Sep 4, 2007.

PHYS 643 lecture 1

I : Fluid basics

We start off by asking, what is a fluid?
Quite often we are in a situation where the mean free path of particles λ is much smaller than the distances over which bulk properties such as temperature, density are varying.

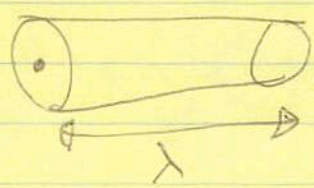
eg. center of the Sun

$$T_c \simeq 10^7\ K \qquad M_\odot = 2\times 10^{33}\,g$$
$$\rho_c \simeq 150\ g/cm^3 \qquad R_\odot = 7\times 10^{10}\,cm$$
$$\bar\rho = \frac{M}{4\pi R^3/3} = 1.4\ g/cm^3$$

The mean free path is given by

[note cgs units]

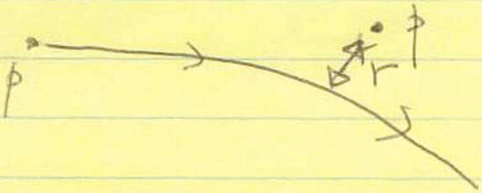
$$n\sigma\lambda = 1$$



this volume must contain 1 particle on average

the number density of particles $n \simeq \rho/m_p \approx 10^{26}\ cm^{-3}$

the Coulomb cross-section is given roughly by



$$\frac{e^2}{r} \simeq kT \qquad \sigma \simeq \pi r^2$$

$$\Rightarrow \text{mfp}\quad \lambda = \frac{1}{n\sigma} \simeq \frac{(kT)^2}{\pi n e^4} \sim \boxed{10^5\,cm\ \frac{T^2}{n}}$$

For solar central conditions, we find $\lambda \sim 10^{-6}$ cm.
This is much smaller than the size of the Sun, or the distances over which temperature or composition vary ($\gtrsim 10^{10}$ cm).
The particles locally have an equilibrium distribution (eg. Maxwell-Boltzmann for ideal gas) - we say they are in local thermodynamic equilibrium (LTE).

We can therefore treat the matter as a continuum, and the equations that describe its behavior - the fluid equations - are a set of conservation laws for mass, momentum, and energy.

There are other situations where this is not such a good approximation, eg. galaxy cluster

$$n \sim \frac{10^{13} M_\odot}{(1\,\mathrm{Mpc})^3} \frac{1}{m_p} \sim 5\times10^{-4}\ \mathrm{cm}^{-3}$$

$$T \sim 10^7\,\mathrm{K}$$

$$\lambda \simeq 20\,\mathrm{kpc} < R \simeq 1\,\mathrm{Mpc},$$

but becoming comparable to R

eg. solar wind $\left.\begin{array}{l} T \sim 10^5\,\mathrm{K} \\ n \sim 10\,\mathrm{cm}^{-3} \end{array}\right\} \lambda \sim 10^{14}$ cm

near Earth's orbit which is several AU.

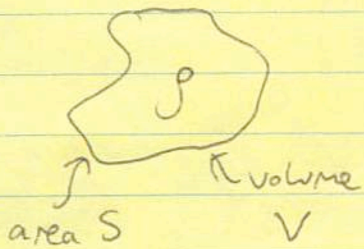
in which case need to worry about individual particles or at least the particle distribution functions.

[Note that even though we can derive the equations governing the behavior of the fluid without worrying about individual particle trajectories, in principle we should be able to arrive at the fluid equations from such a starting point, and indeed we can as we'll see later.]

Fluid Equations (Chapter 4 in Choudhuri)

1) Continuity equation (mass conservation)

Consider a fluid element



$$M = \int \rho \, dV$$

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho \, dV = -\int_S \rho \underline{u} \cdot d\underline{S}$$

$\rho \underline{u}$ = mass flux across the surface (units: $g \, cm^{-2} s^{-1}$)

Apply the divergence theorem

$$\Rightarrow \int \frac{\partial \rho}{\partial t} dV = -\int \underline{\nabla} \cdot (\rho \underline{u}) \, dV$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{u})}$$

or we can rewrite this

$$\underbrace{\left(\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}\right)}_{} \rho = -\rho \underline{\nabla} \cdot \underline{u}$$

this derivative comes up a lot

write $$\boxed{\frac{D\rho}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}\right) \rho = -\rho \underline{\nabla} \cdot \underline{u}}$$

where $\frac{D}{Dt}$ is the advective or Lagrangian derivative.

We distinguish between

$\underline{\text{Eulerian}}$ and $\underline{\text{Lagrangian}}$ points of view.

describe fluid properties at each point in space

describe fluid properties following the fluid element

If we label the trajectory of a fluid element $\underline{r}(\underline{r}_0, t)$

initial position labels fluid element

then the velocity of the fluid element

$$\underline{u}(\underline{r}_0, t) = \frac{d\underline{r}}{dt}$$

and the rate of change of some quantity Q (eg. Q might represent temperature, magnetic field ...) following the fluid is

$$\cancel{\frac{dQ}{dt}} \;\; \frac{DQ}{Dt} = \frac{d}{dt} Q(\underline{r}(\underline{r}_0, t), t)$$

$$= \left.\frac{\partial Q}{\partial t}\right|_{\underline{r}} + \frac{d\underline{r}}{dt} \cdot \left.\underline{\nabla} Q\right|_t$$

$$\Rightarrow \frac{DQ}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}\right) Q.$$

[Perhaps a better way to show this is as in the book

$$DQ = Q(\underline{r} + \underline{u}Dt, t + Dt) - Q(\underline{r}, t)$$

$$= \frac{\partial Q}{\partial t} Dt + \underline{u} \cdot \underline{\nabla} Q \, Dt$$

]

Sep 6, 2007

PHYS 643 lecture 2

Last time:
- the idea of a fluid as having $\lambda \ll L$
- continuity equation (mass conservation)

$$\frac{D\rho}{Dt} = -\rho \underline{\nabla} \cdot \underline{u}$$

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{u})$$

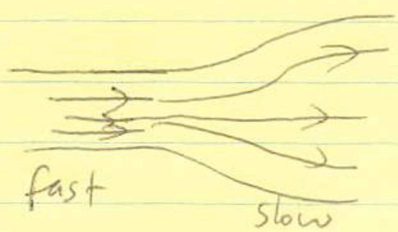
- advective derivative $\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}\right)$

An incompressible fluid has $\frac{D\rho}{Dt} = 0$ and therefore $\boxed{\underline{\nabla} \cdot \underline{u} = 0}$

eg. water

$$\left(\int \underline{u} \cdot d\underline{S} = 0\right)$$

as illustrated for example by the velocity of the flow in a river



$$\begin{pmatrix}\text{Mass}\\ \text{flux}\end{pmatrix} = \rho u A = \text{constant}$$

When the fluid motion $|\underline{u}| \ll c_s$ (sound speed) then $\frac{D\rho}{Dt} = 0$ is a good approximation

- density gradients are rapidly smoothed out when the flow is very subsonic.

-) Momentum equation

now look at

$$\frac{d}{dt} \underbrace{\int_V \rho \underline{u} \, dV}_{\text{momentum of fluid element}} = - \int \underbrace{(\rho \underline{u})}_{\text{momentum flux}} \underline{u} \cdot d\underline{S} + (\text{forces})$$

Again, use the divergence theorem

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + (\text{forces})_i$$

(need to be careful with indices)

The forces could be due to body forces (eg gravity) or a surface stress

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = f_i + \frac{\partial}{\partial x_j}(T_{ij})}$$

$$\left[\text{total force} = \int \underline{f}\, dV + \int \underline{\underline{T}} \cdot d\underline{S}\right]$$

Let's consider different forces that could be important

gravity $\quad \underline{f}_g = \rho \underline{g} = -\rho \underline{\nabla} \Phi$

gravitational potential

note that $\underline{f}$ is a force density (force per unit volume)

(where $\nabla^2 \Phi = 4\pi G \rho$ Poisson's eqn.)

or electromagnetic $\quad \underline{F} = \frac{1}{c}(\underline{J} \times \underline{B}) + \rho_Q \underline{E}$

current density $\quad$ charge density

gas pressure $\quad T_{ij} = -P\,\delta_{ij} \quad$ an isotropic stress

$$\frac{\partial}{\partial x_j} T_{ij} = -(\underline{\nabla} P)_i$$

the force is down the pressure gradient

Example: isothermal atmosphere

$z \uparrow \qquad \downarrow \quad \underline{g} = -\hat{\underline{z}}\,\frac{GM}{R^2} \qquad \rho = \rho(z)$
$\qquad\qquad\qquad\qquad\qquad\qquad P = P(z)$

static: $\frac{\partial}{\partial t} = 0 \qquad \underline{u} = 0$

momentum equation is $\quad 0 = -\frac{\partial P}{\partial z} - \rho g$

or $\boxed{\frac{\partial P}{\partial z} = -\rho g}$ hydrostatic balance

to solve, we need to know how P relates to ρ
(the equation of state)

eg. ideal gas
$$P = n k_B T = \frac{\rho k_B T}{\mu m_p}$$
(μ = mean molecular weight)

if T = constant $\qquad \frac{k_B T}{\mu m_p g}\frac{d\rho}{dz} = -\rho g$

or $\qquad \frac{d\rho}{dz} = -\frac{\rho}{H}$

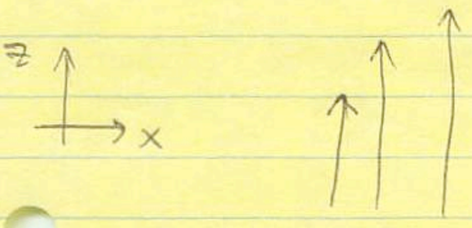
where $H = \frac{k_B T}{\mu m_p g}$ is the scale height
(lengthscale on which ρ changes
$H = \left(\frac{d\ln\rho}{dz}\right)^{-1}$)

$\Rightarrow \boxed{\rho = \rho_0 e^{-z/H}}$ $\qquad$ For Earth $T = 300\,K \quad \mu = 28\ (N_2)$
$g = 1000\ cm/s^2$
$\Rightarrow \boxed{H = 9\,km}$ ($\simeq$ height of Everest]

(Note that H depends on the ratio of thermal to gravitational energy. Compare our result with Boltzmann $\propto e^{-E/kT}$ – would predict this density profile from stat mech!)

In practice, T is not constant, so we need to consider energy equation also.

Go back to the different forces. A different kind of surface stress arises due to <u>viscosity</u>. We'll treat this in detail later, but for now consider a simple case



$$\underline{u} = \hat{\underline{z}}\, u(x)$$

a <u>shearing</u> flow

there is a stress $\quad \tau = \mu \dfrac{du}{dx}$

where μ is the viscosity, a property of the fluid
(depends on density, temperature etc)

For now, we'll focus on non-viscous or <u>inviscid</u> flows.

Summarize momentum eqn:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = F_i + \frac{\partial}{\partial x_j} T_{ij}$$

LHS $\Rightarrow$

$$\rho \frac{\partial u_i}{\partial t} + \underbrace{u_i \cancel{\frac{\partial \rho}{\partial t}} + \rho u_j \frac{\partial}{\partial x_j} u_i + u_i \cancel{\frac{\partial}{\partial x_j}(\rho u_j)}}_{=0 \text{ (continuity)}}$$

$$\Rightarrow \quad \boxed{\rho \frac{D\underline{u}}{Dt} = \underline{f} + \underline{\nabla}\cdot\underline{\underline{T}}}$$

eg. gravity and pressure gradient

$$\rho \frac{D\underline{u}}{Dt} = -\rho \underline{g} - \underline{\nabla} P$$

3) Energy equation

First, we can look at the kinetic energy by $\begin{pmatrix}\text{momentum} \\ \text{equation}\end{pmatrix}\cdot\underline{u}$

$$\rightarrow \quad \underline{u}\cdot\left(\rho\frac{\partial \underline{u}}{\partial t} + \rho\,\underline{u}\cdot\underline{\nabla}\,\underline{u}\right) = \underline{f}\cdot\underline{u} + \underline{u}\cdot(\underline{\nabla}\cdot\underline{\underline{T}})$$

$$u_i\left(\rho\frac{\partial u_i}{\partial t} + \rho\, u_j \partial_j u_i\right) = u_i f_i + u_i \partial_j T_{ij}$$

which leads to

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u_i^2 u_j\right) = u_i f_i + u_i \partial_j T_{ij}} \quad \text{—(1)}$$

or $\begin{pmatrix}\text{rate of change of} \\ \text{k.e. density}\end{pmatrix} = -\begin{pmatrix}\text{flux of ke} \\ \text{across the surface}\end{pmatrix} + \begin{pmatrix}\text{mechanical} \\ \text{work } \underline{u}\cdot\underline{f}\end{pmatrix}$

What about internal energy?

recall 1st law $\quad dU = TdS - PdV$

$$\cancel{= TdS + \frac{Pd\rho}{\rho^2}}$$

if we define U and S per unit mass, this is

$$dU = TdS - P\,d\left(\frac{1}{\rho}\right) = TdS + \frac{Pd\rho}{\rho^2}$$

[Note: we're ignoring magnetic energy for now — we'll put that in next week.]

for a given fluid element

$$\boxed{\underbrace{T\frac{Ds}{Dt}}_{} = \frac{Du}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}} \quad \sim (2)$$

rate of change of heat content

$$\boxed{T\frac{Ds}{Dt} = \varepsilon - \frac{\underline{\nabla}.\underline{F}}{\rho}}$$

ε = local sources or sinks of energy ($\mathrm{erg\ g^{-1}s^{-1}}$)

$\underline{F}$ = heat flux ($\mathrm{erg\ cm^{-2}\ s^{-1}}$)

$= -K\underline{\nabla}T$ typically — heat flows down the temperature gradient

↑ thermal conductivity

For adiabatic flow $\frac{Ds}{Dt} = 0 = \frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right)$

(time for heat transport/generation $\gg$ flow time)

for an ideal gas $\gamma = \frac{c_p}{c_v}$ ratio of specific heats

Adding (1) and (2), you can show that the total energy evolves according to

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho U\right) + \frac{\partial}{\partial x_j}\left(u_j\left(\frac{1}{2}\rho u^2 + \rho U + P\right)\right) = \left(\varepsilon - \frac{1}{\rho}\underline{\nabla}.\underline{F}\right) + \underline{u}.\underline{f}}$$

Note that in the flux term, the enthalpy $h = U + P/\rho$ appears — takes into account the PdV work.

Bernoulli's principle

The momentum equation is

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\,\underline{u} = -\frac{\underline{\nabla} P}{\rho} - \underline{\nabla}\Phi$$

Use the identity

$$(\underline{u}\cdot\underline{\nabla})\,\underline{u} = \underline{\nabla}\left(\tfrac{1}{2}u^2\right) - \underline{u}\times(\underline{\nabla}\times\underline{u})$$

you should be able to prove this identity!

and define $\quad h = \int \frac{dP}{\rho} \quad$ (enthalpy)

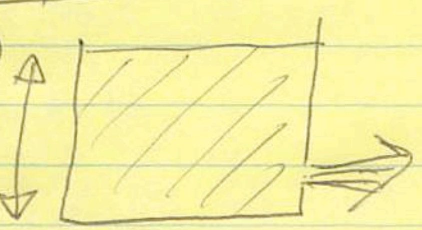
$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} - \underline{u}\times(\underline{\nabla}\times\underline{u}) = -\underline{\nabla}\left(\tfrac{1}{2}u^2 + h + \Phi\right)$$

if the flow is steady ($\partial/\partial t = 0$), then

$$\boxed{\underline{u}\cdot\underline{\nabla}\left(\tfrac{1}{2}u^2 + h + \Phi\right) = 0}$$

or $\quad \tfrac{1}{2}u^2 + h + \Phi \quad$ is constant along streamlines.

Examples

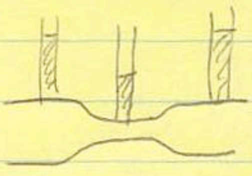
1) 

water flows out of a tank

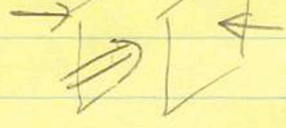
$\rho = \text{const.} \Rightarrow h = \frac{P}{\rho}$

$$\Rightarrow v_{out} = \sqrt{2gH}$$

3) Venturi tube



measure flow rates

2) lift on aeroplane wing or two pieces of paper 

Remind yourself of how to prove vector identities !

eg. Proof of

$$(\underline{u}.\underline{\nabla})\,\underline{u} = \underline{\nabla}\left(\tfrac{1}{2}u^2\right) - \underline{u}\times(\underline{\nabla}\times\underline{u})$$

$$\begin{aligned}
\left[\underline{u}\times(\underline{\nabla}\times\underline{u})\right]_i &= \varepsilon_{ijk}\; u_j \;\varepsilon_{klm}\; \partial_l u_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,(u_j\,\partial_l u_m) \\
&= \tfrac{1}{2}\,\partial_i(u_j^2) - u_j\,\partial_j\,u_i \\
&= \left[\tfrac{1}{2}\,\underline{\nabla}(u^2) - (\underline{u}.\underline{\nabla})\,\underline{u}\right]_i
\end{aligned}$$

PHYS 643 lecture 3

Sep 11, 2007.

(continue with Chapter 4)

Bernoulli's principle

The momentum equation is

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\,\underline{u} = -\frac{\underline{\nabla} P}{\rho} - \underline{\nabla}\Phi$$

use the identity

$$(\underline{u}\cdot\underline{\nabla})\,\underline{u} = \underline{\nabla}\left(\frac{1}{2}u^2\right) - \underline{u}\times(\underline{\nabla}\times\underline{u})$$

you should be able to prove this identity!

and define $h = \int \frac{dP}{\rho}$ (enthalpy)

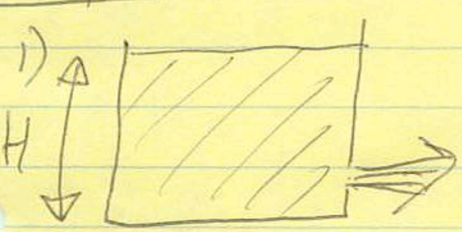
$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} - \underline{u}\times(\underline{\nabla}\times\underline{u}) = -\underline{\nabla}\left(\frac{1}{2}u^2 + h + \Phi\right)$$

if the flow is steady ($\partial/\partial t = 0$), then

$$\boxed{\underline{u}\cdot\underline{\nabla}\left(\frac{1}{2}u^2 + h + \Phi\right) = 0}$$

or $\frac{1}{2}u^2 + h + \Phi$ is constant along streamlines.

Examples

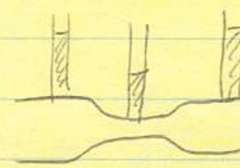
1) 

water flows out of a tank

$\rho = \text{const.} \Rightarrow h = \frac{P}{\rho}$

$$\Rightarrow v_{out} = \sqrt{2gH}$$

3) Venturi tube



measure flow rates

2) lift on aeroplane wing or two pieces of paper

Some extra notes on the enthalpy

In the derivation of Bernoulli's principle, we wrote

$$\frac{\underline{\nabla} P}{\rho} = \underline{\nabla} h$$

When is this appropriate?

— For an adiabatic flow, the enthalpy change is dP/ρ.
To see this, recall that the enthalpy is

$$H = U + PV$$

$$\Rightarrow \quad dH = dU + PdV + VdP = TdS + VdP$$

per unit mass $\quad dh = Tds + \frac{dP}{\rho}$

So for an adiabatic flow $\underline{\nabla} s = 0$
we can write $\underline{\nabla} h = \frac{\underline{\nabla} P}{\rho}$
where h is the enthalpy.

— For a constant density fluid, $h = \frac{P}{\rho}$ obeys $\underline{\nabla} h = \frac{\underline{\nabla} P}{\rho}$.

— For a barotropic fluid more generally (where $P(\rho)$)
eg. $P \propto \rho^{\gamma}$ then $h = \left(\frac{\gamma}{\gamma - 1}\right)\frac{P}{\rho}$
satisfies $\underline{\nabla} h = \frac{\underline{\nabla} P}{\rho}$

— To write $\frac{\underline{\nabla} P}{\rho}$ as $\underline{\nabla}$(scalar) then $\frac{\underline{\nabla} P}{\rho}$ must be curl-free
$(\underline{\nabla} P \times \underline{\nabla} \rho = 0)$

Vorticity

The momentum equation is

$$\frac{\partial \underline{u}}{\partial t} + \underbrace{(\underline{u}\cdot\underline{\nabla})\,\underline{u}}_{\underline{\nabla}\left(\frac{1}{2}u^2\right) - \underline{u}\times(\underline{\nabla}\times\underline{u})} = -\frac{\underline{\nabla} P}{\rho} - \underline{\nabla}\Phi + \underline{F}$$

Take the curl of this equation, defining the vorticity
$\underline{\omega} = \underline{\nabla}\times\underline{u}$

$$\Rightarrow \quad \frac{\partial \underline{\omega}}{\partial t} - \underbrace{\underline{\nabla}\times(\underline{u}\times\underline{\omega})}_{(\underline{\omega}\cdot\underline{\nabla})\underline{u} - (\underline{u}\cdot\underline{\nabla})\underline{\omega} + \underline{u}\,\underline{\nabla}\cdot\underline{\omega} - \underline{\omega}\,\underline{\nabla}\cdot\underline{u}} = -\frac{\underline{\nabla} P\times\underline{\nabla}\rho}{\rho^2} + \underline{\nabla}\times\underline{F}$$

$$\Rightarrow \quad \frac{1}{\rho}\frac{D\underline{\omega}}{Dt} - \frac{(\underline{\omega}\cdot\underline{\nabla})\underline{u}}{\rho} + \underbrace{\frac{\underline{\omega}\,(\underline{\nabla}\cdot\underline{u})}{\rho}}_{-\frac{1}{\rho^2}\frac{D\rho}{Dt}\underline{\omega}} = -\frac{\underline{\nabla} P\times\underline{\nabla}\rho}{\rho^3} + \frac{\underline{\nabla}\times\underline{F}}{\rho} \qquad (*)$$

$$\Rightarrow \quad \boxed{\frac{D}{Dt}\left(\frac{\underline{\omega}}{\rho}\right) = \left(\frac{\underline{\omega}}{\rho}\cdot\underline{\nabla}\right)\underline{u} + \frac{\underline{\nabla}\rho\times\underline{\nabla}P}{\rho^3} + \frac{\underline{\nabla}\times\underline{F}}{\rho}}$$

$\frac{\underline{\omega}}{\rho}$ is advected by the fluid

this quantity is known as the vortensity or potential vorticity.

