. (After Silveston 1938 and Rossby 1969.)

when convection ensues. The onset of instability may also be seen directly with visualization techniques.

Silveston's (1958) measurements of $Nu(R, Pr)$ for various liquids between two horizontal plates at distances d varying from 1.45 to 13 mm, together with Graaf & Held's (1953) measurements for air and Rossby's (1969) for mercury, for values of R up to 10^6 are shown in Fig. 2.6. Note the sudden increase of Nu near 1708 for a wide variety of fluids. In fact Silveston (1958) found the experimental value $R_c = 1700 \pm 51$. Some relevant physical quantities for these fluids are shown in Table 2.2.

Cells are made observable by various visualization techniques and photographed, but measurements of their wavelength are not very accurate. However, the wavelength is close to $2d$ at the onset of instability between two rigid plates (Schmidt & Saunders 1938, Silveston 1958). When the separation of the side walls is much greater than d, hexagons seem to predominate for supercritical R. As R increases they tend to join up, as if forming rolls. Disorder increases with R until the motion seems to be turbulent when $R \approx 5 \times 10^4$ (Schmidt & Saunders 1938), although more recently experimentalists have detected some cellular structure up to much higher values of R. Koschmieder (1966) has found that the side

Table 2.2. *Some physical constants of fluids at 20 °C and 10^5 Pa (i.e. 1000 mbar)*

Copper 10^3

	ρ_0(kg m^{-3})	$10^{-3}c$ (m^2 s^{-2} K^{-1})	$10^7\nu$ (m^2 s^{-1})	$10^7\kappa$ (m^2 s^{-1})	$10^4\alpha$ (K^{-1})
Air	1.19	1.01 ($10^{-3}c_p$)	154	248	34.5
Heptane	684	2.22	6.16	0.875	12.4
Water	998	4.18	10.06	1.433	2.07
Silicone oil AK 3	912	1.61	32.0	0.779	10.6
Ethylene glycol	1113	2.38	191.5	0.942	6.4
Silicone oil AK 350	980	1.50	4670	1.061	9.2
Mercury	13550	0.139	1.15	44.0	1.82

walls affect the cell shapes in deep layers, and he has observed circular rolls within a circular side wall and linear rolls within rectangular side walls. (S. H. Davis's (1967) theory is consistent with these observations of linear rolls in so far as they are comparable, but better confirmed by the experiments of Stork & Müller (1972).) As R increases above R_c the wavelength of the cells increases.

On the basis of the linear theory just discussed, the cell pattern and the direction of flow is in principle uniquely determined by the initial conditions. In practice, however, observations of instability are made at values of the Rayleigh number slightly above the critical, and cell patterns and the direction of flow are largely independent of the unknown initial conditions. The facts that the motion has a preferred direction and is steady suggest that nonlinearity is significant.

It is also found, for example, that a liquid usually rises in the middle of a polygonal cell and a gas falls. Graham (1933) suggested that this is because the viscosity of a typical liquid decreases with temperature whereas that of a typical gas increases. This suggestion was subsequently confirmed by Tippelskirch's (1956) experiments on convection of liquid sulphur, for which the dynamic viscosity has

Drazin and Reid "Hydrodynamic Stability"

Busse "Transition to turbulence in Rayleigh-Bénard Convection"

Table 5.1. Properties of fluids in convection experiments at 20 °C

	Mercury	Air	Water	Glycerin
Prandtl number P	0.027	0.66	6.94	11,460
Depth d [cm] needed for $R_c = 1708$ for $T_2 - T_1 = 1\,°C$	0.773	2.54	0.497	3.47

favored because heat conduction between up- and down-going fluid parcels diminishes the available buoyancy.

There is one case of convection in which κ actually tends to zero. Since this

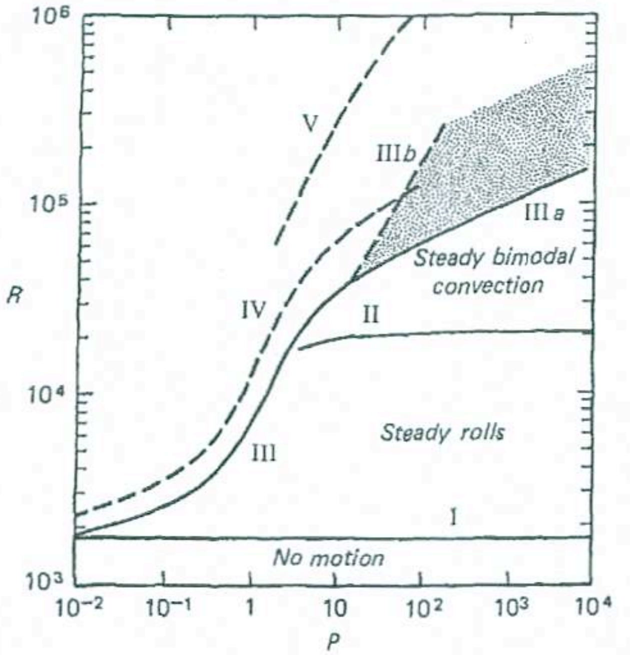


Fig. 5.5. Transitions in thermal convection as a function of Rayleigh and Prandtl numbers according to *Krishnamurti* [5.83] and others

Doppler velocimetry to obtain more detailed information on the time dependence of convection. The picture that emerges from the new experimental data is a complex one because the time dependence does not depend only on the Rayleigh and the Prandtl numbers, but also on the aspect ratio of the convection layer and perhaps even on the geometrical configuration of the sidewalls. In addition, the initial conditions can have a significant effect even in cases when the spatial pattern of convection does not depend much on the history of the experiment. [5.5].

There appears to be general agreement that the evolution of the time dependence of convection is distinctly different in small and in large aspect ratio layers [5.4, 88, 89]. In the latter case, the time dependence of convection

Turbulence

First we'll watch the movie from NCFM about turbulence, and then discuss some points in more detail.

Characteristics of turbulence — "symptoms" in the movie

- irregularity
- diffusivity
- large Re numbers
- 3D vorticity fluctuations
- dissipation

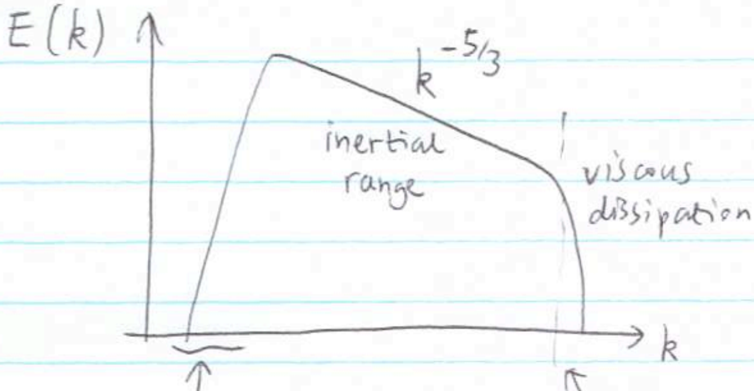
Note that turbulence is a property of the flow not the fluid.

Energy cascade

Turbulence involves a cascade of energy from the largest to smallest scales where viscosity dissipates the energy.