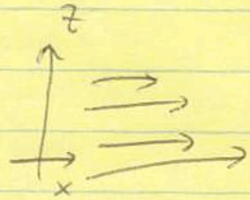
Some simple cases:

1) uniform rotation $\underline{u} = \hat{\underline{\phi}}\, r\Omega$

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \hat{\underline{z}} \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \Omega\right)$$

$$= \hat{\underline{z}}\; 2\Omega$$

2) shear flow $\underline{u} = \hat{\underline{x}}\; u(z)$

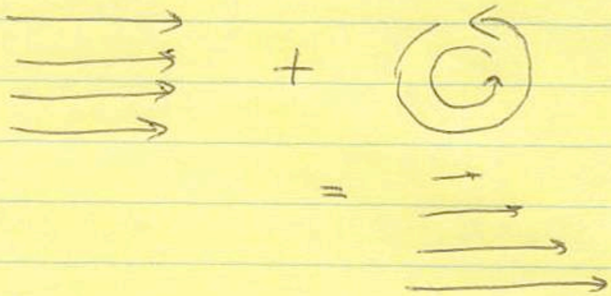


$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \hat{\underline{y}}\; \frac{du}{dz}$$

eg. in this diagram $\frac{du}{dz} < 0$

$\Rightarrow$ $\underline{\omega} \propto -\hat{\underline{y}}$ out of the page

the way to think of it is



A related quantity is the circulation $\Gamma = \oint_C \underline{u}.d\underline{l} = \int \underline{\omega}.d\underline{S}$

Calculate $\frac{D\Gamma}{Dt}$ for a material curve C which moves with the fluid:

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C \underline{u}.d\underline{l} = \oint_C \frac{D\underline{u}}{Dt}.d\underline{l} + \oint_C \underline{u}.\frac{D(d\underline{l})}{Dt}$$

The second term is $\underbrace{\oint_C \underline{u}.d\underline{u}}_{\oint_C du^2 = 0}$ because $\frac{D}{Dt}d\underline{l} = d\underline{u}$

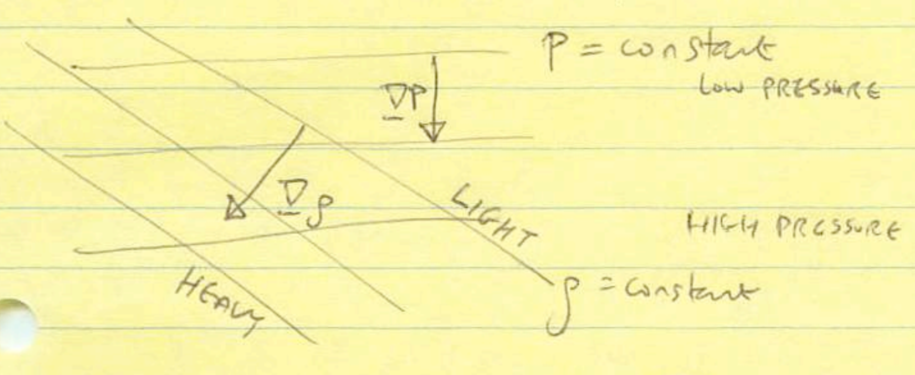
$$\Rightarrow \frac{D\Gamma}{Dt} = \oint \frac{D\underline{u}}{Dt}.d\underline{l}$$

$$= -\oint \frac{\underline{\nabla}P}{\rho}.d\underline{l} - \cancelto{0}{\oint \underline{\nabla}\underline{\Phi}.d\underline{l}} + \oint \underline{F}.d\underline{l}$$

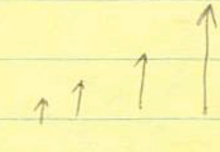
$$\Rightarrow \boxed{\frac{D\Gamma}{Dt} = \int_C \frac{\underline{\nabla}\rho \times \underline{\nabla}P}{\rho^2}.d\underline{S} + \oint_C \underline{F}.d\underline{l}}$$

Baroclinic generation of vorticity

eg. this could be a viscous force. We'll see later that this leads to diffusion of vorticity. - can be either a source or sink of vorticity.



the acceleration is



the resulting circulation is in direction of $\underline{\nabla}\rho \times \underline{\nabla}P$ (baroclinic vector)

A barotropic fluid (pressure only function of density $P(\rho)$) with $\underline{\nabla} \times \underline{F} = 0$ obeys Kelvin's circulation theorem $\dfrac{D\Gamma}{Dt} = 0$.

To get a better feeling for ω, go back to equation (*) for the case $\underline{F} = 0$, $\underline{\nabla} P \times \underline{\nabla} \rho = 0$

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega} \cdot \underline{\nabla})\, \underline{u} - (\underline{u} \cdot \underline{\nabla})\, \underline{\omega}$$

At a local point in the fluid, choose the z-axis to point along $\underline{\omega}$, ie. $\underline{\omega} = \omega \hat{\underline{z}}$.
Write the velocities as $\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}} + w\hat{\underline{z}}$

The RHS is $\quad (\underline{\omega} \cdot \underline{\nabla})\, \underline{u} - (\underline{u} \cdot \underline{\nabla})\, \underline{\omega}$

$$= \left[\omega \frac{\partial w}{\partial z} - \omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] \hat{\underline{z}}$$

$$+ \; \omega \frac{\partial v}{\partial z} \hat{\underline{y}} + \omega \frac{\partial u}{\partial z} \hat{\underline{x}}$$

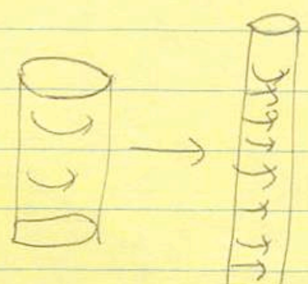
$$= -\omega \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)} \hat{\underline{z}} + \omega \frac{\partial v}{\partial z} \hat{\underline{y}} + \omega \frac{\partial u}{\partial z} \hat{\underline{x}}$$

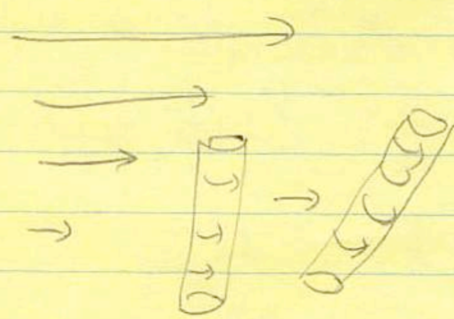
if $\underline{\nabla} \cdot \underline{u} = 0$ this term is $\dfrac{\partial w}{\partial z}$

this is the horizontal divergence $\underline{\nabla}_\perp \cdot \underline{u}_\perp$

"vortex stretching"

"vortex tilting"





The vortex lines are carried by the flow.

PHYS 643 lecture 4 (Choudhuri 14.1, 14.2) Sep 13, 2007.

Magnetized fluids - magnetohydrodynamics (MHD)

Electromagnetism in cgs

Maxwell's equations $\underline{\nabla} \cdot \underline{E} = 4\pi \rho_Q$

$$\underline{\nabla} \times \underline{B} = \frac{4\pi \underline{J}}{c} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}$$

$$\underline{\nabla} \cdot \underline{B} = 0$$

the Lorentz force is $\underline{F} = q\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c}\right)$

charge conservation $\frac{\partial \rho_Q}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0$

potentials $\underline{B} = \underline{\nabla} \times \underline{A}$ $\quad \underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$

Momentum equation

We already included the magnetic force in the momentum equation,

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} P - \rho \underline{\nabla} \underline{\Phi} + \frac{\underline{J} \times \underline{B}}{c}$$

force per unit volume

The current is determined by Ampère's law

$$\underline{J} = \frac{c}{4\pi} \underline{\nabla} \times \underline{B}$$

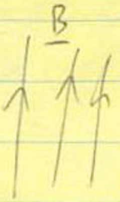
(the displacement current can be neglected for $u \ll c$)

The force is $\dfrac{\underline{J} \times \underline{B}}{c} = \dfrac{(\underline{\nabla} \times \underline{B}) \times \underline{B}}{4\pi} = -\underline{\nabla}\left(\dfrac{B^2}{8\pi}\right) + \dfrac{(\underline{B} \cdot \underline{\nabla})\underline{B}}{4\pi}$

$$\Rightarrow \quad \boxed{\frac{D\underline{u}}{Dt} = -\underline{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \rho \underline{\nabla} \Phi + \frac{(\underline{B} \cdot \underline{\nabla})\underline{B}}{4\pi}}$$

So we can think of the magnetic force as an isotropic pressure plus a term which describes "magnetic tension" due to curvature of the field lines.

To get some intuition, consider a flux tube with $\underline{B} = B\hat{\underline{z}}$



then
$$\frac{\underline{J} \times \underline{B}}{c} = \underbrace{\left(-\frac{d}{dz}\left(\frac{B^2}{8\pi}\right) + \frac{B}{4\pi}\frac{dB}{dz}\right)\hat{\underline{z}}}_{=0} - \frac{d}{dx}\left(\frac{B^2}{8\pi}\right)\hat{\underline{x}} - \frac{d}{dy}\left(\frac{B^2}{8\pi}\right)\hat{\underline{y}}$$

No force along the flux tube

Work needed to confine the field in the horizontal direction.

Evolution of the field

described by the induction equation (Faraday's law)

$$\boxed{\frac{\partial \underline{B}}{\partial t} = -c\underline{\nabla} \times \underline{E}}$$

in the frame of the fluid, Ohm's law $\Rightarrow$ $\underline{J} = \sigma \underline{E}'$

in the inertial frame, $\boxed{\underline{E} + \dfrac{\underline{u} \times \underline{B}}{c} = \dfrac{\underline{J}}{\sigma}}$

$$\Rightarrow \boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla}\times(\underline{u}\times\underline{B}) - c\,\underline{\nabla}\times\left(\frac{\underline{J}}{\sigma}\right)}$$

The first term describes the freezing of field lines into the fluid "flux freezing"

directly analogous to vorticity. Recall that $\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla}\times(\underline{u}\times\underline{\omega})$.

For vorticity we had $\frac{D\Gamma}{Dt} = 0 = \frac{D}{Dt}\int_C \underline{\omega}\cdot d\underline{S}$

surface enclosed by material curve

here, $$\frac{D}{Dt}\underbrace{\int_C \underline{B}\cdot d\underline{S}}_{\text{magnetic flux}} = 0$$

Expand $$\underline{\nabla}\times(\underline{u}\times\underline{B}) = (\underline{B}\cdot\underline{\nabla})\,\underline{u} - (\underline{u}\cdot\underline{\nabla})\,\underline{B} + \underline{u}\,(\cancelto{0}{\underline{\nabla}\cdot\underline{B}}) - \underline{B}(\underline{\nabla}\cdot\underline{u})$$

$$\Rightarrow \frac{\partial \underline{B}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\,\underline{B} + \underbrace{\underline{B}(\underline{\nabla}\cdot\underline{u})}_{-\frac{1}{\rho}\frac{D\rho}{Dt}\underline{B}} = (\underline{B}\cdot\underline{\nabla})\,\underline{u}$$

(same argument as for vorticity)

$$\Rightarrow \boxed{\frac{D}{Dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{(\underline{B}\cdot\underline{\nabla})\underline{u}}{\rho}}$$

flux conservation

shearing along a field line generates new field components.

The second term describes <u>Ohmic diffusion</u>

$$\frac{\partial \underline{B}}{\partial t} = -c\,\underline{\nabla}\times\left(\frac{\underline{J}}{\sigma}\right) = -\frac{c^2}{4\pi}\,\underline{\nabla}\times\left(\frac{\underline{\nabla}\times\underline{B}}{\sigma}\right)$$

$$= \frac{c^2}{4\pi\sigma}\nabla^2\underline{B} - \frac{c^2}{4\pi}\left(\underline{\nabla}\times\underline{B}\right)\times\underline{\nabla}\left(\frac{1}{\sigma}\right)$$

for constant σ, $\boxed{\frac{\partial \underline{B}}{\partial t} = \eta\,\nabla^2\underline{B}}$

where η is the <u>magnetic diffusivity</u> $\eta = \frac{c^2}{4\pi\sigma}$

When $\eta = 0$ $(\sigma \rightarrow \infty)$ we have a "perfect conductor" and the <u>ideal MHD</u> approximation holds - the field lines are frozen into the fluid.
In A plasma with finite η, the field lines are able to diffuse through the fluid.

The relative importance of advection vs. diffusion is measured by the <u>magnetic Reynolds number</u> $R_M = \frac{UL}{\eta}$

Energy

To derive an equation for the evolution of the magnetic energy density, we take

$$\underline{B} \cdot (\text{induction equation})$$

$$\underline{B} \cdot \frac{\partial \underline{B}}{\partial t} = -c\, \underline{B} \cdot (\underline{\nabla} \times \underline{E})$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{1}{2} B^2\right) = -c\left[\underline{\nabla} \cdot (\underline{E} \times \underline{B}) + \underline{E} \cdot (\underline{\nabla} \times \underline{B})\right]$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\underline{\nabla} \cdot \left(\frac{c\, \underline{E} \times \underline{B}}{4\pi}\right) - \underline{E} \cdot \underline{J}}$$

this should look familiar

Poynting flux Ohmic dissipation.

We can write $\underline{J} \cdot \underline{E}$ as $\underline{J} \cdot \left(\frac{\underline{J}}{\sigma} - \frac{\underline{u} \times \underline{B}}{c}\right)$

$$= \frac{J^2}{\sigma} - \frac{\underline{J} \cdot \underline{u} \times \underline{B}}{c}$$

$$= \frac{J^2}{\sigma} + \underline{u} \cdot \frac{\underline{J} \times \underline{B}}{c}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\underline{\nabla} \cdot \left(\frac{c\, \underline{E} \times \underline{B}}{4\pi}\right) - \frac{J^2}{\sigma} - \underline{u} \cdot \left(\frac{\underline{J} \times \underline{B}}{c}\right)$$

in the K.E. equation, we showed that there was a term

$$\underline{u} \cdot \underline{F} = \underline{u} \cdot \frac{\underline{J} \times \underline{B}}{c}$$

$\Rightarrow$ the work done on the fluid by the magnetic force comes from the magnetic energy.

The second term is ohmic dissipation rate per unit volume J^2/σ

(equivalent to $I^2 R$ for a resistor, but written as a local heating rate)

energy lost from the magnetic field goes into heat.

You can show that the total energy equation is

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho U + \frac{B^2}{8\pi}\right) + \underline{\nabla}\cdot\Big(\underline{u}\left[\tfrac{1}{2}\rho u^2 + \rho U + P\right] \\ + c\,\frac{\underline{E}\times\underline{B}}{4\pi}\Big) \\ = \left(\varepsilon - \underline{\nabla}.\underline{F}\right) - J^2/\sigma + \underline{u}.\underline{F}\end{aligned}}$$

but in an isolated fluid element, the total energy
$\int\left(\frac{1}{2}\rho u^2 + \rho U + \frac{B^2}{8\pi}\right)dV$ must be constant

$\Rightarrow$ $-J^2/\sigma$ term is cancelled by the $\varepsilon = J^2/\sigma$ term

ie. the J^2/σ goes into heating the gas.

A better treatment of the $\underline{J} \times \underline{B}$ force

We saw earlier that

$$\underline{J} \times \underline{B} = -\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{(\underline{B}.\underline{\nabla})\underline{B}}{4\pi}$$

Now define the unit vector $\hat{\underline{s}} = \dfrac{\underline{B}}{B}$ which points locally in the field direction.

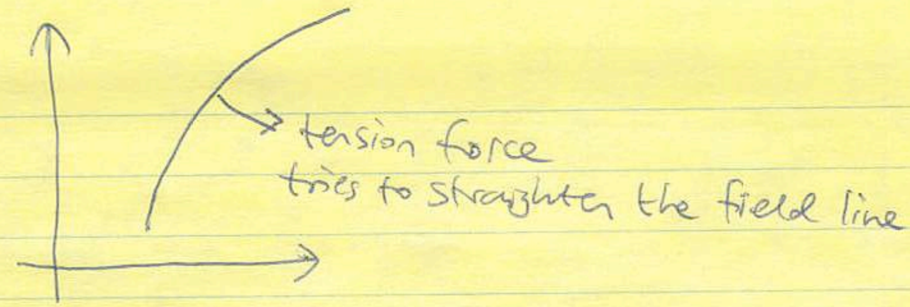
Then
$$\frac{(\underline{B}.\underline{\nabla})\underline{B}}{4\pi} = \frac{B}{4\pi}\frac{d}{ds}\left(B\hat{\underline{s}}\right)$$
$$= \frac{B^2}{4\pi}\frac{d\hat{\underline{s}}}{ds} + \hat{\underline{s}}\frac{d}{ds}\left(\frac{B^2}{8\pi}\right) \quad \text{—}(*)$$

The second term is the gradient of $B^2/8\pi$ along the field — it cancels the magnetic pressure gradient along the field so as we said in class the magnetic pressure only acts perpendicular to the field. (as it must since $\underline{J} \times \underline{B}$ is perpendicular to $\underline{B}$!)

The first term in equation (*) $\dfrac{B^2}{4\pi}\dfrac{d\hat{\underline{s}}}{ds}$ is the magnetic tension term.

The vector $\dfrac{d\hat{\underline{s}}}{ds} = \dfrac{\hat{\underline{n}}}{R_c}$ where $\hat{\underline{n}}$ is the unit vector perpendicular to the field and R_c is the radius of curvature.

The point is that the tension always tries to straighten the field line — so in the example I drew in class :



the tension force pulls to the right, just as an elastic string would behave.

Sep 18, 2007.

PHYS 643 lecture 5

II: Objects in hydrostatic balance

Hydrostatic balance (HB)

Stars and planets are objects in hydrostatic balance in which the pressure gradient from the interior to the surface balances their self-gravity.

To see that this must be the case, look at momentum equation

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\underline{\nabla} P}{\rho} + \underline{g} \qquad \text{—}(*)$$

Imagine we turn off the pressure gradients - the fluid would then accelerate at the local value of $g \simeq GM/R^2$. The time to fall a distance equal to the steller radius is then $\simeq \sqrt{R/g} \simeq \sqrt{R^3/GM}$. For the Sun, this timescale is only 30 minutes! Since the Sun's ~~lifetime is~~ age is 5 billion years, the two terms on the right of eq. (*) must balance to a high degree of accuracy!

Assuming spherical symmetry, the equation of HB is

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

where

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$m(r)$ is the mass contained within a sphere of radius r.

The boundary conditions are $m = 0$ at $r = 0$
$P = 0$ at $r = R$

To solve the equations, we need a relation between P and ρ — the equation of state.

Under the assumption $P \propto \rho^\gamma$ the solutions are known as polytropes. A polytrope of index n has $\gamma = 1 + \frac{1}{n}$.

A "back of the envelope" calculation to get the scalings

We can get a rough estimate of the structure by writing

$$\frac{dP}{dr} \simeq \frac{P_c}{R}$$

P_c ← central pressure
R ← radius

and $\rho g \simeq \left(\frac{M}{R^3}\right)\left(\frac{GM}{R^2}\right)$

$$\Rightarrow \quad P_c \simeq \frac{GM^2}{R^4}$$

For an ideal gas, $P_c \simeq \frac{\rho k T_c}{m_p} \Rightarrow \boxed{T_c \simeq \frac{GM m_p}{k_B R}}$

central temperature

for the Sun, we get $T_c \simeq 2 \times 10^7\,K$
close to the actual value $T_c = 1.5 \times 10^7\,K$.

For a polytropic relation $P \propto \rho^\gamma$ we can get the mass-radius scaling.

$$P_c \simeq \frac{GM^2}{R^4} \propto \rho^\gamma \simeq \left(\frac{M}{R^3}\right)^\gamma$$

$$\Rightarrow \quad \frac{M^2}{R^4} \propto \frac{M^\gamma}{R^{3\gamma}} \quad \Rightarrow \quad \boxed{M^{\gamma-2} \propto R^{3\gamma-4}}$$

eg. white dwarfs non-relativistic degenerate electrons

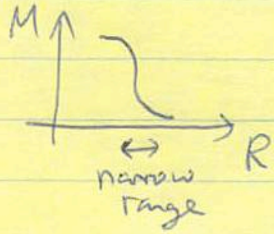
$\gamma = 5/3 \quad \Rightarrow \quad M \propto R^{-3}$

$$\boxed{R \propto M^{-1/3}}$$

as the white dwarf mass increases,
the electrons become relativistic $\gamma \to 4/3$
For $\gamma = 4/3$ M is independent of R!
The corresponding mass is the Chandrasekhar mass $M_{Ch} = 1.4 M_\odot$.

eg. neutron star $\gamma = 2$ $P \propto \rho^2$ interactions between the neutrons stiffen the EOS.

For $\gamma = 2$, $\boxed{\text{R is independent of M.}}$
roughly true in the real calculations



eg. $\gamma \to \infty$ $\boxed{\begin{array}{l}\text{incompressible} \\ M \propto R^3\end{array}}$ "rock"

eg. $\gamma = 1$ $P \propto \rho$ $\boxed{\begin{array}{l}\text{isothermal sphere} \\ M \propto R.\end{array}}$

Equation of state of a Fermi gas

Consider an ideal gas of fermions.