Perhaps the most famous result is the $-5/3$ scaling of the energy spectrum for isotropic homogeneous incompressible turbulence. Let's see how that works.

As mentioned in the movie, the behavior of the flow at a particular point is not predictable, but statistical quantities/averages are. One of these is the energy spectrum $E(k)$ where $E(k)\,dk$ is the kinetic energy density in modes of wavelength $\lambda = 2\pi/k$. It looks like:



outer scale
fluid stirred
with $Re = \frac{UL}{\nu} \gg 1$

inner scale typical velocity v_d
lengthscale l_d
such that $\frac{v_d l_d}{\nu} \sim 1$.

In a steady cascade, the energy transfer rate ε from scale to scale must be constant. Then from dimensional arguments we can write

$$\varepsilon \sim \frac{v^3}{l} \quad \text{at any scale } l$$

where $\boxed{v \sim (\varepsilon l)^{1/3}}$ is a typical velocity on that scale.

In particular this applies at each end of the inertial range

$$\varepsilon \sim \frac{U^3}{L} \sim \frac{v_d^3}{l_d}$$

But we know also that $\frac{v_d l_d}{\nu} \sim 1 \Rightarrow \boxed{\begin{array}{l} l_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \\ v_d \sim (\nu\varepsilon)^{1/4} \end{array}}$

these are the size and velocity of eddies for which the viscous time = turnover time

$$\frac{l_d^2}{\nu} = \frac{l_d}{v_d}$$

The eddy turnover time is $\frac{l}{v} \sim \varepsilon^{-1/3} l^{2/3}$ faster and faster as we go to smaller l.

We can also get the range of lengthscales in the cascade:

$$\left(\frac{L}{\ell_d}\right)^4 = \underset{\uparrow \atop L^3}{L^3} \cdot \underset{\uparrow \atop \times\ L}{\frac{U^3}{\varepsilon}} \cdot \underset{\uparrow \atop \times\ \ell_d^{-4}}{\frac{\varepsilon}{\nu^3}} = \left(\frac{LU}{\nu}\right)^3 = Re^3$$

$$\Rightarrow \quad \boxed{\ell_d = \frac{L}{Re^{3/4}}} \qquad \boxed{v_d \sim \frac{U}{Re^{1/4}}}$$

eg. stirring coffee (in a big cup!)

$$\left.\begin{array}{l} L \sim 10\text{ cm} \quad U \sim 10\text{ cm/s} \\ \nu \sim 0.01\text{ cm}^2/\text{s} \quad (\text{water}) \end{array}\right\} Re = 10^4.$$

$$\Rightarrow \ell_d \approx \frac{10\text{ cm}}{(10^4)^{3/4}} \simeq 0.1\text{ mm}.$$

$$v_d \sim \frac{10\text{ cm/s}}{10} \sim 1\text{ cm/s}.$$

eg. solar convection zone (outer $\simeq$ 30% of Sun)

$$\left.\begin{array}{l} L \sim 10^{10}\text{ cm} \\ U \sim 10^3\text{ cm/s} \Rightarrow Re \sim 10^{12} \end{array}\right\} \ell_d \sim 10\text{ cm}. \quad v_d \sim 1\text{ cm/s}$$

so that $\frac{\ell_d}{v_d} \sim \underline{10 \text{ seconds}}$ compared to $\frac{L}{U} \sim \frac{10^{10}\text{ cm}}{10^3\text{ cm/s}} \sim \underline{\underline{10^7 s}}$.

a difficult problem to simulate!

The scaling for $E(k)$ now follows:

$$\begin{aligned} E(k)\,dk &= \text{kinetic energy density at scale } k \\ &\sim v^2(\ell = 2\pi/k)\ \frac{dk}{k} \\ &\sim \varepsilon^{2/3} k^{-2/3} \frac{dk}{k} \quad \propto \quad \varepsilon^{2/3}\, \underline{\underline{k^{-5/3}}}\, dk \end{aligned}$$

(1941) $\boxed{\text{Kolmogorov spectrum}}$

~~This was~~

This was confirmed for turbulent flow in a tidal channel (which gave high Re ~ 10^8) "Seymour Narrows" by Grant et al. (1961) JFM.

If the stirring is kept the same but the viscosity varied, the inertial range remains fixed but with a different scale for the viscous cutoff. We saw this in the movie where a turbulent jet looks identical at two different Re numbers on large scales, but has much finer structure at larger Re.

We also saw that in freely decaying turbulence the smaller scales are erased first, consistent with the above picture.

Turbulent transport

The other property of turbulence emphasized in the movie was the large increase in transport of momentum and scalars such as temperature in a turbulent flow. Let's try to understand that.

Decompose the fluid motion into a mean flow $\underline{U}$ and fluctuating flow $\underline{u}'$

$$\boxed{\boldsymbol{v} = U + u'}$$

this is the "Reynolds decomposition".

total velocity v

We do this in such a way that $\overline{\boldsymbol{v}} = U$, $\overline{u'} = 0$

where the averaging is

$$\overline{u'} = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} dt\, u' \quad \text{for some large } \tau.$$

Assume incompressible flow. Then $\frac{\partial v_i}{\partial x_i} = 0 \Rightarrow \frac{\partial U_i}{\partial x_i} = 0$

$\left(\text{since } \overline{\frac{\partial v_i}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{v}_i = \frac{\partial}{\partial x_i} U_i\right)$

(The fluctuations and mean flow are seperately incompressible).

The momentum equation is

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

Now split this into mean and fluctuating parts and take the average

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_j}{\partial x_j} = \underbrace{-\overline{u_j' \frac{\partial u_i'}{\partial x_j}}}_{\text{this term can be written } \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)} - \frac{1}{\rho}\frac{\partial P}{\partial x_j} \leftarrow \text{Mean part of the pressure}$$

using incompressibility

$\Rightarrow$ the momentum equation for the mean flow is

$$\rho\left(\frac{\partial}{\partial t} + \underline{U}\cdot\underline{\nabla}\right)\underline{U} = \underline{\nabla}\cdot\underline{\underline{T}}$$

where $$\boxed{T_{ij} = -\delta_{ij} P - \underbrace{\rho\,\overline{u_i' u_j'}}_{}}$$

the turbulence gives rise to a new stress term – the <u>REYNOLDS STRESS</u>

This term shows that <u>correlated velocity fluctuations</u> can lead to transport of momentum.

eg. in the pipe

the term $\tau_{xz} = -\rho\,\overline{u_z' u_x'}$ acts to even out the flow, vertically transporting the x-momentum.

Now we see the central problem in trying to write down equations to describe turbulent flow — the closure problem. We need a closure relation between $\overline{u_i' u_j'}$ and the mean flow.

It is often assumed for simplicity that

$$\overline{u_i' u_j'} = -D_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

"Eddy viscosity"

ie. the same kind of relation as for microscopic viscosity

Note the crucial difference with a viscous fluid, however: even if such a relation were valid (probably not – the dependence of the Reynolds stress on the mean flow is likely much more complex) the Eddy viscosity D_T is a property of the flow unlike the microscopic viscosity which is a property of the fluid!

We can treat the transport of a scalar also using the Reynolds decomposition:

eg. $$\rho c_p \left(\frac{\partial T}{\partial t} + \underline{v}.\underline{\nabla} \right) T = \frac{\partial}{\partial x_j} \left(K \frac{\partial T}{\partial x_j} \right)$$

Decompose $\underline{v}$ and T:

$$\underline{v} = \underline{U} + \underline{u}'$$
$$T = \overline{T} + T'$$

where $$\overline{T'} = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} T'(t)\, dt = 0$$

$$\Rightarrow \quad \rho c_p \left(\frac{\partial T}{\partial t} + \underline{U}.\underline{\nabla} T \right) = \frac{\partial}{\partial x_j} \left(\underbrace{- \rho c_p \overline{T' u_j'}} + K \frac{\partial T}{\partial x_j} \right)$$

again correlated fluctuations lead to enhanced transport, this time of thermal energy. $\leftarrow$ $\boxed{\text{turbulent heat flux } \rho c_p \overline{T' u_j'}}$

Lorenz model

In a famous 1963 paper "Deterministic Non-Periodic Flow" Lorenz solved a highly simplified model of Rayleigh-Benard convection which takes the form of 3 coupled ODEs

$$\left.\begin{aligned}\dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ\end{aligned}\right\} (*)$$

[These eqns come from keeping only the first terms in a Fourier series expansion of the Ra-Benard equations.

the variables $X \propto$ intensity of convective motion

$Y \propto$ temperature difference between ascending and descending currents

$Z \propto$ temperature perturbation away from the background.

The time units are $\tau = \left(\frac{\pi^2}{d^2} + a^2\right) \kappa t$

σ is the Prandtl number $= \nu/\kappa$

$r = \dfrac{Ra}{Ra_c}$ and $b = \dfrac{4}{1 + a^2 d^2/\pi^2}$

Lorenz chooses $\boxed{\sigma = 10}$ $(\nu = 10\kappa)$

and $\dfrac{a^2 d^2}{\pi^2} = \dfrac{1}{2}$ (ie. a_c corresponding to the stress free bound'ary we solved earlier)

$\Rightarrow \boxed{b = 8/3}$

(see handout for phase space sketches)

The behavior of the system then depends on r:

1) $\underline{r < 1}$ The steady state solution of equations $(*)$ is $X = Y = Z = 0$ ie. no convection. Starting from some other location in

Deterministic Nonperiodic Flow[1]

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(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

1. Introduction

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and, even when observed for long periods of time, do not appear to repeat their previous history.

These modes of behavior may all be observed in the familiar rotating-basin experiments, described by Fultz, *et al.* (1959) and Hide (1958). In these experiments, a cylindrical vessel containing water is rotated about its axis, and is heated near its rim and cooled near its center in a steady symmetrical fashion. Under certain conditions the resulting flow is as symmetric and steady as the heating which gives rise to it. Under different conditions a system of regularly spaced waves develops, and progresses at a uniform speed without changing its shape. Under still different conditions an irregular flow pattern forms, and moves and changes its shape in an irregular nonperiodic manner.

Lack of periodicity is very common in natural systems, and is one of the distinguishing features of turbulent flow. Because instantaneous turbulent flow patterns are so irregular, attention is often confined to the statistics of turbulence, which, in contrast to the details of turbulence, often behave in a regular well-organized manner. The short-range weather forecaster, however, is forced willy-nilly to predict the details of the large-scale turbulent eddies—the cyclones and anticyclones—which continually arrange themselves into new patterns. Thus there are occasions when more than the statistics of irregular flow are of very real concern.

In this study we shall work with systems of deterministic equations which are idealizations of hydrodynamical systems. We shall be interested principally in nonperiodic solutions, i.e., solutions which never repeat their past history exactly, and where all approximate repetitions are of finite duration. Thus we shall be involved with the ultimate behavior of the solutions, as opposed to the transient behavior associated with arbitrary initial conditions.

A closed hydrodynamical system of finite mass may ostensibly be treated mathematically as a finite collection of molecules—usually a very large finite collection—in which case the governing laws are expressible as a finite set of ordinary differential equations. These equations are generally highly intractable, and the set of molecules is usually approximated by a continuous distribution of mass. The governing laws are then expressed as a set of partial differential equations, containing such quantities as velocity, density, and pressure as dependent variables.

It is sometimes possible to obtain particular solutions of these equations analytically, especially when the solutions are periodic or invariant with time, and, indeed, much work has been devoted to obtaining such solutions by one scheme or another. Ordinarily, however, nonperiodic solutions cannot readily be determined except by numerical procedures. Such procedures involve replacing the continuous variables by a new finite set of functions of time, which may perhaps be the values of the continuous variables at a chosen grid of points, or the coefficients in the expansions of these variables in series of orthogonal functions. The governing laws then become a finite set of ordinary differential

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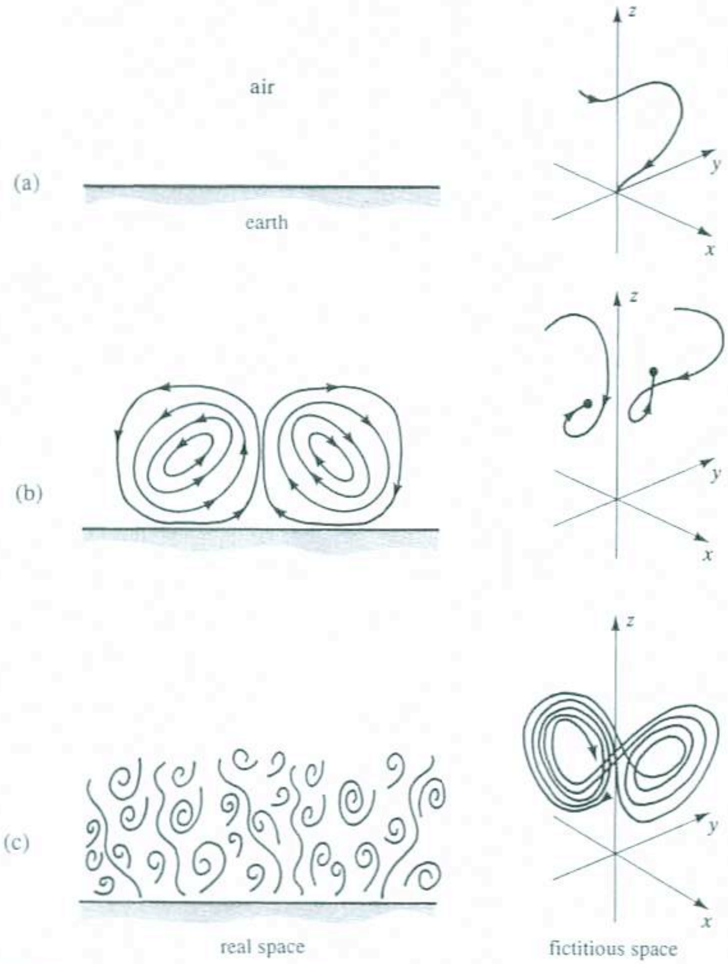
from "The Physics of Chance" by C. Ruhla



r < 1

1 < r < 24.74

r > 24.74

Fig. 6.7 The Lorenz theory of the Earth's atmosphere, illustrated in three typical regimes: (a) atmosphere at rest; (b) atmosphere in ordered convection; (c) atmosphere in turbulent convection.

twists into a nearly plane double spiral, having fractal dimension 2.06, this fractal structure being a characteristic of the strange attractor (Fig. 6.7c).†

Once we recognize the presence of a strange attractor we know that the atmosphere is a dissipative system very sensitive to the initial conditions. Two

† Note that Fig. 6.7c is not a two-dimensional Poincaré section, but a representation of our fictitious three-dimensional space in perspective. It can be shown that the fractal dimension is less than the number of degrees of freedom of the system. In the present case, the fractal dimension 2.06 implies at least three degrees of freedom, conformably with the facts. (Of course, these are degrees of freedom ordinary space.)

phase space, the trajectory is towards the fixed point at $(0,0,0)$ – ie the motion damps out, the system is stable.