Quantum mechanics tells us that the density of states in six-dimensional phase space is $\frac{1}{h^3} = \frac{dn}{d^3p\, d^3x}$

The number of states with energy between E and $E + dE$ is

$$g(E)\,dE = \frac{1}{h^3}\ 2\ 4\pi p^2 \frac{dp}{dE}\, dE$$

per unit spatial volume

density of states

spin ½ particles

convert momentum space to energy

We will consider arbitrary relativity ($v \ll c$ and $v \sim c$), so use the relativistic relation $E^2 = p^2c^2 + (m_e c^2)^2$

$$\Rightarrow \cancel{\frac{dp}{dE}} \quad \frac{dE}{dp} = v = \frac{pc^2}{E} = \frac{\gamma m v c^2}{\gamma m c^2}$$

$$\Rightarrow \quad \boxed{g(E) = \frac{8\pi p^2}{h^3 v}}$$

For a Fermi gas, the occupation number of the state with energy E is

$$\boxed{f(E) = \frac{1}{1 + e^{(E-\mu)/kT}}}$$

μ = chemical potential

The number density of particles is

$$\boxed{n = \int f(E)\, g(E)\, dE} \quad \text{—(1)}$$

the internal energy density is $\boxed{U = \int E\, f(E)\, g(E)\, dE}$

—(2)

The pressure is

$$P = \int f(E)\, g(E)\, dE \int_0^1 d(\cos\theta)\,(p\cos\theta)(v\cos\theta)$$

(momentum flux across a unit area)

or $$\boxed{P = \frac{1}{3}\int p v\, f(E)\, g(E)\, dE.} \quad \text{— (3)}$$

Non-degenerate gas (Assume particles are non-relativistic $kT \ll mc^2$)

For a non-degenerate gas, $\left(\frac{\mu}{kT}\right)$ is large and negative

$$n = \int \frac{8\pi p^2}{h^3 v} e^{\mu/kT} e^{-E/kT}\, dE$$

which corresponds to a Maxwell-Boltzmann distribution of particle energies $f(E) \propto \exp(-E/kT)$.

Integrating, $$\boxed{\mu = kT \ln\left(\frac{n}{2n_Q}\right)}$$

where $$n_Q = \left(\frac{2\pi m kT}{h^2}\right)^{3/2}$$

defines the chemical potential for a non-degenerate gas.

Non-degenerate limit applies for $n \ll n_Q$ (which gives $\frac{\mu}{kT} \ll -1$)

It is straightforward to show that (2) and (3) give

$$\boxed{\begin{aligned} P &= n k_B T \\ U &= \tfrac{3}{2} n k_B T \end{aligned}} \qquad P = \frac{2}{3} U$$

writing $U = \frac{nm\langle v^2\rangle}{2}$ gives $\langle v^2\rangle = \frac{3kT}{m}$

the mean speed is $\langle v\rangle = \left(\frac{8kT}{\pi m}\right)^{1/2}$.

Completely-degenerate gas

For a degenerate gas, $\mu \gg kT$ giving

$$f(E) = \begin{cases} 1 & E<\mu \\ 0 & E>\mu \end{cases}$$

In this limit, μ is refered to as the Fermi energy E_F.

Integrating (1) over momentum rather than energy,

$$n = \int_0^{p_F} \frac{8\pi p^2\,dp}{h^3} = \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3$$

p_F = Fermi momentum = $\hbar k_F$ k_F = Fermi wavevector

$$\Rightarrow \boxed{k_F = (3\pi^2 n)^{1/3}}$$ should memorize this!

The Fermi energy is

$$E_F = \frac{p_F^2}{2m} \propto n^{2/3}$$ (non-relativistic particles)

or $$E_F = p_F c \propto n^{1/3}$$ (relativistic particles).

The pressure is

$$P = \tfrac{2}{5} n E_F \quad \text{(non-relativistic)}$$
$$= \tfrac{1}{4} n E_F \quad \text{(relativistic)}$$

internal energy density $\quad U = \frac{3}{5} n E_F = \frac{3}{2} P \quad (NR)$

or $\quad \frac{3}{4} n E_F = 3P \quad (R)$

Notice that these expressions are similar to those for an ideal gas, but with $\simeq E_F$ replacing kT. For a degenerate gas, the Fermi energy E_F sets the energy scale of the particles.

Partially degenerate gas

Consider a gas of non-relativistic particles. The integrals (1) - (3) can be written in terms of Fermi integrals
$F_n(\mu/kT)$

where

$$F_n(x) = \int_0^\infty \frac{t^n\,dt}{1+e^{t-x}}$$

$$n = \frac{\sqrt{2}(mkT)^{3/2}}{\hbar^3\pi^2} F_{1/2}\left(\mu/kT\right)$$

$$U = \frac{\sqrt{2}(mkT)^{3/2}}{\hbar^3\pi^2} kT\, F_{3/2}\left(\mu/kT\right)$$

$$P = \frac{2}{3}U.$$

For arbitrary relativity, there is an additional parameter $\frac{kT}{mc^2}$.

See ~~Potekhin~~ Chabrier & Potekhin (1998, PRE) for this case.

Antia (1993 ApJS 84 101) provides fitting formulae for the Fermi integrals, and there are analytic expressions given in eg. Clayton's book for the limits $x \to 0$ or $x \to \infty$.

Paczynski (1983) gives a fitting formula for the electron pressure that is quite accurate ($\lesssim$ few percent).

Sep 20, 2007.

PHYS 643 lecture 6

Last time, we looked at the equation of state of a Fermi gas.
Different regimes

non-degenerate $\quad P = nk_BT \qquad U = \frac{3}{2}nk_BT = \frac{3}{2}P$

$$\mu = kT\ln\left(\frac{n}{n_Q}\right)$$

degenerate $\quad k_F = (3\pi^2 n)^{1/3}$

non-relativistic $\quad E_F = \frac{p_F^2}{2m} = \frac{\hbar k_F^2}{2m} \propto n^{2/3}$

$$P = \frac{2}{5}nE_F \qquad U = \frac{3}{2}P$$

relativistic $\quad E_F = p_F c = \hbar k_F c \propto n^{1/3}$

$$P = \frac{1}{4}nE_F \qquad U = 3P$$

Radiation

A photon gas in thermal equilibrium has a distribution function

$$f(E) = \frac{1}{e^{E/kT} - 1}$$

$\uparrow$ bosons

$$\Rightarrow \quad n = \int_0^\infty \frac{8\pi\nu^2}{\cancel{h}c^3}\frac{1}{e^{h\nu/kT}-1}\,d\nu$$

Riemann zeta function $\zeta(3) \simeq 1.202$

$$= \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\int_0^\infty \frac{x^2\,dx}{e^x-1} = \frac{8\pi(k_BT)^3}{(hc)^3}2\zeta(3) \propto T^3$$

$$U = \int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}\,d\nu = \left(\frac{8\pi}{c^3}\right)h\left(\frac{kT}{h}\right)^4\underbrace{\int_0^\infty \frac{x^3\,dx}{e^x-1}}_{\pi^4/15}$$

$$\Rightarrow \quad u = \left(\frac{8\pi^5}{15}\frac{k_B^4}{h^3c^3}\right)T^4 \equiv \boxed{aT^4 = u}$$

$a = 7.5657 \times 10^{-15}$ cgs.
radiation constant.

Similarly, can show that $\boxed{P = \frac{1}{3}aT^4 = \frac{1}{3}u}$

A mixture of ions, electrons and radiation

We add the contributions of the different components,

eg. $\quad$ (pressure) = (ion pressure) + (electron pressure) + (radiation pressure)

We just need to keep track of the different number densities. To do this we define mean molecular weights (μ's) and number fractions (Y's) as follows.

eg. $\quad \boxed{\rho = \mu_e n_e m_p}$ or $\boxed{Y_e \rho = n_e m_p}$

defines μ_e, Y_e $\quad$ mean molecular weight per electron

$\boxed{\rho = \mu_i n_i m_p}$ $\quad \boxed{Y_i \rho = n_i m_p}$ $\quad$ defines μ_i, Y_i for ion species i

note that $\boxed{\mu_e = \frac{1}{Y_e} \quad \mu_i = \frac{1}{Y_i}}$

~~eg. solar composition gas hydrogen mass fraction X~~

~~$\rho X = m_p$~~

For the ions, we also define the mass fraction X_i by

$$\boxed{\rho X_i = A_i n_i m_p}$$

Y_i and X_i are related: $\boxed{Y_i = \frac{X_i}{A_i}}$

eg. Solar composition gas 70% H by mass (mass fraction X_H)
30% He " (" X_{He})
fully-ionized.

$$\rho X_H = m_p n_H$$
$$\rho X_{He} = 4 m_p n_{He}$$

The ion pressure is
$$P_{ion} = n_H kT + n_{He} kT$$
$$= \frac{\rho kT}{m_p}\left(X_H + \frac{X_{He}}{4}\right)$$

or $\boxed{P_{ion} = \frac{\rho kT}{\mu_{ion} m_p}}$

where the mean molecular weight per ion is $\boxed{\frac{1}{\mu_{ion}} = X_H + \frac{X_{He}}{4}}$

(for a general mixture of N ions, $\boxed{Y_{ion} = \frac{1}{\mu_{ion}} = \sum Y_i = \sum \frac{X_i}{A_i}}$)

The electrons also contribute to the pressure $P_e = n_e kT$ if they are in the ideal gas (non-degenerate) limit.

Write this as $P_e = n_e kT = kT \sum n_i Z_i$

charge of ion species i

$$= \frac{\rho kT}{m_p} \sum Y_i Z_i$$

or $\boxed{P_e = \frac{\rho kT}{m_p} \sum \frac{X_i Z_i}{A_i} = \frac{\rho kT}{\mu_e m_p} = \frac{\rho Y_e kT}{m_p}}$

For our H/He mixture, $\frac{1}{\mu_e} = X_H + \frac{X_{He}}{2}$.

The total pressure is $P = (n_e + n_{ion})kT$

$$= \frac{\rho kT}{m_p}\left(\frac{1}{\mu_{ion}} + \frac{1}{\mu_e}\right)$$

or $\boxed{P = \frac{\rho kT}{\mu m_p}}$

which defines the mean molecular weight $\boxed{\frac{1}{\mu} = \frac{1}{\mu_e} + \frac{1}{\mu_{ion}}}$

for the solar mixture, $\frac{1}{\mu} = 2X_H + \frac{3X_{He}}{4}$

$\Big($ or $\mu = \frac{4}{8X_H + 3X_{He}} \simeq 0.6$ similarly $\mu_e = \frac{2}{2X_H + X_{He}} \approx 1.2$

$\mu_{ion} = \frac{4}{4X_H + X_{He}} \simeq 1.3$ $\Big)$

Pure H has $\mu_e = \mu_i = 1$ $\mu = \frac{1}{2}$

Pure He has $\mu_e = 2$ $\mu_i = 4$ $\mu = 4/3$

(heavier elements also have $\mu_e \approx 2$ since $A \simeq 2Z$ for all nuclei except H)

<u>Regimes in the (ρ, T) plane</u>

Now we can look at different regimes in the (ρ, T) plane — when does radiation dominate the pressure, when do degenerate electrons dominate the pressure etc?

First of all, when do electrons become degenerate?

set $E_F = kT$ or $\frac{\hbar^2}{2m_e}(3\pi^2 n_e)^{2/3} = kT$

but $n_e = \frac{Y_e \rho}{m_p}$ $\Rightarrow$ $\underline{T = 3 \times 10^5 K \, (\rho Y_e)^{2/3}}$

(for ions to become degenerate need to lower the temperature further by a factor of $\sim \frac{m_p}{m_e} \simeq 2000$.)

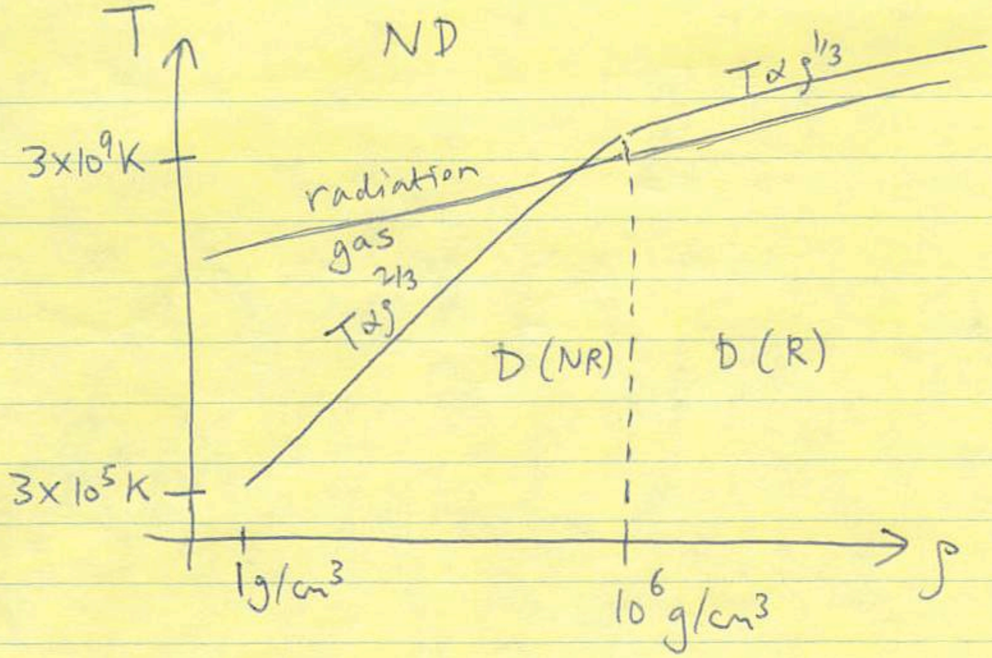
When do degenerate electrons become relativistic?

$$p_F = m_e c$$
$$\hbar (3\pi^2 n_e)^{1/3} = m_e c$$
$$\hbar \left(\frac{3\pi^2 Y_e \rho}{m_p}\right)^{1/3} = m_e c$$
$$\Rightarrow \quad \underline{(\rho Y_e) = 10^6 \, g/cm^3}$$

Radiation pressure compared to ideal gas pressure

$$a T^4 = \frac{\rho k T}{\mu m_p}$$

$$\Rightarrow \quad T = \left(\frac{\rho k_B}{\mu m_p a}\right)^{1/3} = 2 \times 10^7 K \left(\frac{\rho}{\mu}\right)^{1/3}$$



The mass-radius relation for white dwarfs

White dwarfs are stars held up by degenerate electron pressure. For low masses, the electrons are ~~the~~ non-relativistic giving $P \propto \rho^{5/3}$ whereas as the mass approaches the Chandrasekhar mass, the equation of state becomes closer to $\gamma = 4/3$.

We can use results for polytropes to help us. Integrating the equation of hydrostatic balance for $\gamma =$ constant (in which case the two equations $\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$ and $\frac{dm}{dr} = 4\pi r^2 \rho$ can be written as the so-called Lane-Emden equation) gives

$\gamma = \frac{5}{3}$ $(n = \frac{3}{2})$ $\quad \rho_c = 5.99 \langle \rho \rangle \quad P_c = 0.77 \frac{GM^2}{R^4}$

$\gamma = \frac{4}{3}$ $(n = 3)$ $\quad \rho_c = 54.18 \langle \rho \rangle \quad P_c = 11.05 \frac{GM^2}{R^4}$

we will write $\quad \rho_c = \beta \langle \rho \rangle \quad P_c = \alpha \frac{GM^2}{R^4}$

(a) low mass WDs the $\gamma = \frac{5}{3}$ solution is appropriate

at the center of the star $P_c = K_{nr} \rho_c^{5/3}$

or $$\alpha_{nr} \frac{GM^2}{R^4} = K_{nr} \left(\beta_{nr} \frac{M}{\frac{4\pi}{3} R^3} \right)^{5/3}$$

$$\Rightarrow \quad R = M^{-1/3} \left(\frac{K_{nr}}{\alpha_{nr} G} \right) \left(\frac{3\beta_{nr}}{4\pi} \right)^{5/3}$$

$$\Rightarrow \quad \boxed{R = 8.7 \times 10^8 \, \mathrm{cm} \left(\frac{M}{M_\odot} \right)^{-1/3} \left(\frac{Y_e}{0.5} \right)^{5/3}}$$

(b) Chandrasekhar mass $\gamma = 4/3$

$$P_c = K_r \rho_c^{4/3}$$

$$\alpha_r \frac{GM^2}{R^4} = K_r \left(\beta_r \frac{M}{\frac{4\pi}{3} R^3} \right)^{4/3}$$

$$\Rightarrow \quad M_{ch} = \left(\frac{K_r}{\alpha_r G} \right)^{3/2} \left(\frac{3\beta_r}{4\pi} \right)^2$$

$$\boxed{M_{ch} = 1.45 \, M_\odot \left(\frac{Y_e}{0.5} \right)^2}$$

This gives us the limiting behavior, but can we do better? We can use the fitting formula derived by Paczynski (1983)

$$\frac{1}{P_e^2} = \frac{1}{P_{e,nr}^2} + \frac{1}{P_{e,r}^2}$$

which is a good approximation (~ few %) for any $\frac{v}{c}$.

Then we write

$$\left(\frac{GM^2}{R^4}\right)^{-2} \approx \left(\frac{K_r M^{4/3}}{R^4}\right)^{-2} + \left(\frac{K_{nr} M^{5/3}}{R^5}\right)^{-2}$$

[For clarity I've dropped the α and β's and I've written $\langle\rho\rangle = M/R^3$ not $\langle\rho\rangle = \dfrac{M}{\frac{4\pi}{3}R^3}$ — could put these back in later, but for now focus on the scalings.]

$$\Rightarrow \quad \frac{\cancel{R^8}}{G^2 \cancel{M^4} M^{2/3}} = \frac{\cancel{R^8}}{K_r^2 \cancel{M^{8/3}} M^{-2/3}} + \frac{\cancel{R^{10}}\, R^2}{K_{nr}^2 \cancel{M^{10/3}}}$$

$$\Rightarrow \quad R^2 = K_{nr}^2 \cancel{M^{10/3}} \left[\frac{1}{G^2 M^{2/3}} - \frac{M^{2/3}}{K_r^2}\right]$$

$$= \frac{K_{nr}^2}{G^2 M^{2/3}} \left[1 - \frac{G^2 M^{4/3}}{K_r^2}\right]$$

$$\Rightarrow \quad R = \underbrace{\frac{K_{nr}}{G M^{1/3}}}_{\text{this is the } \gamma = 5/3 \text{ value}} \underbrace{\left[1 - \left(\frac{M}{M_{ch}}\right)^{4/3}\right]^{1/2}}_{\text{this correction term allows for the } \gamma \to 4/3 \text{ behavior}}$$

~~pattin~~

Hansen & Kawaler give a similar formula in their book [referenced as Eggleton 1982 priv comm but no derivation!]
but with the prefactor fitted to models

$$\boxed{R = 7.85 \times 10^8 \text{ cm} \left(\frac{M}{M_\odot}\right)^{-1/3} \left[1 - \left(\frac{M}{M_{ch}}\right)^{4/3}\right]^{1/2}}$$

"Testing the WD M-R relation with Hipparcos" 1998 ApJ

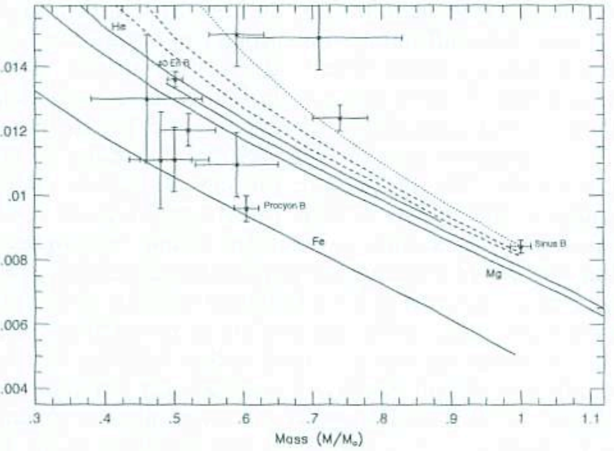


FIG. 2.—Observational support for the white dwarf mass-radius relation after *Hipparcos*, showing revised positions for the visual binaries and including results from the common proper-motion systems.

Figure 2 presents the revised visual binary positions, as well as the results for the CPM pairs. These objects test the mass-radius relation using the absolute minimum of physical assumptions. The physics underlying this figure is Kepler's third law, the gravitational redshift, and some general assumptions regarding the ability of model atmospheres to predict a value of the emergent flux H_λ. There are a considerably greater number of data points than presented in Figure 1, although many of the additions are somewhat uncertain. The important binaries, Sirius B, Procyon B, and 40 Eri B, are plotted with improved accuracy. Figure 3 repeats Figure 2, but also includes the field white dwarfs from Table 6. In addition to the physics underlying Figure 2, broadening theory must be included in the underlying assumptions for Figure 3.

Our first conclusion is that the mass-radius relation is now more firmly supported on observational grounds. For readers who like high-precision data points, Sirius B and 40 Eri B fit the theoretical relation quite precisely. For readers who enjoy an abundance of data points, Figure 3 more than quadruples the number of observed points, the majority of which lie between 1 and 2 σ from the Wood models. We discuss the discrepant points below.

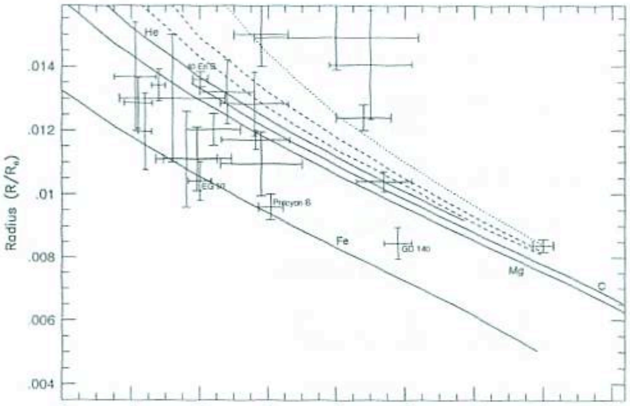


FIG. 3.—Observational support for the white dwarf mass-radius relation, showing the positions of the visual binaries, common proper-motion systems, and field white dwarfs. The field white dwarf masses were derived using published surface gravity measurements and radii based on *Hipparcos* parallaxes.