2) <u>$r > 1$ but < 24.74</u> Now convection appears. There are two new steady state solutions given by

$$Z = r-1, \quad X = Y, \quad X^2 = \frac{8}{3}(r-1).$$

} convective rolls in different directions

Again, starting the system away from these points, the system will evolve to settle down in one of the steady solutions. (The steady solution at $X=Y=Z=0$ still exists but it is now linearly unstable.)

3) <u>for $r > 24.74$</u> you can show that the stable solutions at $Z = r-1$ and $X = Y = \pm\sqrt{\frac{8}{3}}\sqrt{r-1}$ are now unstable. The evolution is now <u>chaotic</u> – irregular oscillations with <u>sensitivity to initial conditions</u> – two slightly different initial values lead to completely different behavior.

The critical value of r can be found by a linear stability analysis – it is

$$r_c = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

for $\sigma = 10$ and $b = 8/3 \rightarrow r_c = \frac{470}{19} = 24.74$.

In phase space the system sketches out the "<u>Lorenz attractor</u>" an example of a <u>strange attractor</u>.

The important idea here is that chaotic behavior and unpredictability emerge from a deterministic system.

J. Atmos Sci 20 130 (1963)

puting machine. Approximately one second per iteration, aside from output time, is required.

For initial conditions we have chosen a slight departure from the state of no convection, namely (0,1,0). Table 1 has been prepared by the computer. It gives the values of N (the number of iterations), X, Y, and Z at every fifth iteration for the first 160 iterations. In the printed output (but not in the computations) the values of X, Y, and Z are multiplied by ten, and then only those figures to the left of the decimal point are printed. Thus the states of steady convection would appear as 0084, 0084, 0270 and -0084, -0084, 0270, while the state of no convection would appear as 0000, 0000, 0000.

The initial instability of the state of rest is evident. All three variables grow rapidly, as the sinking cold fluid is replaced by even colder fluid from above, and the rising warm fluid by warmer fluid from below, so that by step 35 the strength of the convection far exceeds that of steady convection. Then Y diminishes as the warm fluid is carried over the top of the convective cells, so that by step 50, when X and Y have opposite signs, warm fluid is descending and cold fluid is ascending. The motion thereupon ceases and reverses its direction, as indicated by the negative values of X following step 60. By step 85 the system has reached a state not far from that of steady convection. Between steps 85 and 150 it executes a complete oscillation in its intensity, the slight amplification being almost indetectable.

The subsequent behavior of the system is illustrated in Fig. 1, which shows the behavior of Y for the first 3000 iterations. After reaching its early peak near step 35 and then approaching equilibrium near step 85, it undergoes systematic amplified oscillations until near step 1650. At this point a critical state is reached, and thereafter Y changes sign at seemingly irregular intervals, reaching sometimes one, sometimes two, and sometimes three or more extremes of one sign before changing sign again.

Fig. 2 shows the projections on the X-Y- and Y-Z-planes in phase space of the portion of the trajectory corresponding to iterations 1400–1900. The states of steady convection are denoted by C and C'. The first portion of the trajectory spirals outward from the vicinity of C', as the oscillations about the state of steady convection, which have been occurring since step 85, continue to grow. Eventually, near step 1650, it crosses the X-Z-plane, and is then deflected toward the neighborhood of C. It temporarily spirals about C, but crosses the X-Z-plane after one circuit, and returns to the neighborhood of C', where it soon joins the spiral over which it has previously traveled. Thereafter it crosses from one spiral to the other at irregular intervals.

Fig. 3, in which the coordinates are Y and Z, is based upon the printed values of X, Y, and Z at every fifth iteration for the first 6000 iterations. These values determine X as a smooth single-valued function of Y and Z over much of the range of Y and Z; they determine X