4.1. *Tentative Suggestions of Iron-rich Cores*

Procyon B, EG 50, and GD 140 (labeled in Fig. 3) all lie significantly below the mass-radius relation for the expected carbon interior composition of white dwarfs. While the plot on the mass-radius relation may disguise the robust character of our result, a close look at Table 7 and Figures 3 and 4 shows the source of our suggestion that at least two of these three stars have iron-rich cores.

The masses predicted by the zero-temperature carbon-core mass-radius relation for GD 140 and EG 50, using the radii from Table 6, are considerable larger than the masses we observe, with 4 and 7 σ deviations (Fig. 3). GD 140 is a well-studied white dwarf (BSL) with ample spectroscopic evidence suggesting that it is massive. EG 50 is a more mysterious case. While at a similar temperature to GD 140, a comparison of the optical spectra presented in BSL shows that GD 140's Balmer lines are wider and shallower than EG 50's, arguing that GD 140 is more massive. Our radii from Table 6, combined with published values of log g, result in masses of 0.50 ± 0.02 $M_\odot$ for EG 50, and 0.79 ± 0.02 $M_\odot$ for GD 140, further supporting this comparison. BSL finds higher spectroscopic masses, assuming a carbon core and log He $= -4$ mass-radius relation, of 0.66 and 0.90 ± 0.03 $M_\odot$ for EG 50 and GD 140, respectively. Our radii, combined with this same mass-radius relation, imply even higher masses of ≈ 0.8 $M_\odot$ (EG 50) and ≈ 0.95 $M_\odot$ (GD 140) (Fig. 4).

In essence, both EG 50 and GD 140 have radii that are significantly smaller than predicted by their observed masses, assuming the carbon-core mass-radius relation. The only way we can see of explaining the observations is by assume an iron, or an iron-rich, core composition. It is then possible to fit the observed radii, masses, and surface gravities consistently. It is conceivable that GD 140 harbors a core heavier than carbon. If, however, EG 50 is really a garden variety white dwarf with an average mass, we find it difficult to explain an iron core with current theories of white dwarf formation.

We discuss the problematic situation of Procyon B separately (Provencal et al. 1997). Even though our discussion does not incorporate the *Hipparcos* parallax, we

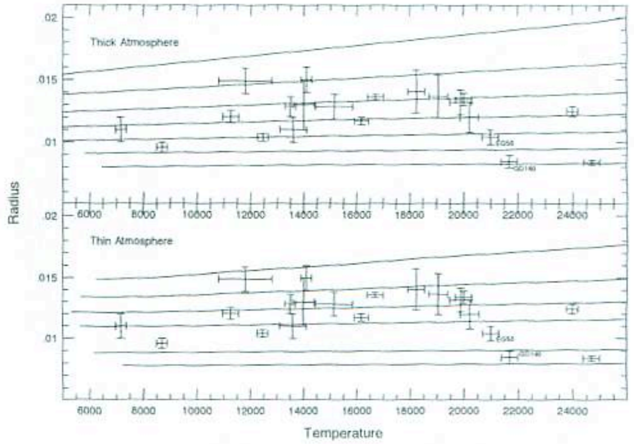


FIG. 4.—Predicted masses for our white dwarf sample based on the model used. The top panel uses models with thick [log q(H) $= -4$] surface layers, and the second has log q(H) $= 0$. The solid lines are white dwarf cooling curves at constant mass, beginning at 0.4 $M_\odot$ and increasing by 10ths sequentially downward. The error bars mark the 1 σ error positions for our observed points.

Neutron stars

We saw that
the radius of a $\gamma = 5/3$ star is $R \propto M^{-1/3} \dfrac{K_{nr}}{G}$

The constant K_{nr} is $K_{nr} = \dfrac{P}{\rho^{5/3}} = \dfrac{2}{5} \dfrac{n E_F}{\rho^{5/3}}$

$$= \frac{2}{5} \left(\frac{n}{\rho}\right)^{5/3} \frac{\hbar}{2m} (3\pi^2)^{2/3}$$

$\Rightarrow$ K_{nr} is $\propto \dfrac{1}{m}$

where m is the mass of the degenerate particle.

A neutron star is held up by degeneracy pressure of protons and neutrons – therefore we expect

$$R_{N.S} \approx R_{WD} \left(\frac{m_e}{m_p}\right) \approx \frac{R_{WD}}{2000} \approx \frac{10^9 \text{cm}}{2000}$$
$$= 5\,\text{km}.$$

which is about right – detailed neutron star models give radii of around 10-15 km.

As I mentioned earlier, the equation of state in neutron stars is affected by interactions between the particles which make the EOS stiffer. Roughly, $P \propto \rho^2$
which gives radius almost independent of mass.

Lattimer & Prakash 2001 ApJ 550 426

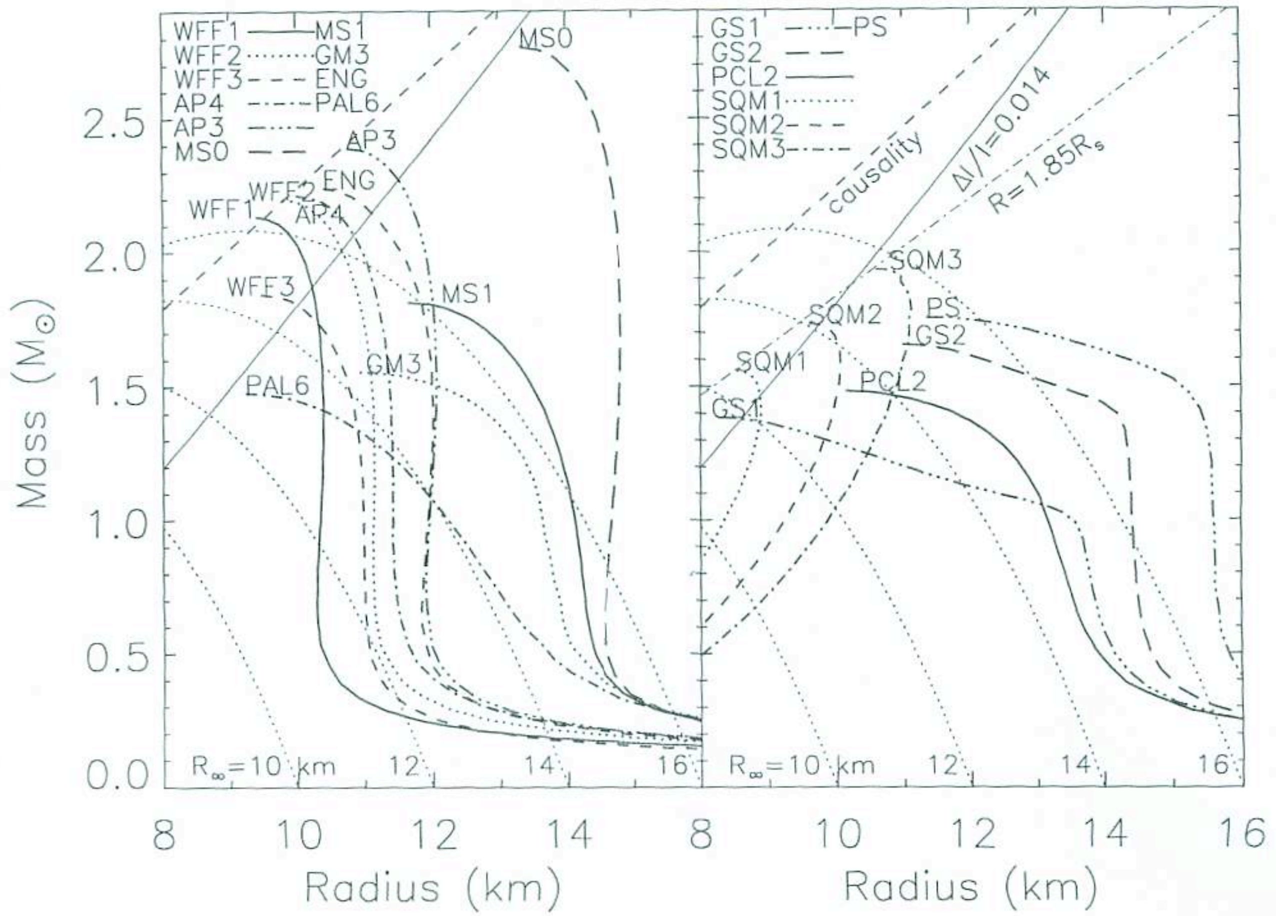


FIG. 2.—Mass-radius curves for several EOSs listed in Table 1. The left-hand panel is for stars containing nucleons and, in some cases, hyperons. The right-hand panel is for stars containing more exotic components, such as mixed phases with kaon condensates or strange quark matter, or pure strange quark matter stars. In both panels, the lower limit causality places on R is shown as a dashed line, a constraint derived from glitches in the Vela pulsar is shown as the solid line labeled $\Delta I/I = 0.014$, and contours of constant $R_\infty = R/(1-2GM/Rc^2)^{1/2}$ are shown as dotted curves. In the right-hand panel, the theoretical trajectory of maximum masses and radii for pure strange quark matter stars is marked by the dot-dashed curve labeled $R = 1.85R_s$.

theoretical perspective, it appears that values of R_∞ in the range of 12–20 km are possible for normal neutron stars whose masses are greater than 1 $M_\odot$.

Corresponding to the two general types of EOSs, there are two general classes of neutron stars. *Normal* neutron stars are configurations with zero density at the stellar surface and have minimum masses, of about 0.1 $M_\odot$, that are primarily determined by the EOS below n_s. At the minimum mass, the radii are generally in excess of 100 km. The second class of stars are the so-called *self-bound* stars, which have finite density, but zero pressure, at their surfaces. They are represented in Figure 2 by strange quark matter stars (SQM1–3).

Self-bound stars have no minimum mass, unlike the case of normal neutron stars for which pure neutron matter is unbound. Unlike normal neutron stars, the maximum mass self-bound stars have nearly the largest radii possible for a given EOS. If the strange quark mass $m_s = 0$ and interactions are neglected ($\alpha_c = 0$), the maximum mass is related to the bag constant B in the MIT-type bag model by $M_{\max} = 2.033(56 \text{ MeV fm}^{-3}/B)^{1/2}$ $M_\odot$. Prakash et al. (1990) and Lattimer et al. (1990) showed that the addition of a finite strange quark mass and/or interactions produces larger maximum masses. The constraint that $M_{\max} > 1.44$ $M_\odot$ is thus automatically satisfied for all cases by the condition that the energy ceiling is 939 MeV. In addition, models satisfying the energy ceiling constraint, with any values of m_s and α_c, have larger radii for every mass than the case SQM1. For the MIT model, the locus of maximum masses of self-bound stars is given simply by $R \cong 1.85R_S$ (Lattimer et al. 1990), where $R_S = 2GM/Rc^2$ is the Schwarzschild radius, which is shown in the right-hand panel of Figure 2. Strange quark stars with electrostatically supported normal-matter crusts (Glendenning & Weber 1992) have larger radii than those with bare surfaces. Coupled with the additional constraint $M > 1$ $M_\odot$ from proto–neutron star models, MIT-model strange quark stars cannot have $R < 8.5$ km or $R_\infty < 10.5$ km. These values are comparable to the possible lower limits for a Bose (pion or kaon) condensate EOS.

Although the M-R trajectories for normal stars can be strikingly different, in the mass range from 1 to 1.5 $M_\odot$ or more, it is usually the case that the radius has relatively little dependence upon the stellar mass. The major exceptions illustrated are the model GS1, in which a mixed phase containing a kaon condensate appears at a relatively low density, and the model PAL6, which has an extremely small nuclear incompressibility (120 MeV). Both of these have considerable softening and a large increase in central density for $M > 1$ $M_\odot$. Pronounced softening, while not as

Sep 25, 2007.

PHYS 643 lecture 7

First, finish off the mass-radius relations for white dwarfs and neutron stars from last time.

Now, ask what happens as we go to lower masses? The white dwarf M-R relation predicts about the right radius for Jupiter

$$R = 8.7\times 10^{8}\ \mathrm{cm}\left(\frac{M}{M_\odot}\right)^{-1/3}$$

$$\Rightarrow \text{ for } M = M_J \simeq \frac{1}{1000} M_\odot \qquad R = \frac{1}{10} R_\odot$$

which is about right since $R_J = 7\times 10^{9}$ cm.

But clearly at lower masses, the M-R relation of cold objects must turnover and make smaller and smaller radii objects

e.g. { the radius of the Earth $R_\oplus = 6.4\times 10^{8}$ cm is 10 times smaller than Jupiter &.

For a rocky object we'd guess $M \propto R^3$ (roughly constant density)

Scaling from Earth to Jupiter, we'd guess

$$R_J \simeq R_\oplus\left(\frac{M_J}{M_\oplus}\right)^{1/3} \simeq 6.8 R_\oplus = 4.4\times 10^{9}\ \mathrm{cm}$$

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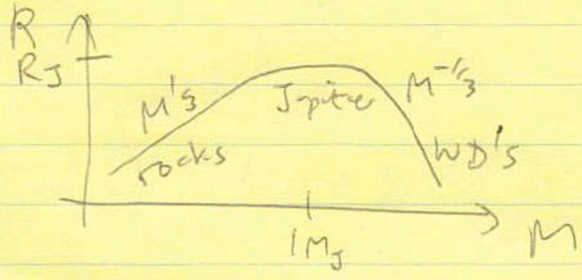
almost right!

This suggests that the M-R relation must look like



Indeed this is the case, and we can understand it by looking at the interactions between the electrons and ions

Coulomb pressure in a degenerate gas

The electrons and ions interact with each other through Coulomb forces.

To calculate the size of this effect, first note that it's a good approximation to assume the electrons are uniformly distributed in space, since you can show that

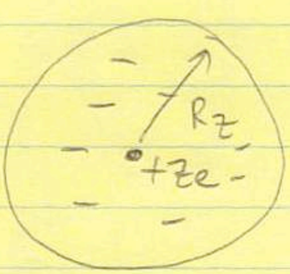
$$E_F \gg \frac{Ze^2}{\langle r \rangle}$$

The electrons in a degenerate gas are perturbed only mildly by the ions.

Then we use the Wigner-Seitz approximation – we divide the plasma into cells containing one ion and the nearest Z electrons. Each cell is electrically neutral, so we can calculate the energy of each separately.

To make the calculation simple, assume the cells are spherical.

We have therefore a sphere of radius R_Z



where $\frac{4\pi}{3} R_Z^3 \, n_e = Z$

[NB same as interion spacing if $n_i = n_e/Z$ $\quad \frac{4\pi}{3} a^3 n_i = 1$]

Two contributions to the energy:

electron-electron $\quad U_{ee} = \frac{3}{5} \frac{(Ze)^2}{R_Z}$ ← (positive – it takes work to assemble the electron sphere)

electron-ion $\quad U_{eZ} = -\frac{3}{2} \frac{(Ze)^2}{R_Z}$

(negative since the nucleus attracts the surrounding electrons) the total energy is $\quad U_{ee} + U_{eZ} = -\frac{9}{10}\frac{(Ze)^2}{R_Z}$

Per unit volume (divide by Z to get the number per electron then multiply by n_e)

$$\boxed{U_{\text{Coulomb}} = -n_e \frac{9}{10}\frac{Ze^2}{R_Z} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3}}$$

Notice that U_{Coulomb} becomes more negative as density increases $\Rightarrow$ there must be a <u>negative pressure</u>!

$$\cancel{P = -\frac{\partial U}{\partial V}}$$

To calculate the pressure write the Coulomb energy per gram

$$U_C = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3}\frac{Z^{2/3} e^2 n_e^{4/3}}{\rho} \propto \rho^{1/3}$$

then

$$P_C = \rho^2 \frac{\partial U_c}{\partial \rho} = \rho U_c \frac{\partial \ln U_c}{\partial \ln \rho} = \frac{1}{3}\rho U_c$$

$$\Rightarrow \quad P_c = -\frac{1}{3}\,\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 \left(\frac{\rho Y_e}{m_p}\right)^{4/3}$$

$$\boxed{P_c = -2.2 \times 10^{12}\ \text{erg/cm}^3\ \rho^{4/3}\left(\frac{Y_e}{0.5}\right)^{4/3} Z^{2/3}}$$

Zero pressure solid

Now consider the total pressure (Assume ion pressure negligible; NR electrons)

$$P_{tot} = K_e \rho^{5/3} - K_c \rho^{4/3}$$

There is a zero-pressure solution
$$\rho = \left(\frac{K_c}{K_e}\right)^3$$
$$= 0.4\,g/cm^3\ Z^2 \left(\frac{Y_e}{0.5}\right)^{-1}$$
$$= 0.2\,g/cm^3\ ZA$$

eg. terrestrial metal iron $A=56$
$Z = 1$ or 2 (?)
$$\rho = (10.6\ g/cm^3)\,Z$$

Central pressure of Jupiter

Jupiter has
$$M \approx \frac{1}{1000} M_\odot \approx 300\, M_\oplus$$
$$R \approx 0.1\, R_\odot \simeq 10\, R_\oplus$$
$\Rightarrow$ it's mean density is about the same as the sun $\langle\rho\rangle \sim 1\, g\, cm^{-3}$

$\gamma = 5/3$ polytrope
$$P_c = 0.77\frac{GM^2}{R^4}$$
$$= 8.6\times10^{13}\ cgs.$$

(also just for information $T_c \simeq 20{,}000K$)

$$\left(\sim 10^{14}\ cgs = 10^{13}\ Pa = 10^4\ GPa = 100\ Mbar = 100\ Matm\right)$$

$\rho_c = 6\langle\rho\rangle \Rightarrow P_c = 2.2\times10^{12}\times 6^{4/3} = 2.4\times10^{13}$ cgs

Significant!

Mass-radius relation (for NR electrons)

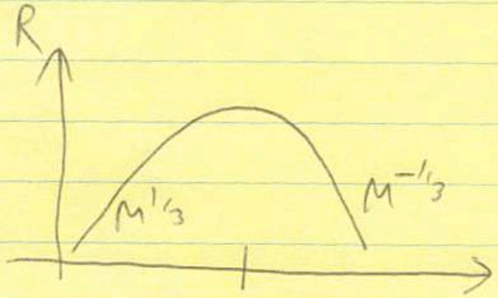
Simple model $\quad P_e = K_e \rho^{5/3} \qquad P_C = -K_C \rho^{4/3}$

$$\frac{GM^2}{R^4} = K_e \left(\frac{M}{R^3}\right)^{5/3} - K_C \left(\frac{M}{R^3}\right)^{4/3}$$

$$\Rightarrow \quad R = K_e M^{5/3} \left(GM^2 + K_C M^{4/3} \right)^{-1}$$

$$\boxed{R = \frac{K_e}{GM^{1/3} + K_C M^{-1/3}}}$$

two limits $\quad R = \begin{cases} \dfrac{K_e}{G} \dfrac{1}{M^{1/3}} & \text{"WD"} \\ \left(\dfrac{K_e}{K_C}\right) M^{1/3} & \text{"rock"} \end{cases}$



$$\left(\langle \rho \rangle \propto \begin{cases} M^2 & \text{WD} \\ \text{const.} & \text{rock} \end{cases} \right)$$

the maximum is where $\quad M = \left(\dfrac{K_C}{G}\right)^{3/2} \simeq 0.1\, M_J$

See Fortney et al (2007) for recent calculations of M-R for rocky and gaseous planets.

Fortney et al. 2007 ApJ, 659, 1661

(Guillot 2005), and we find that in our cooling calculations that the model planets reach a T_{int} of $\sim$102–110 K at 4.5 Gyr, which is only weakly dependent on stellar irradiation.

A deep external radiative zone is found in the most highly irradiated models. For the planets at $\lesssim$0.05 AU convection does not begin until $P > 1$ kbar. From 0.1 to 2 AU the deep internal adiabat for all models begins at 300+ bar, but there is a second, detached convective zone at pressures close to 1 bar. This detached convective zone grows at stellar distance increases, and by 3 AU the convective zones have merged. Only when these convective zones merge is the interior adiabat cooler as a function of orbital distance. The models from 0.1 to 2 AU have essentially the same internal adiabat, meaning the planets would have the same radius at a given mass. As we will see, a striking consequence of this effect is that stellar irradiation at 2 AU has approximately the same effect on retarding cooling and contraction as at 0.1 AU, even though the incident fluxes vary by a factor of 400!

5. RESULTS: ICE-ROCK-IRON PLANETS

5.1. *Planetary Radii*

It seems likely that planets with masses within an order of magnitude of the Earth's mass will be composed primarily of more refractory species, like the planetary ices, rocks, and iron. Within our solar system, objects of similar radius can differ by over a factor of 3 in mass, due to compositional differences. A planet with the radius of Mercury, which is potentially detectable with *Kepler*, could indicate a mass of 0.055 $M_\oplus$, like Mercury itself, or a mass of 1/3 this value, like Callisto, which has a radius that differs by only 30 km. With our equations of state, we are able to explore the radii of objects with any possible combination of ice, rock, and iron. In order to keep this task manageable, we have limited our calculations to several illustrative compositions. These include pure ice and ice/rock mixtures, which could be described as "water worlds" or "Ocean planets." Such objects in our solar system, like the icy satellites of the outer planets, generally have small masses. However, Kuchner (2003) and Léger et al. (2004) have pointed out water-rich objects could reach many Earth masses (perhaps as failed giant planet cores) and migrate inward to smaller orbital distances. We also consider planets composed of pure rock, rock and iron mixtures, and pure iron, more similar to our own terrestrial planets. The ice/rock and rock/iron mixtures are computed for 75/25, 50/50, and 25/75 percentages by mass, with ice always overlaying rock, and rock always overlaying iron.

Our results are shown in Figure 4. Since we make few assumptions regarding what is a reasonable planet, we have computed radii from masses of 0.01 to 1000 $M_\oplus$. For all compositions, the radii initially grow as $M^{1/3}$, but at larger masses, compression effects become important. As a greater fraction of the electrons become pressure ionized, the materials begin to behave more like a Fermi gas, and there is a flattening of the mass-radius curves near 1000 $M_\oplus$. Eventually the radii shrink as mass increases, with radii falling with $M^{-1/3}$ (see Zapolsky & Salpeter 1969).

At the top left of Figure 4 we also show the size of various levels of uncertainty in planetary mass, as a percentages of a given mass, from 10% to 200%. For instance, if one could determine the mass of a 1 $M_\oplus$ planet to within 50%, even a radius determination accurate to within 0.25 $R_\oplus$ would lead to considerable ambiguity concerning composition, ranging from 50/50 ice/rock to pure iron. The shallow slope of the mass-radius curves below a few $M_\oplus$ makes accurate mass determinations especially important for understanding composition. In Table 1 we give the mass and radius for a subset of these planets. We note that from 1 to 10 $M_\oplus$ we find

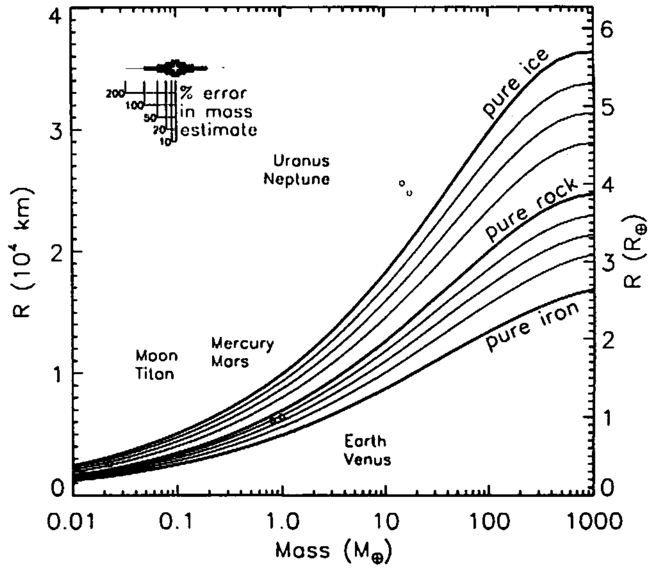


FIG. 4.—Mass (in $M_\oplus$) vs. radius (in km and $R_\oplus$) for planets composed for ice, rock, and iron. The topmost thick black curve is for pure "warm" water ice. (See text.) The middle thick curve is for pure rock (Mg_2SiO_4). The bottommost thick curve is for pure iron (Fe). The three black thin curves between pure ice and pure rock, are from top to bottom, 75% ice/25% rock, 50/50, and 25/75. The inner layer is rock and the outer layer is ice. The gray dotted lines between rock and pure warm ice are the same pure ice and ice/rock curves, but for zero-temperature ice. The three black thin curves between pure rock and iron, are from top to bottom, 75% rock/25% iron, 50/50, and 25/75. The inner layer is iron and the outer layer is rock. Solar system objects are open circles. At the upper left we show the horizontal extent of mass error bars, for any given mass, from 10% to 200%.

excellent agreement between our models and the more detailed "Super-Earth" models of Valencia et al. (2006).

5.2. *Validation in the Solar System*

On Figure 4 we have also plotted, in open circles, the masses and radii of solar system planets and moons. These planets can be used to validate our methods. For instance, detailed models of the Earth's interior indicate that the Earth is approximately 33% iron by mass with a core-mantle boundary at 3480 km (Dziewonski & Anderson 1981). This composition is readily recovered from Figure 4, where Earth plots between the 25% and 50% iron curves, but closer to 25%. Our simple Earth model, with a iron/rock boundary at 3480 km yields a planetary radius within 100 km (1.5% smaller) of the actual Earth. Given that our model lacks thermal corrections to EOSs that are found in detailed Earth models, and that we ignore lower density species such as sulfur that are likely mixed with iron into the Earth's core, we regard this agreement as excellent, and entirely sufficient with regard to the expected radii uncertainties as measured by transit surveys.

Elsewhere in the solar system, one can see that we recover ice/rock or rock/iron ratios of other bodies, which are derived by more complex models. A brief overview of the structure of the terrestrial planets and icy moons is given in de Pater & Lissauer (2001). Earth's Moon is composed almost entirely of rock, with a very small iron core of radius $\lesssim$400 km. Here, the Moon (the leftmost circle) plots on top of the line for pure rock. Mercury is calculated to be $\sim$60% iron by mass, and with our models Mercury falls between the 50/50 (rock/iron) and 25/75 curves, but again, closer to 50/50, which shows excellent agreement. Titan is calculated to be composed of $\sim$35% ices, and again we find excellent agreement, as Titan falls between the 50/50 (ice/rock) and 25/75 curves, slightly

Sep 27, 2007

$\boxed{\text{PHYS 643 lecture 8}}$

We now turn to "<u>hot" objects</u>, by which we mean objects in which the pressure is provided by particles whose energy is determined by (kT) rather than E_F.

eg. main sequence stars $\quad P_c \simeq \dfrac{GM^2}{R^4} \simeq \dfrac{\rho_c k T_c}{\mu m_p} \simeq \dfrac{M}{R^3}\dfrac{kT_c}{\mu m_p}$

$$\Rightarrow \boxed{k_B T_c \simeq \frac{GM\mu m_p}{R}}$$

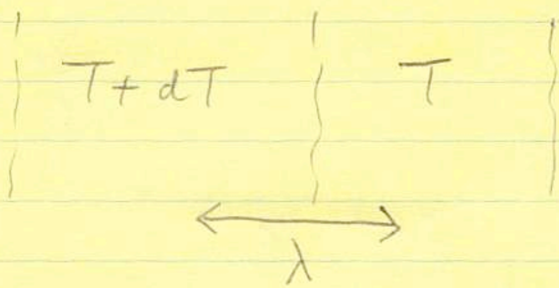
$$\left.\begin{aligned} M &= M_\odot = 2\times 10^{33}\,\mathrm{g} \\ R &= R_\odot = 7\times 10^{10}\,\mathrm{cm} \end{aligned}\right\} \quad T_c = 2\times 10^7\,\mathrm{K}$$

Given the T_c and R, we can calculate the luminosity. The heat flux is

$$F = -K\frac{dT}{dr}$$

erg/cm^2/s $\qquad$ thermal conductivity

Photons transport the heat



kinetic theory type argument gives $\quad F \simeq \frac{1}{3}c\,\Delta(aT^4)$

or $$F \simeq \frac{1}{3} c \lambda \frac{d}{dr}(aT^4)$$

Recall that the mfp is $\frac{1}{n\sigma} = \lambda$

we define the radiative opacity κ such that $\lambda = \frac{1}{n\sigma} = \frac{1}{\rho\kappa}$

(so that κ is the "cross-section per gram")

$$\Rightarrow \quad F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

or the luminosity is $$\boxed{L = -\frac{16\pi r^2 acT^3}{3\kappa\rho}\frac{dT}{dr}}$$

To get the scalings, we can write $\frac{dT}{dr} \approx \frac{T_c}{R} \simeq \frac{\mu m_p GM}{k_B R^2}$

and $\rho \simeq \frac{M}{R^3}$

$$\Rightarrow \quad L \propto \frac{R^2T^3}{\kappa\rho}\frac{dT}{dr} \propto \frac{R^2}{\kappa}\left(\frac{R^3}{M}\right)\left(\frac{M}{R}\right)^3\frac{M}{R^2}$$

$$\propto \frac{M^3}{\kappa}$$

For constant opacity, $\boxed{L \propto M^3}$

e.g. for main sequence stars with $M \gtrsim M_\odot$ the radiative opacity is set by electron scattering (Thompson cross-section $\sigma_T = 6.65 \times 10^{-25}\ \mathrm{cm}^2$), giving $\kappa = 0.40\ \frac{\mathrm{cm}^2}{\mathrm{g}}\ Y_e$

= constant.

so the mass of the star determines its luminosity!
What about the radius? The radius adjusts so that the central temperature ($T_c \propto M/R$) is at the right value for nuclear burning to provide the required luminosity.

Nuclear reaction rates are v. temperature sensitive (why?)
$\Rightarrow$ $T_c \simeq$ constant with mass for $M \gtrsim M_\odot$

$\Rightarrow$ $\boxed{R \propto M}$ is expected. ~~Roughly true~~ (bit shallower).
In fact $R \propto M^{1/2}$ is closer (because T_c increases with M)

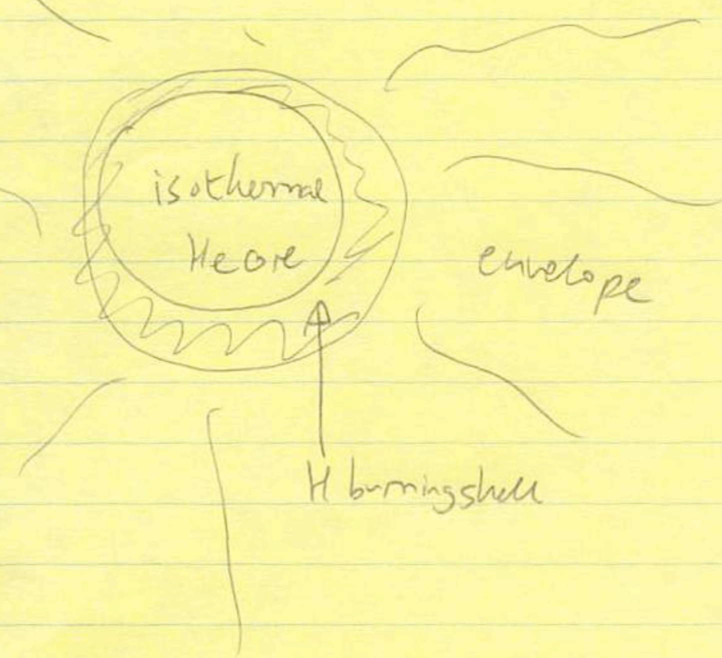
For $M < M_\odot$, free-free opacity dominates $\kappa \propto \rho T^{-7/2}$
which leads to a steeper dependence ~~$L \propto M^{5.5}$~~
$$L \propto M^{5.5}$$

Also, the heat transport is partially by convection - more on that later.

[If you want to play around with stellar models, look at the ~~Paxton's~~ EZ stellar evolution code by Bill Paxton,
See http://theory.kitp.ucsb.edu/~paxton/EZ-intro.html]

A core can support only a finite-size envelope

A common situation is to have a core + envelope structure. For example, at the end of a star's main sequence lifetime, hydrogen is exhausted in the core. The star evolves to a state in which hydrogen burns in a shell surrounding an isothermal helium core.



If the core is non-degenerate, there is a maximum ~~mass envelope~~ ~~as~~ core mass beyond which the envelope cannot be supported.

To see this, calculate the pressure at the edge of the core ($r=R_c$)

$$P_c = P(r=R_c) \lesssim \frac{kT_c}{\mu_c m_p}\frac{M_c}{\frac{4\pi}{3}R_c^3} - \frac{1}{4\pi}\frac{GM_c^2}{R_c^4} \qquad (*)$$

μ_c = core mean molecular weight = $4/3$ for pure He.

~~As~~ There is a maximum value of P_c which occurs for

$$\boxed{R_{c,max} = \frac{4}{9}\frac{GM_c\,\mu_c m_p}{k_B T_c}}$$

and is $$\boxed{P_{c,max} = 0.68\left(\frac{k_B T_c}{\mu m_p}\right)^4 \frac{1}{G^3 M_c^2}}$$

The idea is that a given mass core can supply a surface pressure no larger than $P_{c,max}$. If the pressure due to the overlying layers is larger than this value then there is no hydrostatic solution $\Rightarrow$ core collapse!

The pressure at the base of the envelope will be

$$P_{base} \simeq \frac{GM^2}{R^4} \qquad (M \gg M_c)$$

the temperature must be the same as the core temperature

$$T_{base} = T_c = \cancel{\frac{GM_c \mu_c m_p}{k_B R_c}} = \frac{GM m_p \mu_{env}}{k_B R}$$

$$\Rightarrow \quad R \simeq \frac{GM m_p \mu_{env}}{k_B T_c}$$

$$\text{or} \quad P_{base} = \frac{GM^2 (k_B T_c)^4}{(GM)^4 (\mu_{env} m_p)^4} = \frac{(k_B T_c)^4}{G^3 M^2 (\mu_{env} m_p)^4}$$

there is a stable solution if

$$P_{base} < P_{c,max}$$

$$\text{or} \quad \left(\frac{kT_c}{\mu_{env} m_p}\right)^4 \frac{1}{G^3 M^2} < 0.68 \left(\frac{k_B T_c}{\mu_c m_p}\right)^4 \frac{1}{G^3 M_c^2}$$

$$\Rightarrow \quad \frac{M_c}{M} < 0.17 \qquad \left(\begin{array}{l}\mu_{env} = 0.6 \\ \mu_c = 4/3\end{array}\right)$$

in numerical models the critical ratio is $\simeq 10\%$.
As the H shell burns, M_c increases until it reaches the Schönberg-Chandrasekhar limit then the core must collapse!

Another example is runaway accretion of gas onto a rocky core to make a gas giant planet (eg. Pollack & Bodenheimer 1986). When the rocky core (which grows by accreting planetesimals) reaches $\simeq 10 M_\oplus$ there is no longer a hydrostatic solution for the envelope and runaway accretion occurs. A simple model is presented by Stevenson (1982) that nicely illustrates the physics.

The Schönberg-Chandrasekhar limit does not apply for degenerate cores - in that case the first term in equation (*) is $\propto \frac{1}{R^5}$ rather than $\frac{1}{R^3}$, so there is always a solution for small enough R_c one can match the external pressure.

Evolution of stars in the (ρ_c, T_c) plane

Overall the star contracts and heats — stopping at nuclear burning stages — until it becomes degenerate.

Evolution of stars in the (ρ, T) plane 7.

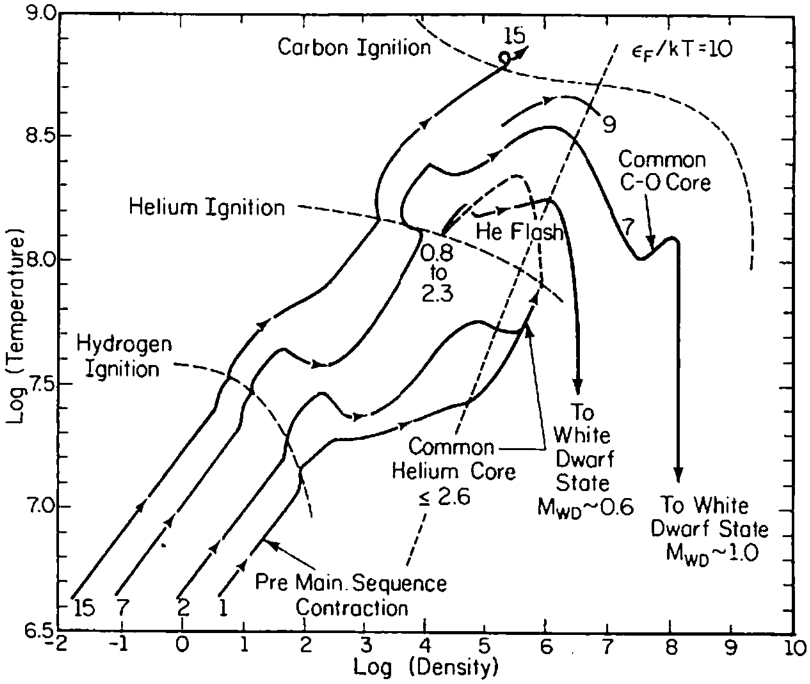


FIG. 2. Tracks in the density–temperature plane followed by matter at the centres of single stars of masses 1, 2, 7 and 15 $M_\odot$. Density is in g cm^{-3} and temperature is in degrees Kelvin. Also shown are the loci of points where hydrogen, helium and carbon are ignited and the locus of points where the electron Fermi energy equals $10 \times kT$. Note that stars of mass larger than those under primary consideration in this paper ignite carbon at the centre before electrons there become degenerate, whereas intermediate mass stars (1–9 $M_\odot$) develop an electron-degenerate core before the carbon ignition temperature is reached. In such stars, the core first cools rapidly and then heats as more mass is added to the core in consequence of hydrogen and helium burning in shells above the core. In stars less massive than about 2·3 $M_\odot$, matter in the hydrogen-exhausted core becomes electron-degenerate before helium is ignited. Such stars experience a helium core flash which lifts the degeneracy and helium thereafter burns quiescently at higher temperatures and lower densities than at ignition. After developing a carbon-oxygen core and while burning hydrogen and helium alternately in shells, all intermediate mass stars lose matter from their surfaces via a strong wind which may ultimately grow into a planetary nebula ejection event which abstracts most of the hydrogen-rich matter above the CO core. The remnant contracts to become the central star of a planetary nebula and then contracts further to become a white dwarf. Adapted from Iben, I. (Jr), 1974. *Ann. Rev. Astr. Astrophys.*, **12**, 215.

examples of the evolution of central density and temperature are shown in Fig. 2 for intermediate mass stars of mass below and above the critical limit for experiencing a helium core flash. Also shown is the track followed by matter at the centre of a model star that eventually develops a core composed of iron-peak elements before undergoing core collapse due to photodisintegration of these elements, exploding as a type II supernova, and leaving a neutron star or black hole remnant.

2.2 *The core helium-burning phase, the Cepheid phenomenon, and comparisons with the observations*

The rate at which helium burns and, therefore, the lifetime of the core helium-burning phase are determined by the mass of the helium core at the moment of helium ignition. Since core mass is nearly the same for all stars

From Iben (1985) "The Life and Times of an Intermediate Mass Star"

Oct 2, 2007

PHYS 643 lecture 9

III: Compressible Fluids

Sound waves

Pressure disturbances propagate at the sound speed.

Consider a perturbation to a background state. As before, we can take an Eulerian or Lagrangian approach.
eg. density perturbation

$$\delta\rho(\underline{r},t) = \rho(\underline{r},t) - \rho_0(\underline{r},t) \qquad \text{Eulerian}$$

$$\Delta\rho(\underline{r},t) = \rho(\underline{r}+\underline{\xi},t) - \rho_0(\underline{r},t)$$

$$\text{or} \quad \Delta\rho = \delta\rho + \underline{\xi}\cdot\underline{\nabla}\rho_0 \qquad \text{Lagrangian}$$

Start by considering a uniform fluid, $\underline{\nabla}\rho_0 = 0$, which means that $\delta\rho = \Delta\rho$. (also at rest $\underline{u}_0 = 0$)

The continuity equation is to first order

$$\frac{\partial\,\delta\rho}{\partial t} + \rho_0\,\underline{\nabla}\cdot\delta\underline{u} = 0$$

The momentum equation is $\dfrac{\partial}{\partial t}\delta\underline{u} = -\dfrac{\underline{\nabla}\,\delta p}{\rho_0}$

To solve these, we need to connect δP with $\delta\rho$.
If the perturbations occur quickly enough that there is no time for heat transfer, then they are adiabatic (constant entropy)
$P \propto \rho^\gamma$

$$\Rightarrow \quad \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}$$

We can then derive a single equation for $\delta \underline{u}$:

$$\frac{\partial^2 \delta \underline{u}}{\partial t^2} = -\frac{\gamma p_0}{\rho_0^2} \underline{\nabla}\left(\frac{\partial \delta \rho}{\partial t}\right) = +\frac{\gamma p_0}{\rho_0^2} \underline{\nabla}\left(\rho_0 \underline{\nabla} \cdot \delta \underline{u}\right)$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \delta \underline{u}}{\partial t^2} = \frac{\gamma p_0}{\rho_0} \nabla^2 \delta \underline{u} = c_s^2 \nabla^2 \delta \underline{u}} \quad (*)$$

a wave equation. $\qquad c_s^2 = \frac{\gamma p_0}{\rho_0}$

c_s is the adiabatic sound speed

In general, c_s^2 is $\left(\frac{\partial P}{\partial \rho}\right)$ where the derivative is taken under whatever conditions are appropriate for the perturbations, eg. isothermal $\Rightarrow c_s^2 = \frac{P}{\rho}$, the isothermal sound speed.

It is often useful to look for perturbations $\propto e^{i\underline{k}.\underline{r}} e^{i\omega t}$ (ie. Fourier modes)

Writing $\delta \underline{u} = \delta \underline{u}\, e^{i\underline{k}.\underline{r}} e^{i\omega t}$, equation (*) then

$$\Rightarrow \quad -\omega^2 \delta \underline{u} = -c_s^2 k^2 \delta \underline{u}$$

$\Rightarrow$ the dispersion relation for the waves is $\boxed{\omega^2 = c_s^2 k^2}$

The phase and group velocities for the waves are $\frac{\omega}{k}$ and $\frac{\partial \omega}{\partial k}$ as usual.

Waves in a magnetized fluid

Now we add a uniform magnetic field – how does this change the wave properties?

Note: From now on we'll drop the "0" subscript for the background quantities – ie. ρ instead of ρ_0.

The continuity equation doesn't have any terms containing $\underline{B}$ so as before

$$\boxed{i\omega\,\delta\rho + \rho\, i\underline{k}\cdot\delta\underline{u} = 0} \quad \text{—(1)}$$

The momentum equation is

$$i\omega\rho\,\delta\underline{u} = -i\underline{k}\,\delta p + \delta\left(\frac{\underline{J}\times\underline{B}}{c}\right)$$

There is no background current $\underline{J}=0$ so the last term is

$$\frac{\delta\underline{J}\times\cancel{\delta}\underline{B}}{c} = \frac{(\underline{\nabla}\times\delta\underline{B})\times\underline{B}}{4\pi\,\cancel{c}} \qquad \left[\delta\underline{J} = \frac{c}{4\pi}\underline{\nabla}\times\delta\underline{B}\right]$$

$$= \frac{(i\underline{k}\times\delta\underline{B})\times\underline{B}}{\cancel{c}\,4\pi}$$

$$\Rightarrow \quad \boxed{\omega\rho\,\delta\underline{u} = -\underline{k}\,\delta p - \frac{\underline{B}\times(\underline{k}\times\delta\underline{B})}{4\pi}} \quad \text{—(2)}$$

We also need the induction equation

$$i\omega\,\delta\underline{B} = \underline{\nabla}\times(\delta\underline{u}\times\underline{B})$$

$$\text{or} \quad \boxed{\cancel{i}\omega\,\delta\underline{B} = \underline{k}\times(\delta\underline{u}\times\underline{B})} \quad \text{—(3)}$$

To proceed further, we combine eqs (1) → (3) to derive the dispersion relation for the waves.