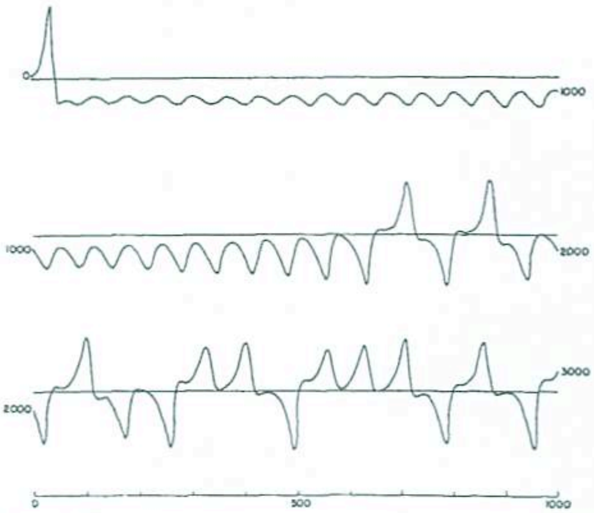


FIG. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

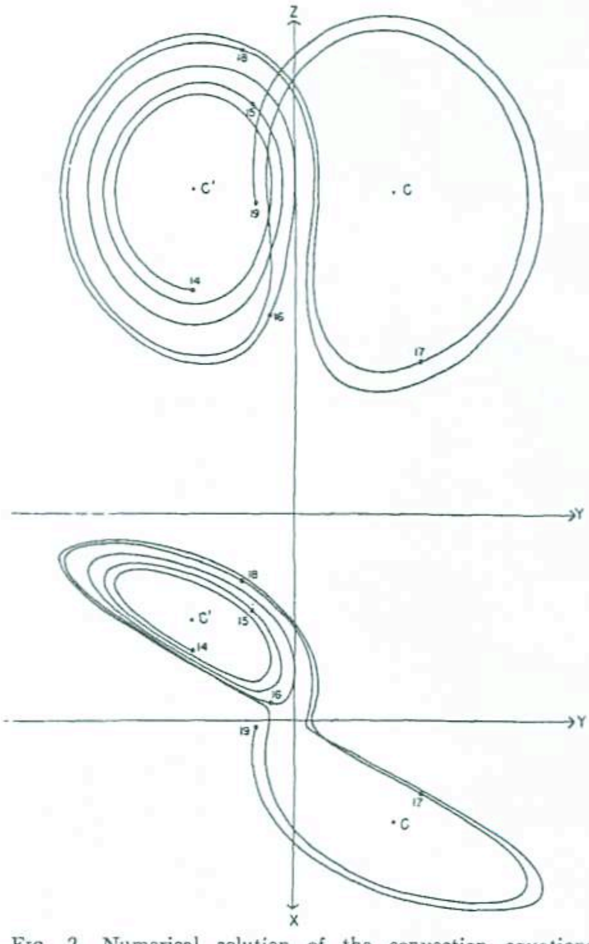
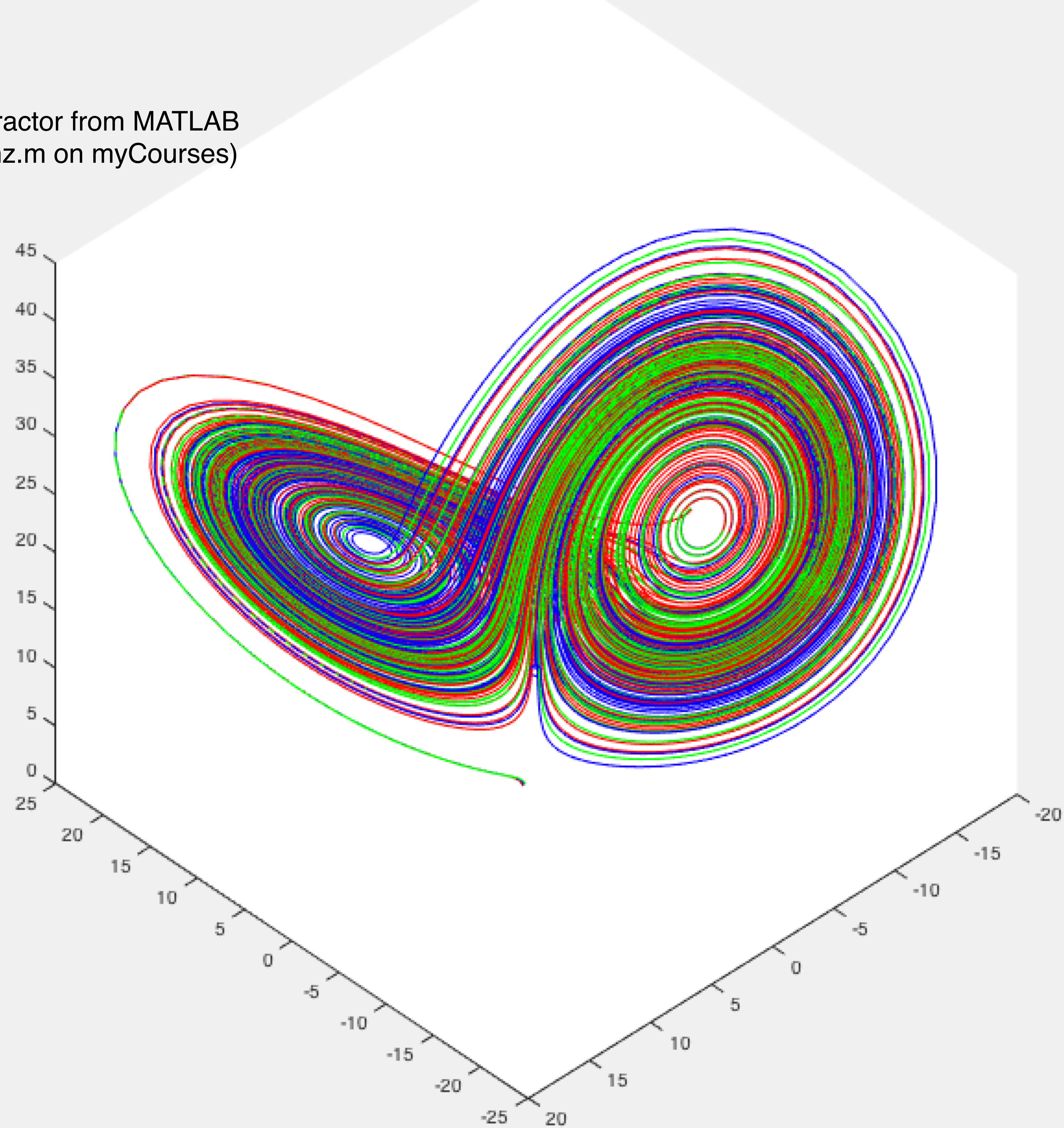


FIG. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

Lorenz attractor from MATLAB
(see lorenz.m on myCourses)

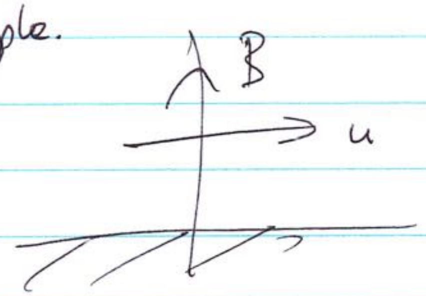


$\boxed{\text{Magnetohydrodynamics Part 1}}$ Start with the response of the flow to $J \times B$ forces.

$\boxed{\text{Hartmann layer}}$ is a classic example.

$$\vec{u} = \vec{e}_x\, u(y)$$

$$\vec{B} = B\, \vec{e}_y$$



$$\vec{J} = \sigma\left(\vec{u} \times \vec{B}\right)$$

$$\vec{J} \times \vec{B} = -\sigma B^2 \vec{u} \qquad \text{drag force}$$

$$\rho\nu \frac{\partial^2 u}{\partial y^2} - \sigma B^2 u = +\frac{\partial p}{\partial x} \qquad — (*)$$

drag timescale: $\rho \dfrac{\partial u}{\partial t} \sim \sigma B^2 u$

$$t_{drag} \sim \frac{\rho}{\sigma B^2} \sim \frac{1}{\sigma v_A^2}$$

Write $(*)$ in a simpler way:

$$u'' - \underbrace{\frac{\sigma B^2}{\rho\nu}\, u}_{\text{write as } \frac{u}{\delta^2}} = \underbrace{\frac{1}{\rho\nu}\frac{\partial P}{\partial x}}_{\equiv \beta \text{ (a constant)}}$$

Hartmann layer

$$u'' - \frac{u}{\delta^2} = \beta$$

homogeneous eqn: $\quad u'' - \dfrac{u}{\delta^2} = 0$

$$u = e^{\pm y/\delta}$$

PI: $\quad u = a + by + cy^2$

$$u' = b + 2cy$$

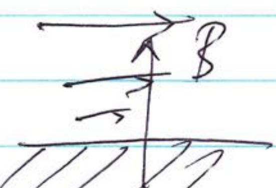
$$u'' = 2c$$

$$2c - \frac{a}{\delta^2} - \frac{by}{\delta^2} - \frac{cy^2}{\delta^2} = \beta$$