Look at the RHS of the momentum equation (2)

1st term $\quad -\underline{k}\,\delta p = -\underline{k}\,c_s^2\,\delta\rho \qquad [\delta p = c_s^2\,\delta\rho]$

$$= -\underline{k}\,c_s^2\left(-\frac{\rho\,\underline{k}.\delta\underline{u}}{\omega}\right) \qquad [\text{using (1)}]$$

$$= \frac{\rho c_s^2}{\omega}\,\underline{k}\,(\underline{k}.\delta\underline{u})$$

2nd term $\quad -\dfrac{\underline{B}\times(\underline{k}\times\delta\underline{B})}{4\pi} = -\dfrac{1}{4\pi\omega}\,\underline{B}\times\left(\underline{k}\times\left[\underline{k}\times(\delta\underline{u}\times\underline{B})\right]\right)$

[using (3)]

$$= -\frac{1}{4\pi\omega}\,\underline{B}\times\left(\underline{k}\times\left[\delta\underline{u}\,(\underline{k}.\underline{B}) - \underline{B}\,(\underline{k}.\delta\underline{u})\right]\right)$$

$$= -\frac{1}{4\pi\omega}\,\underline{B}\times\left[(\underline{k}\times\delta\underline{u})(\underline{k}.\underline{B}) - (\underline{k}\times\underline{B})(\underline{k}.\delta\underline{u})\right]$$

$$= -\frac{1}{4\pi\omega}\Big[(\underline{k}.\underline{B})\,\underline{k}\,(\underline{B}.\delta\underline{u}) - \delta\underline{u}\,(\underline{k}.\underline{B})^2 - \underline{k}\,B^2(\underline{k}.\delta\underline{u}) + \underline{B}\,(\underline{k}.\underline{B})(\underline{k}.\delta\underline{u})\Big]$$

Now multiply by $\dfrac{\omega}{\rho}$, and write $\underline{v}_A \equiv \dfrac{\underline{B}}{\sqrt{4\pi\rho}}$ Alfven velocity.

$$\Rightarrow \quad \omega^2 \delta \underline{u} = c_s^2 \underline{k} (\underline{k} \cdot \delta \underline{u}) - (\underline{k} \cdot \underline{v}_A) \underline{k} (\underline{v}_A \cdot \delta \underline{u}) + \delta \underline{u} (\underline{k} \cdot \underline{v}_A)^2$$
$$+ \underline{k} v_A^2 (\underline{k} \cdot \delta \underline{u}) - \underline{v}_A (\underline{k} \cdot \underline{v}_A)(\underline{k} \cdot \delta \underline{u})$$

gather terms:

$$\Rightarrow \quad \boxed{\delta \underline{u} \left(\omega^2 - (\underline{k} \cdot \underline{v}_A)^2\right) - (\underline{k} \cdot \delta \underline{u}) \left[\underline{k} (c_s^2 + v_A^2) - \underline{v}_A (\underline{k} \cdot \underline{v}_A) \right] + \underline{k} (\underline{k} \cdot \underline{v}_A)(\underline{v}_A \cdot \delta \underline{u}) = 0}$$

Quite a complicated result! It helps to break it down into <u>special cases</u>:

① $\underline{k} \cdot \underline{v}_A = 0$ <u>k perpendicular to the field</u>

then $\omega^2 \delta \underline{u} - (\underline{k} \cdot \delta \underline{u}) \underline{k} (c_s^2 + v_A^2) = 0$ —(*)

This tells us two things: $\underline{k} \times \delta \underline{u} = 0$ or $\boxed{\delta \underline{u} \parallel \underline{k}}$

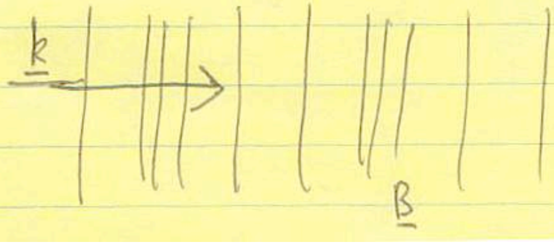
the velocity perturbations are along $\underline{k}$

and taking $\underline{k} \cdot (*)$, we find

$$\boxed{\omega^2 = k^2 (c_s^2 + v_A^2)}$$

This the <u>fast magnetosonic mode</u>

Physically, just like a sound wave, but ~~the~~ with extra (magnetic) pressure as the wave squeezes magnetic field lines together.



② $\underline{k} \parallel \underline{v}_A$ — $\underline{k}$ directed along the field

then $\delta\underline{u}\,(\omega^2 - k^2 v_A^2) - (\underline{k}\cdot\delta\underline{u})\,\underline{k}\,c_s^2 + \underline{k}\,v_A^2\,(\underline{k}\cdot\delta\underline{u}) = 0$

or $$\boxed{\delta\underline{u}\,(\omega^2 - k^2 v_A^2) = (\underline{k}\cdot\delta\underline{u})\,\underline{k}\,(c_s^2 - v_A^2)} \quad (*)$$

There are two kinds of waves here

— Alfven waves $\boxed{\underline{k}\cdot\delta\underline{u} = 0}$ $\boxed{\omega^2 = k^2 v_A^2}$

incompressible
these waves don't know about the sound speed

restored by magnetic tension

from the induction equation $\omega\,\delta\underline{B} = \underline{k}\times(\delta\underline{u}\times\underline{B})$
$= \delta\underline{u}\,(\underline{k}\cdot\underline{B}) - \underline{B}\,(\underline{k}\cdot\delta\underline{u})$

$\Rightarrow$ $\delta\underline{B} \parallel \delta\underline{u}$

$\Rightarrow$ transverse waves $\delta\underline{B} \perp \underline{B}, \underline{k}$
$\delta\underline{u} \perp \underline{B}, \underline{k}$

— Slow magnetosonic mode $\underline{k}\cdot\delta\underline{u} \neq 0$ compressive

now take $\underline{k}\cdot(*)$: $\cancel{(\underline{k}\cdot\delta\underline{u})}(\omega^2 - k^2 v_A^2) = \cancel{(\underline{k}\cdot\delta\underline{u})}\,k^2(c_s^2 - v_A^2)$

$$\Rightarrow \boxed{\omega^2 = k^2 c_s^2}$$

looks just like the sound waves we had before!
When the wave travels along the field, there is no extra magnetic restoring force!

Some further comments:

1. it can be useful to work with the fluid displacement $\underline{\xi}$ rather than $\delta \underline{u}$, where $\delta \underline{u} = i\omega \underline{\xi}$

e.g. The continuity equation is $i\omega \frac{\delta \rho}{\rho} + i\underline{k}.\delta\underline{u} = 0$

or $\frac{\delta \rho}{\rho} + i\underline{k}.\underline{\xi} = 0$

$$\boxed{\frac{\delta \rho}{\rho} = -\underline{\nabla}.\underline{\xi}}$$

which makes sense physically.

2. For displacements perpendicular to B, ie. $(\delta\underline{u}.\underline{B}) = 0$,

$$\frac{\underline{B}.\delta\underline{B}}{4\pi} = \delta\left(\frac{B^2}{8\pi}\right) = -\frac{B^2}{4\pi}\frac{\underline{k}.\delta\underline{u}}{\omega} = +\frac{B^2}{4\pi}\frac{\delta\rho}{\rho}$$

$$\Rightarrow \quad \frac{\delta\left(\frac{B^2}{8\pi}\right)}{B^2/8\pi} = 2\frac{\delta\rho}{\rho}$$

The magnetic field behaves as a fluid with $P \propto \rho^2$

5th October 2007

PHYS 643 lecture 10

General solution to the linearized wave equation

We return to the non-magnetized fluid, and consider linear perturbations in a 1D flow.

(homogeneous, static background)

$$\frac{\partial \delta\rho}{\partial t} = -\rho \frac{\partial \delta u}{\partial x}$$

$$\rho \frac{\partial \delta u}{\partial t} = -\frac{\partial \delta p}{\partial x} = -c_s^2 \frac{\partial \delta\rho}{\partial x}$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \frac{\partial^2 \delta\rho}{\partial x^2}}$$

A wave equation describing propagation of sound waves as we found last time.

The general solution to this equation is

$$\begin{pmatrix} \delta\rho \\ \delta \underline{u} \end{pmatrix} = f(x - c_s t) + g(x + c_s t)$$

propagates to the left — propagates to the right

To see this, change variables to $\eta = x - ct$ and $\xi = x + ct$

which gives $$\frac{\partial^2 \delta\rho}{\partial\eta\,\partial\xi} = 0$$

$$\Rightarrow \quad \delta\rho = f(\eta) + g(\xi)$$

This solution implies a relation between δu and $\delta\rho$?

$$\delta u(x \pm ct) \Rightarrow \frac{\partial}{\partial x}\delta u = \pm \frac{1}{c}\frac{\partial \delta u}{\partial t}$$

NB: sometimes I drop the subscript on c_s. So if I write c I mean c_s.

The continuity equation is then

$$\frac{\partial \delta\rho}{\partial t} = \mp \frac{\rho}{c} \frac{\partial \delta u}{\partial t}$$

$$\Rightarrow \quad \boxed{\frac{\delta\rho}{\rho} = \mp \frac{\delta u}{c_s}}$$

wave propagating to the right $(x-ct)$ has $\frac{\delta\rho}{\rho} = \frac{\delta u}{c_s}$

" " left $(x+ct)$ " $\frac{\delta\rho}{\rho} = -\frac{\delta u}{c_s}$

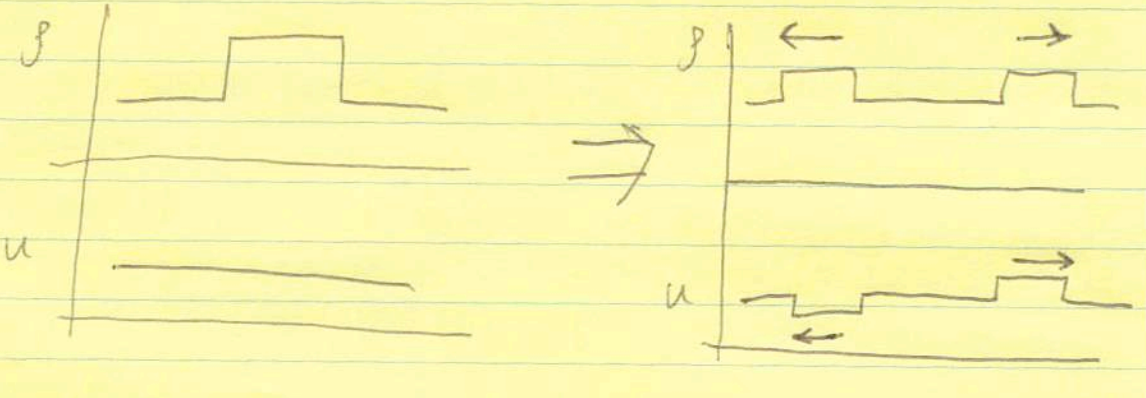
Therefore in general we can write

$$\boxed{\begin{aligned} \frac{\delta\rho}{\rho} &= f_1(x-ct) + f_2(x+ct) \\ \frac{\delta u}{c_s} &= f_1(x-ct) - f_2(x+ct) \end{aligned}}$$

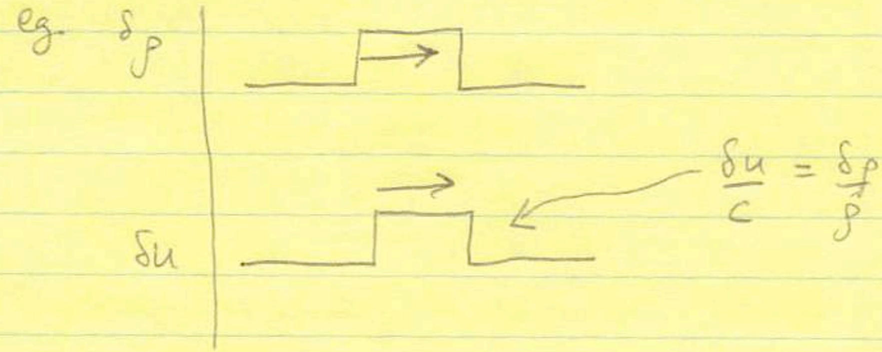
The functions f_1 and f_2 are determined by initial conditions:

$$f_{1,2} = \frac{1}{2}\left[\frac{\delta\rho}{\rho}(x, t=0) \pm \frac{\delta u}{c}(x,0) \right]$$

For example,

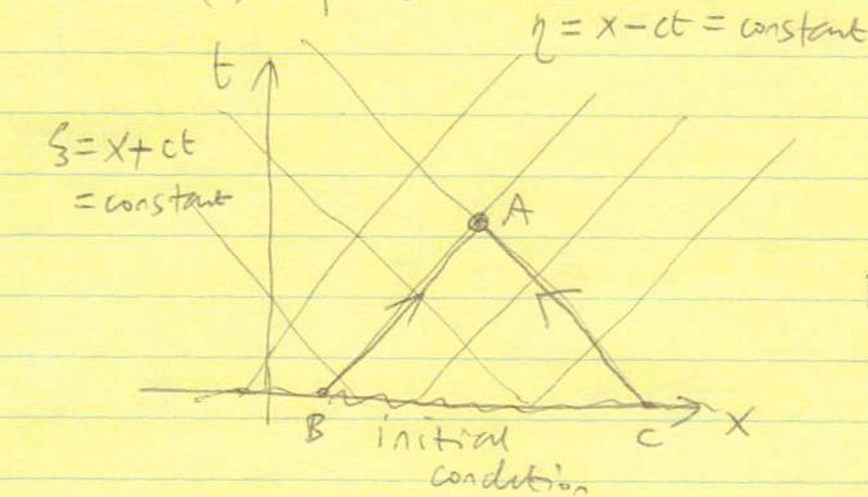


if we arrange the initial conditions correctly, we can get only a right or left solution.



the pulse moves off to the right.

In the (x,t) plane

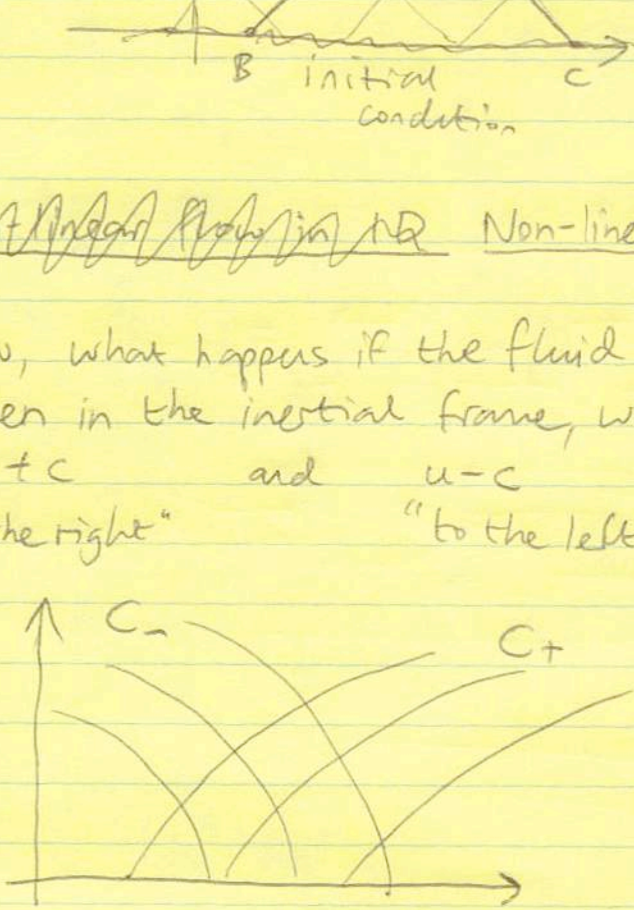


the solution at point A is determined by f_1 from point B and f_2 at point C.

Non-linear disturbances: method of characteristics

Now, what happens if the fluid is moving?
Then in the inertial frame, waves propagate at velocities

$u + c$ and $u - c$
"to the right" "to the left"



The C_+ characteristics are described by $\frac{dX_+}{dt} = u + c$

$$C_- : \frac{dx_-}{dt} = u - c$$

We'll see that the same idea applies – the initial conditions propagate into the fluid along the characteristics.

Consider an isentropic flow – everywhere $P = K\rho^\gamma$.

Continuity $\Rightarrow$ $\quad \frac{D\rho}{Dt} + \rho \frac{\partial u}{\partial x} = 0$

or $\quad \frac{1}{c_s^2} \frac{DP}{Dt} + \rho \frac{\partial u}{\partial x} = 0$

momentum $\Rightarrow$ $\quad \frac{Du}{Dt} + \frac{1}{\rho} \frac{\partial P}{\partial x} = 0$

$$\Rightarrow \quad \frac{1}{\rho}\frac{\partial P}{\partial t} + \frac{u}{\rho}\frac{\partial P}{\partial x} + c_s^2 \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

add and subtract:

$$\left[\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho c_s}\left[\frac{\partial P}{\partial t} + (u \pm c)\frac{\partial P}{\partial x}\right] = 0$$

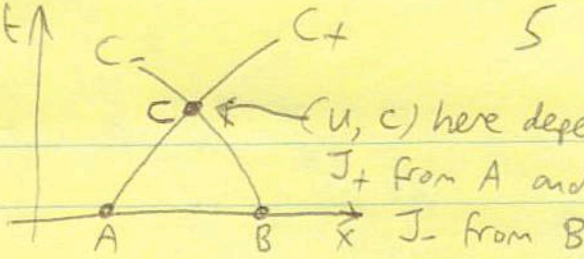
Now define the <u>Riemann invariants</u>

$$J_\pm = u \pm \int \frac{dP}{\rho c_s} = u \pm \frac{2c_s}{\gamma - 1}$$

$$\Rightarrow \quad \boxed{\frac{\partial J_\pm}{\partial t} + (u \pm c)\frac{\partial J_\pm}{\partial x} = 0}$$

$\Rightarrow$ $J_\pm$ is constant along the characteristic curve $x_\pm$

where $\frac{dx_\pm}{dt} = u \pm c.$



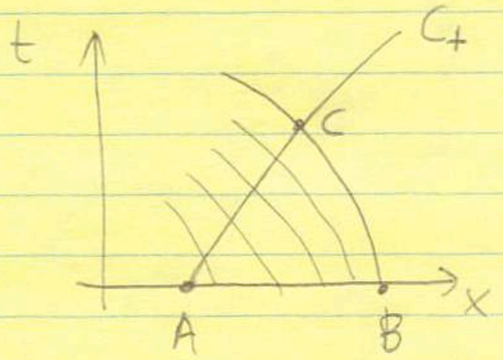
Given J_+ and J_- at a particular location, we can reconstruct u and c:

$$u = \frac{1}{2}(J_+ + J_-) \qquad c = \left(\frac{\gamma-1}{4}\right)(J_+ - J_-)$$

It is instructive to rewrite the equations for the characteristics in terms of J_+, J_-

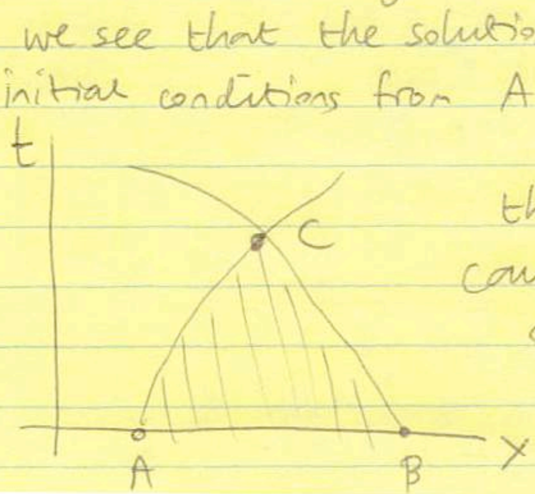
$$\frac{dx_+}{dt} = u + c = \left(\frac{\gamma+1}{4}\right) J_+ + \left(\frac{3-\gamma}{4}\right) J_-$$

$$\frac{dx_-}{dt} = u - c = \left(\frac{3-\gamma}{4}\right) J_+ + \left(\frac{\gamma+1}{4}\right) J_-$$



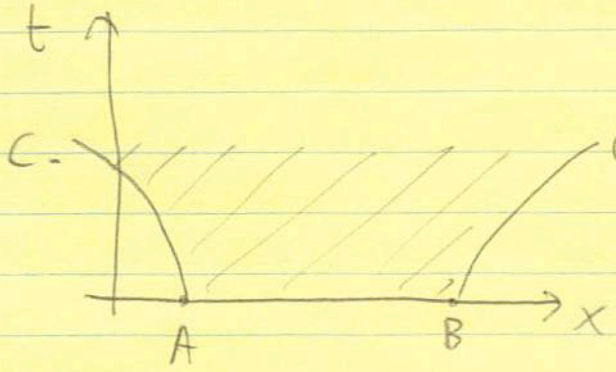
The shape of the C_+ characteristic between points A and C is determined by J_- along the path from A to C.

These values of J_- are set by the initial conditions along AB. Same argument for the J^- characteristic BC.
So we see that the solution at C is influenced by the initial conditions from A to B.



therefore we see that the idea of causality arises naturally
only points in the shaded region can communicate information to C.

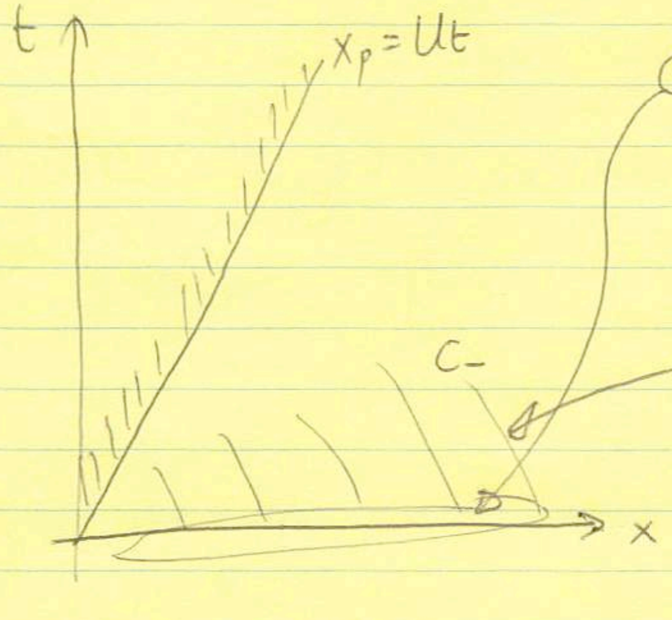
Similarly, we can think of the region of influence of points between A and B



region of influence grows with time

Piston propagating into shock tube

Now consider a simple example: A piston is pushed into a semi-infinite tube of gas, with constant ~~acceleration~~ velocity so that the position of the piston is $x_p = Ut$.
What happens?



① A time $t=0$ $J_- = \text{constant} = -\frac{2c_0}{\gamma-1}$

since the fluid is at rest initially

② but J_- is carried into the fluid by the C_- characteristics

$\Rightarrow$ $\boxed{J_- \text{ constant everywhere}}$

$$J_- = u - \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1} = \text{constant}$$

$$\Rightarrow \boxed{c = c_0 + \left(\frac{\gamma-1}{2}\right)u}$$

the sound speed is a function of u only.

Now, along the characteristic C_+, J_+ must be constant, but we see that in this example J_- is also constant! $\Rightarrow$

$$\frac{dx_+}{dt} = \left(\frac{\gamma+1}{4}\right) J_+ + \left(\frac{3-\gamma}{4}\right) J_- = \text{constant}$$

$\Rightarrow$ $\boxed{\text{the } C_+ \text{ characteristics are straight lines}}$

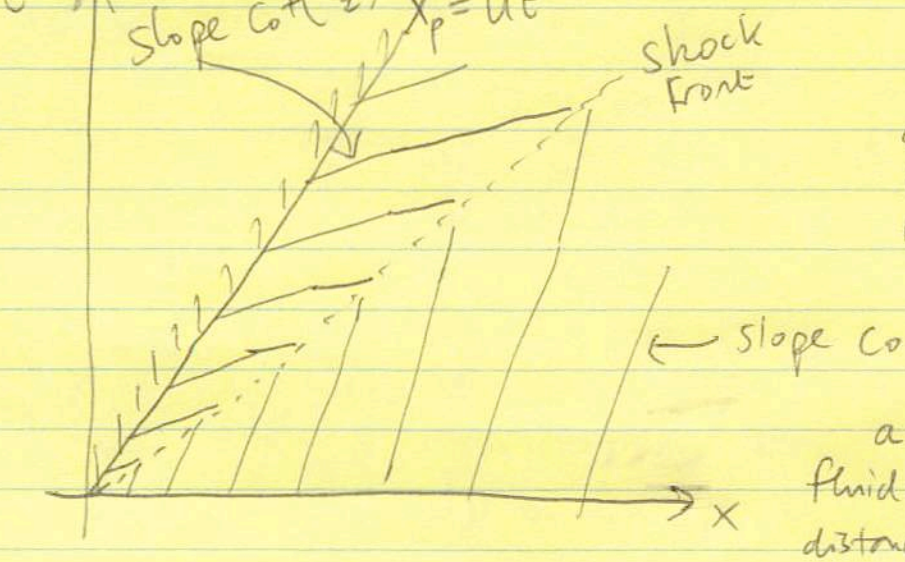
Furthermore, the slope depends on u only, since
$$\frac{dx_+}{dt} = c+u$$
$$= c_0 + \left(\frac{\gamma+1}{2}\right) u$$

$\Rightarrow$ $\boxed{u \text{ is constant along the } C_+ \text{ characteristics}}$

There are two sets of C_+ curves:

1) emerging from the undisturbed gas at $t=0$
   they have $\frac{dx_+}{dt} = c_0$, $u = 0$

2) emerging from the piston with velocity $u = U$
   starting from $x_0 = U t_0$
   The slope is $\frac{dx_+}{dt} = c_0 + \left(\frac{\gamma+1}{2}\right) U.$



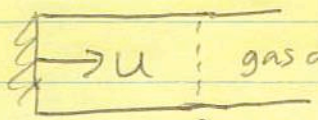
We can see that there's a problem! The two sets of C_+ curves intersect! The solution is <u>overdetermined</u> if there are multiple values of J_+ at any point.

In fact what happens is that a <u>shock</u> develops at which the fluid properties change over a small distance from one J_+ to the other.

11th October 2007

PHYS 643 lecture 11

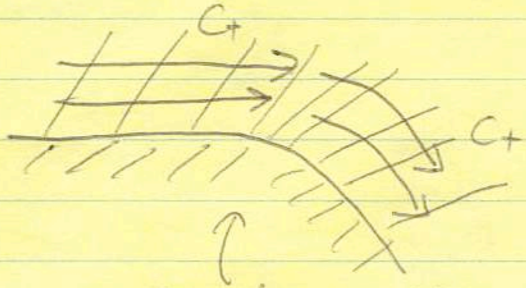
Last time, we discussed a shock tube



a discontinuity or shock develops associated with characteristics that cross

In this example, one of the Riemann invariants (J_-) was constant everywhere and so the motion depends only on how the other (J_+) varies — a flow like this is known as a simple wave.

A similar situation arises in flow of gas around a bend



C_+ characteristics diverge

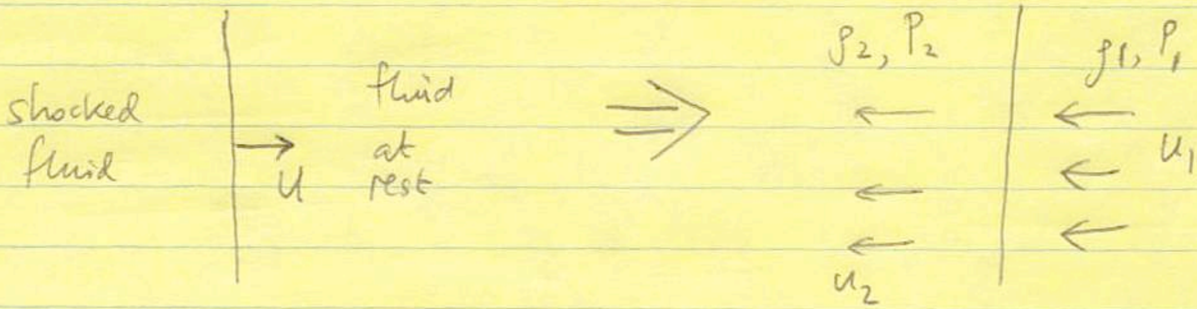
characteristics converge and intersect $\Rightarrow$ shock

At the shock, the fluid velocity and thermodynamic variables (P, ρ, c) change over a very short lengthscale — we will approximate the shock as a discontinuity and derive the behavior across it.

Shock jump conditions (Rankine-Hugoniot relations)

We relate quantities on each side of the shock using conservation relations for mass, momentum, and energy.

First move into the frame of the shock



The continuity equation is $\frac{\partial}{\partial x}(\rho u) = 0$

(for a time-independent 1D flow)

integrate this across the shock

$$\int_{-\varepsilon}^{\varepsilon} dx \, \frac{d}{dx}(\rho u) = [\rho u]_{-\varepsilon}^{\varepsilon}$$

$$\Rightarrow \quad \boxed{\rho u_1 = \rho u_2} \quad \text{—(1)}$$

momentum

$= \rho u \frac{du}{dx}$

$$\frac{d}{dx}(\rho u^2) = -\frac{dP}{dx}$$

integrate this $\Rightarrow$ $\boxed{P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2}$ —(2)

~~total~~ energy

$$\cancel{\frac{d}{dx}\left(\frac{1}{2}\rho u^2 u\right) = -u\frac{dP}{dx}}$$

$$\frac{d}{dx}\left(u\left(\frac{1}{2}\rho u^2 + \rho U + P\right)\right) = 0$$

(no energy sources or sinks)

$$\Rightarrow \quad \frac{1}{2}u_1^2 + U_1 + \frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + U_2 + \frac{P_2}{\rho_2}$$

For a perfect gas, $P = (\gamma - 1) U$ giving $U + \frac{P}{\rho} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho}$

$$\Rightarrow \boxed{\tfrac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \tfrac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2}} \quad -(3)$$

Equations (1), (2) and (3) give us the ~~upstream~~ downstream conditions (ρ_2, P_2, u_2) if we know the ~~downstream~~ upstream (u_1, ρ_1, P_1) ones.

To solve, use (1) and (2) to eliminate P_2 and u_2 from (3)
This gives

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2} = \frac{u_1}{u_2}$$

where M_1 is the upstream Mach number $M_1 = \frac{u_1}{c_1}$

(in other words $\frac{\text{shock velocity}}{\text{undisturbed sound speed}}$)

Notes ~~that~~ 1) there is a maximum compression. In the limit $M_1 \to \infty$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$$
$$= 4 \text{ for } \gamma = 5/3$$

2) The pressure jump is $\frac{P_2}{P_1} = \frac{-(\gamma-1) + 2\gamma M_1^2}{\gamma+1}$

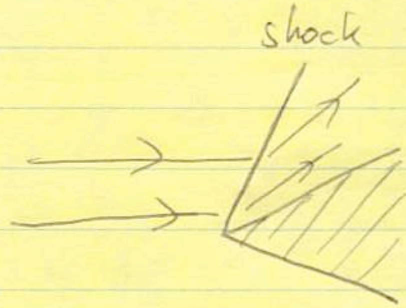
The $P_2 - \rho_2$ relation is known as the shock adiabat or Hugoniot curve

[* Exercise: you can use these relations to calculate the position of the shock in the shock tube from last time]

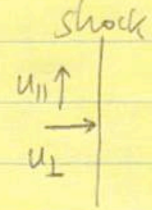
but be careful! The flow across the shock is definitely not adiabatic — there is a large jump in the entropy as the ordered kinetic energy in the bulk flow is converted into heat in the compressed gas.

Oblique shock

eg. supersonic flow past a wedge



In this case, the jump conditions are



$$[\rho u_{\perp}] = 0$$

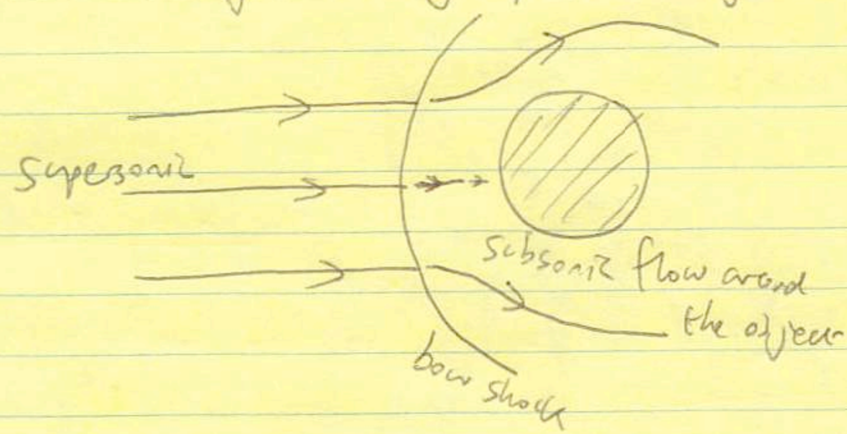
$$[\rho u_{\perp}^2 + P] = 0$$

$$[\rho u_{\perp} u_{\parallel}] = 0 \Rightarrow [u_{\parallel}] = 0$$

$$0 = \left[\rho u_{\perp}\left(\frac{1}{2}u_{\perp}^2 + \frac{1}{2}u_{\parallel}^2 + h\right)\right] \Rightarrow \left[\frac{1}{2}u_{\perp}^2 + h\right] = 0$$

the same as previously, but with the additional relation $u_{\parallel,1} = u_{\parallel,2}$ or $u_{\parallel}$ is constant across the shock.
The constancy of $u_{\parallel}$ means that we can always transform to a local frame in which the flow is normal to the shock.

Shu (p221) has an interesting discussion calculating the shock angle for the flow past a wedge. An attached shock only forms if the object has a sharp enough "nose" and is moving quickly enough; otherwise a detached bow shock forms eg blunt object moving supersonically



Radiative shock (optically thin) (again, we'll follow Shu p 226)

cold gas moving rapidly →	hot gas moving slowly	radiative relaxation layer	cold gas moving v. slowly!
ρ_1, u_1, T_1	$\rho_2 T_2 u_2$	$L > 0$	$\rho_3 \ T_3 \ u_3$
$L = 0$			$L = 0$

The equations are (for steady 1D normal flow)

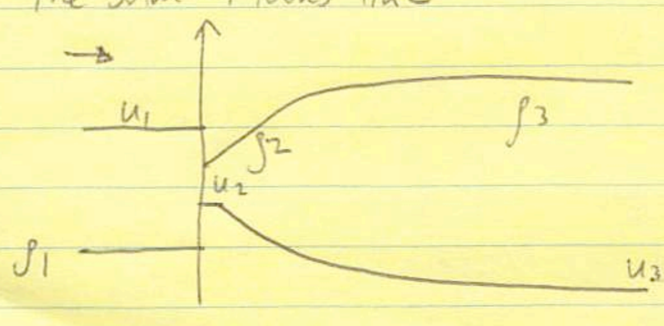
$$\rho u = \text{constant}$$

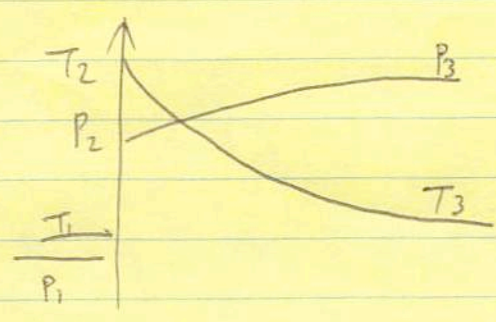
$$u(\rho u) + P = \text{constant}$$

$$\rho u \frac{dU}{dx} = -P \frac{du}{dx} - \rho L(\rho, T)$$

cooling function

The solution looks like





if we jump from 1 to 3 without worrying about the details

$$\rho_3 u_3 = \rho_1 u_1$$
$$\rho_3 u_3^2 + P_3 = \rho_1 u_1^2 + P_1$$

where $P_3 = \frac{\rho_3 k T_3}{\mu m_p}$ with $L(T_3, \rho_3) = 0$

If the shock radiates the excess energy $T_3 \simeq T_1$ isothermal shock
then we get the isothermal jump conditions

$$u_3 u_1 = c_T^2 \qquad \frac{\rho_3}{\rho_1} = \left(\frac{u_1}{c_T}\right)^2 \qquad c_T^2 = \frac{P}{\rho} = \frac{kT}{\mu m_p}$$

isothermal sound speed

Notice that the compression can be arbitrarily large $\frac{\rho_3}{\rho_1} = M_1^2$.

~~Spherical~~

Spherical blast wave in uniform medium Taylor-Sedov solution

eg. application to SN remnants

Initially there is an input of energy E in a small volume at the origin. A shock wave propagates outwards into the rest of the gas. While the ram pressure $P_2 \simeq \rho_1 U_{sh}^2 \gg P_1$ and total radiated energy is small compared to E, we have a "blast wave". The flow is in the "energy conserving phase".

What characteristic lengthscale should we expect for this problem? The only relevant parameters are E and ρ_1 (the density of the undisturbed medium), so dimensional analysis $\Rightarrow$

at time t $\quad r \propto t^{2/5}\left(\frac{E}{\rho_1}\right)^{1/5}$

$\Rightarrow$ 1) we expect the ~~shock radius to be~~ solution for physical quantities inside the blast wave should depend on r and t only through the combination

$$\xi = r\left(\frac{\rho_1}{E t^2}\right)^{1/5}$$

2) the shock front itself will correspond to some characteristic value of $\xi = \xi_s$

$$\Rightarrow \quad r_{shock} = \xi_s \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5}$$

3) velocity of the shock $\quad u_s = \frac{dr_s}{dt} = \frac{2}{5}\frac{r_s}{t} = \frac{2}{5}\xi_s\left(\frac{E}{\rho_1 t^3}\right)^{1/5}$

So the shock weakens over time.

Next time, we'll write the equations for the flow in terms of ξ (see Taylor 1950 Proc Roy Soc London A 201, 1065).

PHYS 643 lecture 12

Taylor-Sedov similarity solution for spherical blast wave

Choudhuri follows the standard treatment of using the similarity variable $\xi = r\left(\frac{\rho_0}{Et^2}\right)^{1/5}$. However the problem with this for numerical integration is that we don't know in advance the value of ξ corresponding to the shock front — one has to try different values of ξ_{shock} until we obtain the correct total energy. (see Choudhuri p121).

I prefer to follow the procedure of Taylor's 1950 paper (Proc Roy Soc London A201, 159). I'll use the same notation.

We look for a solution

$$\frac{p}{p_0} = R^{-3} f_1(\eta)$$

$$\frac{\rho}{\rho_0} = \psi(\eta)$$

$$u = R^{-3/2}\phi_1(\eta)$$

[the motivation for these scalings with R is to conserve total E see later]

where $\eta = r/R$ and $R(t)$ is the location of the shock front.

The momentum equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\Rightarrow \quad -\frac{3}{2}R^{-5/2}\dot{R}\phi_1 - R^{-3/2}\phi_1'\eta\frac{\dot{R}}{R} + R^{-3/2}\phi_1 R^{-3/2}\frac{\phi_1'}{R}$$

$$= -\frac{p_0}{\rho_0\psi}R^{-3}\frac{f_1'}{R}$$

$$\Rightarrow \quad \boxed{-\dot{R}R^{-5/2}\left(\frac{3}{2}\phi_1 + \eta\phi_1'\right) + R^{-4}\left(\phi_1\phi_1' + \frac{p_0 f_1'}{\rho_0\psi}\right) = 0}$$

R is a function only of t; the other terms are functions only of η
$\Rightarrow$ (just as in separation of variables)

$$\boxed{\frac{dR}{dt} = A R^{-3/2}} \quad \Rightarrow \quad R = \left(\tfrac{5}{2}At\right)^{2/5} \qquad \dot{R} = \frac{2}{5}\frac{R}{t}$$

and $$\boxed{-A\left(\tfrac{3}{2}\phi_1 + \eta\phi_1'\right) + \phi_1\phi_1' + \frac{f_1' p_0}{\rho_0 \psi} = 0}$$

Continuity $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0$$

$$\Rightarrow \quad \boxed{-A\eta\psi' + \psi'\phi_1 + \psi\left(\phi_1' + \frac{2}{\eta}\phi_1\right) = 0}$$

Energy assume the evolution is adiabatic

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\left(\frac{p}{\rho^\gamma}\right) = 0$$

$$\frac{\partial}{\partial t}\left(p\rho^{-\gamma}\right) = (\rho_0\psi)^{-\gamma} p_0\left(-3R^{-4}\dot{R}f_1 - R^{-3}\frac{f_1' r\dot{R}}{R^2}\right)$$
$$+ p_0 R^{-3} f_1 (-\gamma)\,\rho_0^{-\gamma}\,\psi^{-(\gamma+1)}\,\psi'\left(-\frac{r}{R^2}\right)\dot{R}$$
$$= p_0\rho_0^{-\gamma}\psi^{-\gamma}\left\{\left[-\dot{R}R^{-4}(3f_1 + \eta f_1')\right] + \frac{f_1\gamma R^{-4}}{\psi}\eta\psi'\dot{R}\right\}$$
$$= \dot{R}R^{-4}p_0\rho_0^{-\gamma}\psi^{-\gamma}\left[-3f_1 - \eta f_1' + \frac{\gamma\eta\psi' f_1}{\psi}\right]$$

$$u\frac{\partial}{\partial r}\left(p\rho^{-\gamma}\right) = R^{-3/2}\phi_1\left[p_0\rho_0^{-\gamma}\psi^{-\gamma}R^{-3}\frac{f_1'}{R} + p_0 f_1 R^{-3}\rho_0^{-\gamma}(-\gamma)\psi^{-(\gamma+1)}\frac{\psi'}{R}\right]$$
$$= \phi_1 R^{-4}\rho_0^{-\gamma}\psi^{-\gamma}p_0\left[R^{-3/2}f_1' + (-\gamma)f_1 R^{-3/2}\frac{\psi'}{\psi}\right]$$

$$\Rightarrow \quad -3f_1 - \eta f_1' + \frac{\gamma\eta\psi' f_1}{\psi} + \underbrace{\frac{R^{-3/2}}{\dot{R}}}_{\frac{1}{A}}\phi_1\left(f_1' - \frac{\gamma f_1\psi'}{\psi}\right) = 0$$

$$\Rightarrow \quad \boxed{A(3f_1 + \eta f_1') + \frac{\gamma f_1 \psi'}{\psi}\left(-A\eta + \phi_1\right) - \phi_1 f_1' = 0}$$

Now make dimensionless: $\quad f = \frac{f_1 c_s^2}{A^2} \qquad \phi = \frac{\phi_1}{A} \qquad c_s^2 = \frac{\gamma p_0}{\rho_0}$

which means that

$$\boxed{\begin{aligned} \rho &= \rho_0 \psi\left(\tfrac{r}{R}\right) \\ p &= \frac{A^2}{R^3} f\left(\tfrac{r}{R}\right) \frac{\rho_0}{\gamma} = f\left(\tfrac{r}{R}\right) \frac{\rho_0 \dot{R}^2}{\gamma} \\ u &= R^{-3/2} A\, \phi\left(\tfrac{r}{R}\right) = \dot{R}\, \phi\left(\tfrac{r}{R}\right) \end{aligned}}$$

where ψ, f and ϕ obey

$$\boxed{\begin{aligned} \phi'(\eta - \phi) &= \frac{f'}{\gamma \psi} - \frac{3}{2}\phi \\ \frac{\psi'}{\psi} &= \frac{\phi' + 2\phi/\eta}{\eta - \phi} \\ f'\left\{(\eta-\phi)^2 - \frac{f}{\psi}\right\} &= f\left\{-3\eta + \phi\left(3 + \frac{\gamma}{2}\right) - 2\gamma\phi^2/\eta\right\} \end{aligned}}$$

which can be integrated. The boundary conditions at $\eta = 1$ $(r = R)$ are set by the conditions at the shock. For a strong shock the boundary conditions are consistent with the self-similar solution and are

$$\frac{\rho_1}{\rho_0} = \frac{\gamma+1}{\gamma-1} \qquad \frac{u_1}{U} = \frac{2}{\gamma+1} \qquad \frac{U^2}{c_s^2} = \frac{2\gamma}{\gamma+1}\frac{p_1}{p_0}$$

$$\Rightarrow \quad \boxed{\psi = \frac{\gamma+1}{\gamma-1} \qquad f = \frac{2\gamma}{\gamma+1} \qquad \phi = \frac{2}{\gamma+1} \qquad \text{at } \eta = 1}$$

Given $\psi(\eta)$, $f(\eta)$ and $\phi(\eta)$, we can find ρ, p, u if we know R and $\dot{R}$ — but this depends on A.