$$\Rightarrow \quad b = 0 \quad c = 0 \quad a = -\beta\delta^2$$

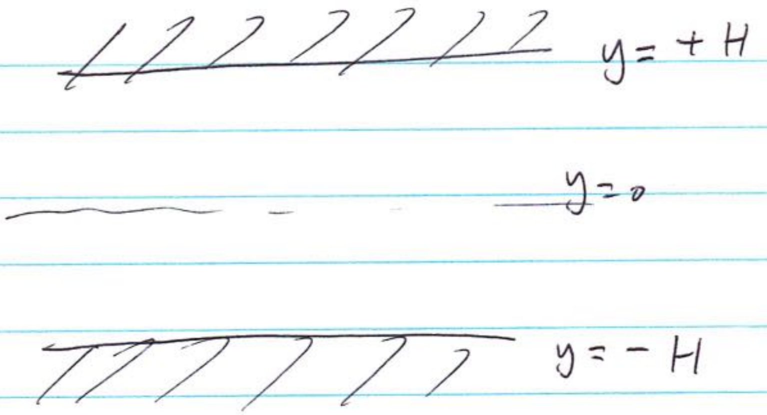
$\Rightarrow$ general solution: $\quad \boxed{u = a + be^{-y/\delta} + ce^{+y/\delta}}$

eg. semi-infinite



$u = 0$ at $y = 0 \Rightarrow a = -b$
$U = \text{const}$ at $y \to \infty \Rightarrow c = 0$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \underline{\underline{u = a\left(1 - e^{-y/\delta}\right)}}$$



$$0 = a + b e^{-H/\delta} + c e^{H/\delta}$$
$$0 = a + b e^{H/\delta} + c e^{-H/\delta}$$

add: $0 = 2a + b\left(e^{H/\delta} + e^{-H/\delta}\right) + c\left(e^{H/\delta} + e^{-H/\delta}\right)$

$$-a = b\cosh H/\delta + c\cosh H/\delta$$

Subtract: $0 = b\sinh H/\delta \cancel{+} - c\sinh H/\delta$

$$\Rightarrow b = c \Rightarrow b = \frac{-a}{2\cosh H/\delta}$$

$$\Rightarrow u = a\left[1 \cancel{+} - \frac{\left(e^{y/\delta} + e^{-y/\delta}\right)}{2\cosh H/\delta}\right]$$

$$\boxed{u = a\left[1 - \frac{\cosh(y/\delta)}{\cosh(H/\delta)}\right]}$$

at the mid-plane

$$\boxed{u = a\left[1 - \frac{1}{\cosh H/\delta}\right]}$$

So if $H \gg \delta$ $\quad u \to a$ at $y = 0$.

otherwise $H \ll \delta$ $\quad \cancel{2}\cosh\frac{H}{\delta} \approx e^{H/\delta} + e^{-H/\delta}$

$$\simeq 1 + \left(\frac{H}{\delta}\right)^2 \frac{1}{2}$$

$$u = a\left[1 - \frac{1}{1 + \frac{1}{2}(H/\delta)^2}\right] \simeq \underline{\frac{a}{2}}$$

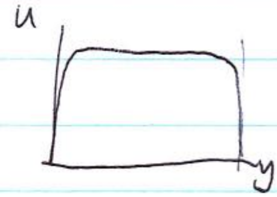
$$\simeq a\left[1 - 1 - \frac{1}{2}\left(\frac{H}{\delta}\right)^2\right]$$

$$u \simeq \underline{\frac{a}{2}\left(\frac{H}{\delta}\right)^2}$$

Hartmann #

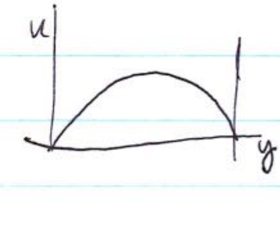
$$Ha = \frac{H}{\delta} \qquad \delta = \left(\frac{\rho\nu}{\sigma B^2}\right)^{1/2} \qquad a = \left(-\frac{\partial P}{\partial x}\right)\frac{\delta^2}{\rho\nu}$$

$Ha \gg 1$ $\qquad u_{max} = \underline{\frac{\delta^2}{\rho\nu}\left(-\frac{\partial P}{\partial x}\right)}$



$Ha \ll 1$ $\qquad u_{max} \simeq \frac{\delta^2}{\rho\nu}\left(-\frac{\partial P}{\partial x}\right)\frac{1}{2}\frac{H^2}{\delta^2}$

$$= \underline{\frac{1}{2} H^2 \left(-\frac{\partial P}{\partial x}\right)\frac{1}{\rho\nu}}$$



terminal vel. : drag force/vol. $= \sigma B^2 u$

$$= \left(-\frac{\partial P}{\partial x}\right)$$

$$\Rightarrow \quad u_{terminal} = \frac{1}{\sigma B^2}\left(-\frac{\partial P}{\partial x}\right)$$

✓ $(Ha \gg 1)$

otherwise acc^n × time

$$\frac{1}{\rho}\left(-\frac{\partial P}{\partial x}\right) \times \frac{H^2}{\nu} = u \quad ✓ \quad (Ha \ll 1)$$

So in the $H \ll \delta$ limit, we can think of the limiting velocity as being the velocity reached in a viscous time, ie. limited by viscous drag,

whereas when $H \gg \delta$ $(Ha \gg 1)$ the flow velocity is determined by the magnetic drag.

Hartmann layer thickness: $\cancel{\left(-\frac{1}{\rho}\frac{\partial P}{\partial x}\right) \times \left(\text{drag time}\right) =}$

drag time = viscous time — adjusts B.L. thickness until $t_{visc} = t_{bulk}$.

sets v in the bulk

$$\frac{\rho}{\sigma B^2} = \frac{\delta^2}{\nu} \quad \Rightarrow \quad \delta^2 = \left(\frac{\rho\nu}{\sigma B^2}\right) \quad ✓$$

$$\left[\frac{\delta^2}{\nu} \approx \frac{1}{\sigma v_A^2}\right]$$

$$R = \frac{\text{viscous stress}}{\text{drag force}} = \frac{\rho\nu (u'')^2}{-u\sigma B^2}$$

$$= \underbrace{\left(\frac{\rho\nu}{\sigma B^2}\right)}_{\delta^2} \frac{u''}{(-u)} = \frac{\cosh^2(y/\delta) / \cosh^2(H/\delta)}{1 - \frac{\cosh(y/\delta)}{\cosh(H/\delta)}}$$

$$u = a\left[1 - \frac{\cosh(y/\delta)}{\cosh(H/\delta)}\right]$$

$$\frac{du}{dy} = -\frac{1}{\cosh(H/\delta)} \frac{1}{\delta} \sinh(y/\delta)$$

$$\frac{d^2u}{dy^2} = -\frac{1}{\cosh(H/\delta)} \frac{1}{\delta^2} \cosh(y/\delta)$$

at $y = 0$ $\quad R = \dfrac{1}{\cosh^2 H/\delta} \dfrac{1}{1 - \frac{1}{\cosh H/\delta}}$

$$\frac{H}{\delta} \gg 1 \quad \cosh\frac{H}{\delta} \simeq \frac{1}{2}e^{H/\delta} \quad R \simeq 4e^{-2H/\delta}$$

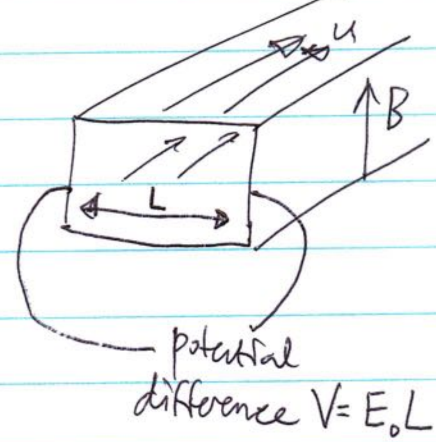
$$\frac{H}{\delta} \ll 1 \quad \underbrace{\frac{1}{\left(1 + \frac{1}{2}\frac{H^2}{\delta^2}\right)^2}}\ \frac{1}{1 - \frac{1}{\left(1 + \frac{1}{2}\frac{H^2}{\delta^2}\right)}} \qquad \frac{2\delta^2}{H^2}$$

$$R = 2\left(\frac{\delta}{H}\right)^2$$

Applications:

large Ha

$$J_z = \sigma(\underset{\text{applied } E_0}{E_0} + u_0 B)$$

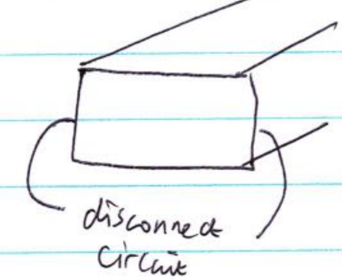


$$u_0 B = - E_0 - \frac{1}{\sigma B}\frac{dP}{dx}$$

different choices for E_0, J:

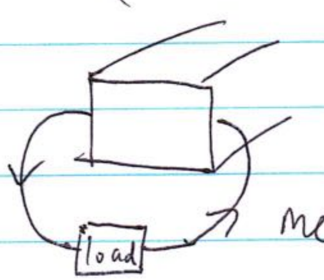
(1) $J_z = 0$ $\quad E_0 = -u_0 B \quad$ Lorentz force vanishes

MHD flowmeter.



(2) $E_0 \simeq 0$ $\quad J \simeq \sigma u_0 B$

$$\frac{dP}{dx} \simeq \sigma B^2 u_0$$

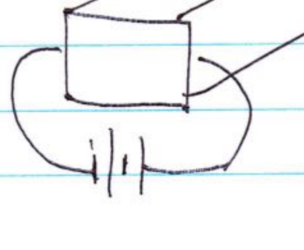


mechanical energy → electrical energy + heat.

MHD power generation.

(3) $E_0 < 0$ $\quad |E_0| > u_0 B \quad$ MHD pump.

then $\frac{dP}{dx} > 0$ $\quad$ Metallurgy + nuclear industry.



Magnetohydrodynamics Part 2

Let's take a closer look at the $\underline{J} \times \underline{B}$ force. Ampère's law tells us that

$$\underline{J} = \frac{1}{\mu_0} \underline{\nabla} \times \underline{B}$$

(we don't need the displacement current as long as flow time ≪ light crossing time)

$$\Rightarrow \underline{J} \times \underline{B} = \frac{1}{\mu_0} (\underline{\nabla} \times \underline{B}) \times \underline{B}$$

$$= -\underline{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \underline{B}.\underline{\nabla}\left(\frac{\underline{B}}{\mu_0}\right)$$

this looks like the gradient of a pressure
(recall that $\frac{B^2}{2\mu_0}$ is the energy density in the B field)

The magnetic pressure acts only perpendicular to the field lines. To see this, define the unit vector $\hat{\underline{s}} = \frac{\underline{B}}{B}$ which points locally in the direction of $\underline{B}$.

Then
$$\frac{(\underline{B}.\underline{\nabla})\underline{B}}{\mu_0} = \frac{B}{\mu_0}\frac{d}{ds}\left(B\hat{\underline{s}}\right)$$
$$= \frac{B^2}{\mu_0}\frac{d\hat{\underline{s}}}{ds} + \hat{\underline{s}}\frac{d}{ds}\left(\frac{B^2}{2\mu_0}\right)$$

this term is magnetic tension

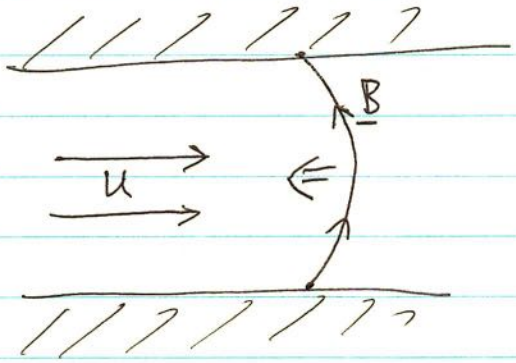
this cancels the component of $\underline{\nabla} B^2/2\mu_0$ along the field

$$\frac{d\hat{\underline{s}}}{ds} = \frac{\hat{\underline{n}}}{R_c}$$ ← radius of curvature

The tension force acts to try to straighten the field line.

Note that both tension and magnetic pressure act perpendicular to $\underline{B}$ — as they must since the total force is $\underline{J} \times \underline{B}$!

For the Hartmann flow we looked at last time, this suggests that the magnetic drag must come from distortion of the field:



To understand this, we need to think about how $\underline{B}$ changes in response to the flow, ie. what is $\partial B/\partial t$?

Faraday's law $\Rightarrow$ $$\frac{\partial \underline{B}}{\partial t} = -\underline{\nabla} \times \underline{E}$$

Ohm's law $\underline{E} = -\underline{v} \times \underline{B} + \underline{J}/\sigma$ (we wrote this down last time)

$$\Rightarrow \quad \boxed{\frac{\partial \underline{B}}{\partial t} = \underbrace{\underline{\nabla} \times (\underline{v} \times \underline{B})}_{\text{"flux freezing"}} - \underbrace{\underline{\nabla} \times \left(\frac{\underline{J}}{\sigma}\right)}_{\text{"magnetic diffusion"}}}$$

"the induction equation"

1) <u>Flux freezing</u> We've seen an equation of the form

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B})$$

before – the vorticity equation

$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{\omega}).$$

(see the Week 2 notes, p 20)

Just like vortex lines are advected by the fluid, magnetic field lines are "frozen" into the fluid when this term dominates in the induction equation.

2) <u>Magnetic diffusion</u>

$$-\underline{\nabla} \times \left(\underline{J}/\sigma\right)$$

$$= -\underline{\nabla} \times \left(\frac{1}{\sigma \mu_0} \underline{\nabla} \times \underline{B} \right)$$

$$= \frac{1}{\sigma \mu_0} \nabla^2 \underline{B} \quad \text{for constant } \sigma \text{ and } \mu_0$$

$$\Rightarrow \quad \frac{\partial \underline{B}}{\partial t} = \frac{1}{\sigma \mu_0} \nabla^2 \underline{B} = \eta \nabla^2 \underline{B}$$

<u>a diffusion equation</u>

magnetic diffusivity $\eta = \dfrac{1}{\sigma \mu_0}$

This term causes field lines to diffuse through the fluid, ie. it "breaks" flux freezing.

The <u>magnetic Reynolds number</u> $R_M = \dfrac{UL}{\eta}$

compares the size of the two terms

$R_M \ll 1$ – diffusion dominates

$R_M \gg 1$ – flux freezing

Let's go back to the Hartmann flow. When magnetic drag dominates, the velocity is $\simeq$ constant with height and given by

$$\sigma v B_y^2 = -\frac{\partial P}{\partial x}$$

$\Rightarrow$ the current density is $J = \sigma v B_y = \dfrac{-\partial P/\partial x}{B_y}$

But $\underline{J} = \dfrac{1}{\mu_0} \underline{\nabla} \times \underline{B}$ $\Rightarrow$ there must be an x-component of $\underline{B}$ that is induced by the flow

Such that $$J_z = -\frac{1}{\mu_0}\frac{dB_x}{dy} = \frac{-\partial P/\partial x}{B_y}$$

$$\Rightarrow \quad B_x = \frac{\mu_0 \, \partial P/\partial x}{B_y} \, y$$

(From the symmetry, set the integration constant so that $B_x = 0$ at $y = 0$)

or $$\frac{B_x}{B_y} = -\frac{\mu_0}{B_y^2} \sigma v B_y^2 \, y$$

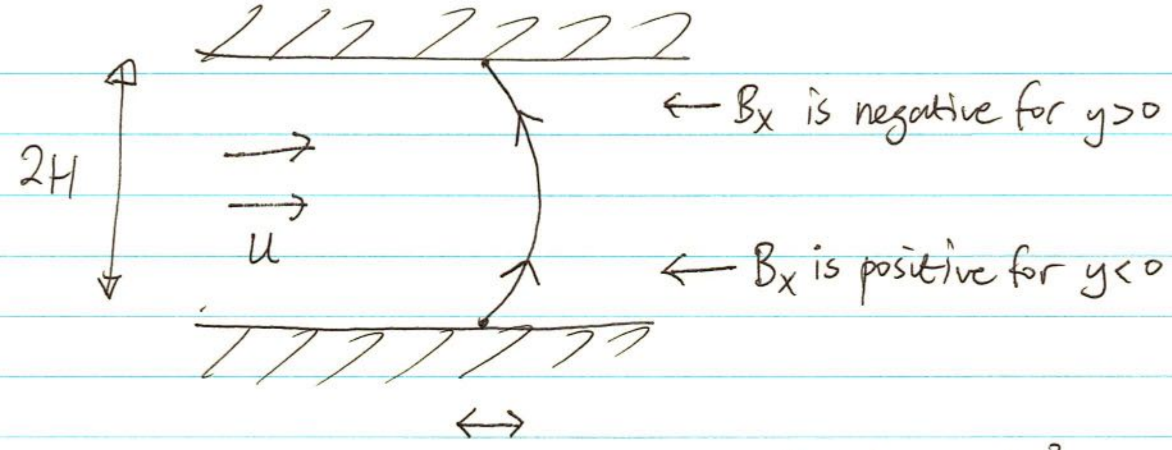
$$= -\left(\frac{y}{H}\right) \frac{vH}{\eta}$$

$$= -\left(\frac{y}{H}\right) R_M$$

We see that $B_x \simeq B_y R_M$

Shape of the field lines: $\dfrac{dx}{dy} = \dfrac{B_x}{B_y} = -R_M \dfrac{y}{H}$

$$\Rightarrow \quad x = \frac{R_M}{2H}\left(H^2 - y^2\right) \quad \text{quadratic}$$



displacement is $\dfrac{R_M H}{2} = \dfrac{1}{2}\dfrac{H^2}{\eta} v$

$\simeq \begin{pmatrix} \text{diffusion} \\ \text{time across} \\ \text{the channel} \end{pmatrix} \times (\text{velocity})$

Laboratory flows (eg. liquid sodium reactor cooling I talked about last time) have $R_M \ll 1$.

Astrophysical flows are the opposite $R_M \gg 1$. These are the conditions under which you can get dynamos — fluid motion acts to create and maintain a magnetic field, eg. Earth's core, convection zone of the Sun.

The magnetic tension and pressure provide restoring force for waves but also can lead to instability.

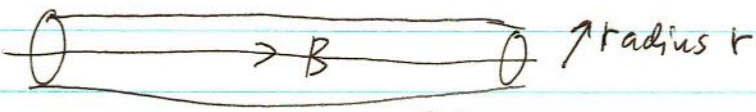
E.g. <u>sound waves</u>

Sound waves travelling along a magnetic field have the usual dispersion relation $\omega^2 = c_s^2 k^2$
but perpendicular to field lines, the magnetic pressure provides an extra restoring force, giving

$$\omega^2 = k^2 (c_s^2 + v_A^2)$$

where $v_A^2 = \dfrac{B^2}{\mu_0 \rho}$, v_A is the <u>Alfven speed</u>.

Can think of the magnetic field as having adiabatic index of 2.
Consider compressing a flux tube



flux conservation $\Rightarrow$ $r^2 B = \text{constant}$, mass cons. $\Rightarrow \rho \propto \dfrac{1}{r^2}$

$$\Rightarrow P_B \propto B^2 \propto \frac{1}{r^4} \propto \rho^2$$

"sound speed" $\dfrac{\gamma P}{\rho} = \dfrac{2 P_B}{\rho} = \dfrac{B^2}{\rho \mu_0}$ ✓.

<u>interchange instability</u>

flux tubes are buoyant and want to rise

eg. Unmagnetized gas $P = \dfrac{\rho k_B T}{m}$

magnetized gas $P = \dfrac{\rho k_B T}{m} + \dfrac{B^2}{2\mu_0}$

pressure balance $\Rightarrow$ density is smaller in magnetized layer
$\Rightarrow$ heavy on light
unstable!

Alfven waves A new kind of wave restored by magnetic tension.
(Analogous to wave on a string)

$\xrightarrow{\underline{B}}$ $\uparrow \delta\underline{B}$ $\uparrow \delta\underline{u}$

$\rightarrow \underline{k}$

$$\frac{\partial}{\partial t}\delta\underline{B} = \underline{\nabla}\times(\delta\underline{u}\times\underline{B}) = ikB\,\delta u$$

$$\rho\frac{\partial}{\partial t}\delta\underline{u} = \frac{(\underline{B}\cdot\underline{\nabla})\,\delta\underline{B}}{\mu_0} = \frac{ikB\,\delta B}{\mu_0}$$

$$-i\omega\,\delta B = ikB\,\delta u$$

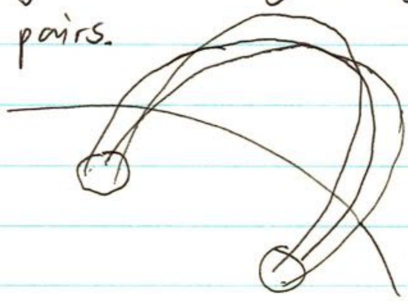
$$-i\omega\rho\,\delta u = \frac{ikB\,\delta B}{\mu_0}$$

$$\Rightarrow \quad \omega^2 = \frac{k^2 B^2}{\rho\mu_0} = k^2 v_A^2.$$

Solar dynamo

An application of high R_m flow with buoyancy of flux tubes is the solar dynamo. We'll look at some slides.

In the solar convection zone, differential rotation stretches out and amplifies field into flux "ropes". Once the magnetic pressure becomes comparable to the gas pressure, the flux rope rises buoyantly. When they emerge from the surface, they make sunspot pairs.



For more info see
Paul Charbonneau (2014) ARAA 52, 251

(or go and talk to him at UdeM!!)