A is related to the energy in the blast.

$$E = \int 4\pi r^2 dr \left(\underset{\text{kinetic}}{\tfrac{1}{2}\rho u^2} + \underset{\text{internal}}{\frac{p}{\gamma - 1}} \right)$$

$$= \underbrace{R^3 \dot{R}^2}_{A^2} \rho_0 \underbrace{\int_0^1 4\pi \eta^2 d\eta \left(\tfrac{1}{2} \psi \phi^2 + \frac{f}{\gamma(\gamma-1)} \right)}_{B(\gamma)}$$

$$\Rightarrow \quad A = \left(\frac{E}{\rho_0}\right)^{1/2} B(\gamma)^{-1/2}$$

$$\Rightarrow \quad \boxed{\dot{R} = \frac{2}{5}\frac{R}{t}} \qquad R = \left(\frac{5}{2}\right)^{2/5} \left(\frac{E}{\rho_0}\right)^{1/5} B^{-1/5} t^{2/5}$$

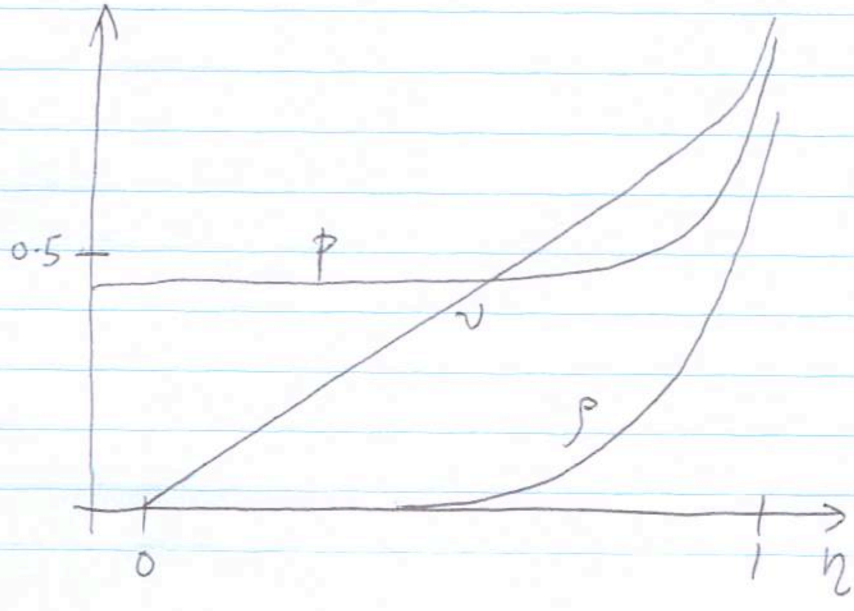
$$\boxed{R = \left(\frac{25}{4B}\right)^{1/5} \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}}$$

Taylor gives $B = 5.36$ for $\gamma = 1.4$. $\Rightarrow \left(\frac{25}{4B}\right)^{1/5} = 1.03$

(which corresponds to $\xi_0 = 1.03$ for the shock position in the similarity variable ξ, see Shu for example).

For $\gamma = 5/3$, $B = ?$

The solution looks like:



see Choudhuri Fig 6.5

Taylor derived some approximations for the solution which we can use to get some of the behaviour.

1) The velocity is $v = \frac{\dot{R}}{\gamma}\frac{r}{R} + \frac{\gamma-1}{\gamma(\gamma+1)}\left(\frac{r}{R}\right)^n \qquad n = \frac{7\gamma-1}{\gamma^2-1}$

it rises linearly at first but then as $v \propto r^n$

for $\gamma = 5/3$ $\quad n = \frac{\frac{35}{3}-1}{\frac{25}{9}-1} = 6 \qquad v \propto r^6$

2) The pressure is constant throughout most of the interior $\Rightarrow$ uniform distribution of internal energy.

3) The density drops rapidly to zero ~~at~~ in the central regions. Taylor found for $\eta \ll 1$ $\quad \rho \propto \eta^{\frac{3}{\gamma-1}}$ or $\rho \propto \left(\frac{r}{R}\right)^{9/2}$ for $\gamma = 5/3$.

But the pressure remains constant $\Rightarrow$ very high central temperatures.

4) Most of the mass ~~for~~ is right behind the shock.

Application to SNe

1) A typical energy is $E = 10^{51}$ ergs, density of ISM $n = 1\,\text{cm}^{-3}$.

$$\Rightarrow \quad R \approx \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} = 5\,\text{pc}\left(\frac{t}{1000\,\text{yrs}}\right)^{2/5}\left(\frac{E_{51}}{n_0}\right)^{1/5}$$

$$\dot{R} = \frac{2}{5}\frac{R}{t} = \cancel{4000}\ 1800\,\text{km/s}\left(\frac{E_{51}}{n_0}\right)^{1/5}\left(\frac{t}{1000\,\text{yrs}}\right)^{-3/5}$$

2) How early can we apply this solution? After the initial explosion we might expect a shock to form when the ejecta has travelled a distance comparable to the mfp. But this is a huge distance! For an ejecta mass of $\simeq 1 M_\odot$, the energy per particle is $\frac{10^{51}\,\text{erg}}{2\times10^{33}\,\text{g}} = 5\times10^{17}$ erg/g

(see lecture 1) $\qquad \sim 0.5$ MeV

$\lambda \sim \frac{1}{n\sigma} \sim \frac{E_{\text{particle}}^2}{ne^4} \simeq$ distances of $\sim$ Mpc! $\qquad$ or $v_{\text{ejecta}} \simeq 10^4$ km/s

But for typical $B_{ISM} \sim 10^{-6}$ G the particles have a radius of gyration $r_B = \frac{m_p v c}{eB} \simeq 10^{11}\text{cm} \left(\frac{v}{10^4 \text{km/s}}\right)\left(\frac{1\mu G}{B}\right) \ll \lambda$

A hydromagnetic shock forms, an example of a "collisionless shock".

How early can we apply our solution? We could set $\dot{R} = v_{ejecta}$ which gives $\underline{t = 60 \text{ yrs}}$ [or we could say $\frac{4\pi}{3} R^3 \rho_0 = M_{ejecta}$

$$\Rightarrow \quad \frac{4\pi R^3}{3} \rho_0 = \frac{2E}{v_{ejecta}^2} \quad \Rightarrow \quad v_{ejecta}^2 = \frac{6}{4\pi} \frac{E}{\rho_0 R^3}$$

$$\simeq \frac{6}{4\pi} \dot{R}^2$$

the same condition!]

For $t \lesssim 100$ yrs $\quad r \simeq v_{ejecta} t \quad$ ballistic

$$= 0.01 \text{ pc} \left(\frac{v_{ejecta}}{10^4 \text{km/s}}\right)\left(\frac{t}{\text{yrs}}\right)$$

3) How late can we apply our solution? Eventually, cooling becomes significant — this occurs when the temperature falls below $\simeq 1$ MK when the cooling function increases significantly because of collisional excitation of CNO.

We can ask how long to radiate E for a given Λ?

$$t = \frac{E}{n^2 \Lambda \frac{4\pi R^3}{3}} = 3 \times 10^8 \text{ yrs} \left(\frac{1\text{pc}}{R}\right)^3 \frac{1}{n^2} \left(\frac{10^{-21} \text{erg cm}^3 \text{s}^{-1}}{\Lambda}\right) E_{51}$$

$\simeq$ peak of cooling function

Plug in R from the Sedov-Taylor solution and solve for the time of the transition

$\Rightarrow \quad t_t \simeq 3 \times 10^4$ yrs at which point the radius is $R_t \simeq 20$ pc.

(assuming $n = 1 \text{ cm}^{-3}$, R and t at the transition decrease with increasing n).

4) When cooling is important, the outward motion of the shock is maintained by the momentum of the gas. "Snowplough phase"
Then

$$M v = \underbrace{M_t v_t}_{\text{values at the transition}}$$

since $M \approx \frac{4\pi}{3} \rho_0 R^3$ we see that $\dot{R} \propto \frac{1}{R^3}$

much steeper than the $\dot{R} \propto \frac{1}{R^{3/2}}$ behaviour we had previously.

In this phase (radiative shock) the gas behind the shock forms a thin compressed shell.

5) See Chevalier (1974) and Mansfield & Salpeter (1974) for classic papers doing numerical simulations of supernova remnant evolution. Notice the "reverse shock" that propagates back into the shell of ejecta.

Mansfield & Salpeter 1974

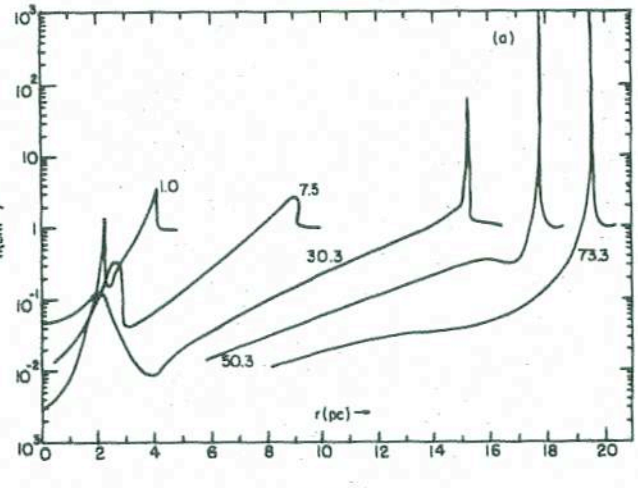


FIG. 1a

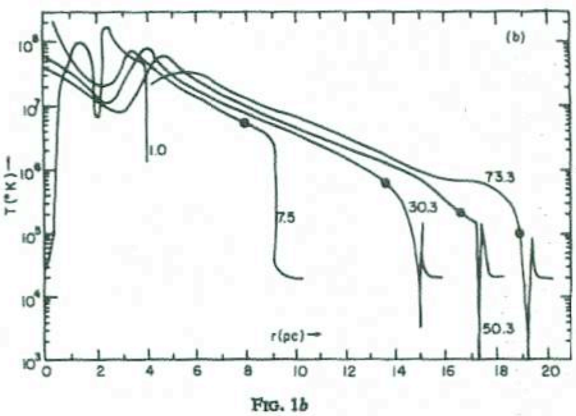


FIG. 1b

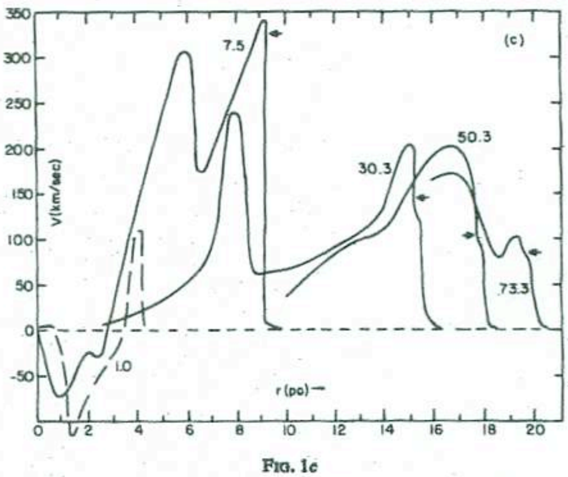


FIG. 1c

FIG. 1.—Various quantities plotted against radial distance r (in parsecs) for a case with $\xi = \eta = 1$. All curves are labeled with the value (in units of 10^3 years) of the age t of the supernova remnant. n is the number-density of hydrogen atoms and T the temperature (filled circles on fig. 1b denote mean temperature $\bar{T}$). In fig. 1c the radial velocity V is plotted, except for the dashed curve which plots $0.1\,V$ for $t = 1.04$ (the small arrows denote the mean velocity V_m).

The expansion actually takes place into an ambient medium of density $n_{0\xi}$; but $n_\xi/n_{0\xi}$ is independent of ξ, and shock velocities and shock temperatures are the same in the two compared cases. If radiation were absent, equation (4A) gives the exact scaling for any mass-shell in the general blast-wave solution, with kinetic energy per unit mass, $E_\xi/n_\xi r_\xi^3$, and temperature independent of ξ.

Now consider the inclusion of radiation, but only in our approximation which omits grain radiation and collisional quenching of forbidden lines. All emissivities are then proportional to the square of the density n_ξ, so that emission rates per particle per second are proportional to $n_\xi = \xi^{-1}n$. However, the time development of internal energy depends on the emission per particle over a time period t_ξ which is proportional to $n_\xi t_\xi$ and thus independent of ξ. The scaling in equation (4) is then still correct at all times t_ξ even with radiative cooling included; in particular the radius R_ξ of the shock front, the swept-up total mass M_ξ, and the power P_ξ radiated per second (in any frequency range, from the whole blast wave) are given by

$$\frac{R_\xi(t_\xi)}{R(t)} = \xi\,, \quad \frac{M_\xi(t_\xi)}{M(t)} = \xi^2\,, \quad \frac{P_\xi(t_\xi)}{P(t)} = \xi\,. \tag{4B}$$

Radiation flux or photon number-density is proportional to $P_\xi/R_\xi^2 \propto \xi^{-1}$, as is the particle density n_ξ, so that ionization equilibrium is independent of ξ. Reabsorption effects depend on optical depths which are proportional to $n_\xi R_\xi$ which is also independent of ξ. The inclusion of radiative transfer (any number of absorptions and reemissions) therefore does not disturb the scaling argument at all (always with the neglect of grain radiation, collisional quenching, or three-body recombinations).

McKee (1974)

When the shell is first ejected into the medium surrounding the supernova, the precipitous density drop at the surface of the supernova leads to the formation of a rarefaction wave that propagates back into the shell, lowering the pressure and accelerating the shell. The pressure is further reduced by the large adiabatic-expansion losses associated with the spherically diverging flow: the pressure $p \propto R^{-4}$ when radiation pressure dominates and $p \propto R^{-5}$ when gas pressure dominates. Even when the effect of heating by a central pulsar is included (Rees 1970), the pressure in the expanding shell inevitably will drop below the pressure behind the blast wave advancing into the circumstellar medium. When this happens a compression wave will begin to propagate back into the shell. The low sound speed in the shell isolates it from the higher pressure outside and prevents it from responding quasi-statistically; hence, the compression wave will rapidly steepen into a shock—the reverse shock wave (see fig. 1). As discussed by Stanyukovich (1960), a similar phenomenon occurs when a spherical explosive is detonated in air, although in that case the formation of the inward-directed shock is delayed, since the explosion products expand to only about 10 times the initial radius.

The numerical models referred to above do not show the reverse shock wave: Rosenberg and Scheuer (1973) did not follow the internal dynamics of the supernova shell; and Gull (1973), presumably for computational reasons, assumed an initial temperature in excess of 10^9 ° K at $R_s \simeq 0.1$ pc, which is orders of magnitude larger than the actual value. Gull's work is the first quantitative study of the effect of the Rayleigh-Taylor instability on the expanding supernova shell.

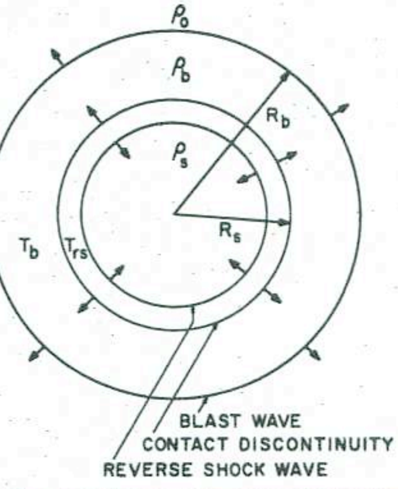


FIG. 1.—The ejected supernova material extends out to R_s, where there is a contact discontinuity. A blast wave at R_b propagates ahead of the ejected shell and shocks the circumstellar medium from an initial density ρ_0 (hydrogen number density n_H) to a density ρ_b and a temperature T_b. The reverse shock wave propagates into the ejected supernova material of density ρ_s and shocks it to a temperature T_{rs}. Until the reverse shock wave has shocked a significant fraction of the supernova shell, it will actually move *outward* with respect to fixed coordinates.

From his results it appears that this instability will not have a significant effect on the dynamics of the reverse shock wave, although it could alter the luminosity. As a first approximation, however, we shall ignore the effects of this instability here.

Our analysis is based on the assumption that the gas shocked by the reverse shock wave will attempt to maintain approximate pressure equilibrium with the gas behind the blast wave. Hence the temperature and the density at a point fixed in the shocked gas will remain roughly constant; this will preserve the large density jump at the contact discontinuity between the stellar and circumstellar material, as found by Gull. It is possible that succeeding, weaker shocks will propagate inward in order to maintain pressure equilibrium. Assuming that the unshocked region of the expanding supernova shell has a uniform density ρ_s, we write the equation of pressure equilibrium as (cf. eq. [1])

$$\rho_s v_{rs}^2 = \beta \rho_0 v_b^2 , \tag{4}$$

where v_{rs} is the velocity of the reverse shock wave relative to the unshocked supernova shell and $\rho_0 = \rho_b/4$ is the density of the circumstellar medium. Since the post-shock temperature varies as v^2 (here we neglect differences in the mean mass μ) the density ratio can be expressed as $\rho_s/\rho_0 = \beta T_b/T_{rs}$. Hence the expansion of the shell and the resulting decrease in ρ_s causes the reverse shock wave to accelerate and the post-shock temperature T_{rs} to increase.

In order to make the problem tractable, we restrict ourselves to the case in which only a small fraction F of the ejected mass M_s has been shocked. This means that the radius of the reverse shock wave is about equal to the radius of the supernova shell R_s. Furthermore, the velocity at the contact discontinuity v_s and the velocity of the blast wave v_b remain approximately constant. Since the compressed circumstellar gas is confined to $\frac{1}{4}$ its initial volume, one has $R_b^3 = \frac{1}{3}(4R_s^3)$ so that

$$\frac{M_s}{M_b} = \frac{3}{4}\frac{\rho_s}{\rho_0} = \frac{3}{4}\frac{\beta T_b}{T_{rs}} . \tag{5}$$

The shocked mass fraction F is determined by the mass flux into the reverse shock wave,

$$M_s \frac{dF}{dt} = 4\pi R_s^2 \rho_s v_{rs} . \tag{6}$$

This equation can be readily integrated by noting that $dr_s = v_s dt \simeq v_b dt$ and that $\rho_s \propto R_s^{-3}$; the result is

$$F = 2\left(\frac{T_{rs}}{T_b}\right)^{1/2} = \left(3\beta \frac{M_b}{M_s}\right)^{1/2} . \tag{7}$$

From Gull's (1973) work, we find that $\beta \simeq \frac{1}{3}$ is a reasonable estimate when $M_b \ll M_s$. Equation (7) then indicates that the reverse shock wave will reach the center ($F = 1$) when $M_b \simeq M_s$; also, $T_{rs} \propto F^2$ until it becomes comparable to the blast-wave temperature T_b.

PHYS 643 lecture 13

Oct 25, 2007

Steady flow of gas through a nozzle

(see Landau & Lifschitz § 80, 90)

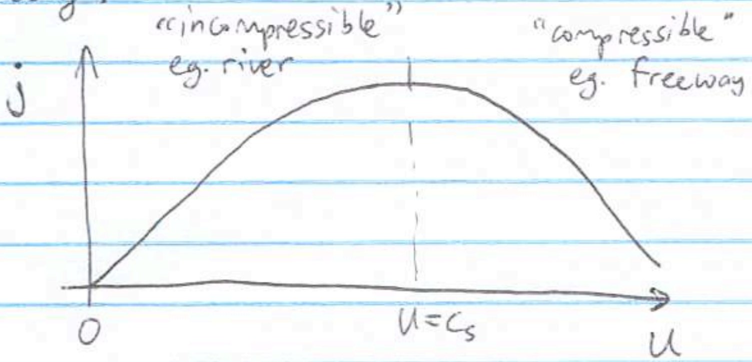
Consider a 1D isentropic flow.
The momentum equation is

$$u \frac{du}{dx} = -\frac{1}{\rho}\frac{dp}{dx} = -\frac{c_s^2}{\rho}\frac{d\rho}{dx}$$

$$\Rightarrow \quad \frac{u}{\rho}\frac{d\rho}{du} = -\frac{u^2}{c_s^2}$$

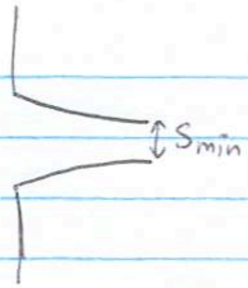
$$\Rightarrow \quad \frac{d}{du}(\rho u) = \rho + u\frac{d\rho}{du} = \rho\left(1 - \frac{u^2}{c_s^2}\right) = \rho(1 - M^2)$$

so for $u < c_s$, the mass flux $j = \rho u$ increases with increasing fluid velocity, whereas for $u > c_s$ j decreases with increasing velocity.



The maximum flux is $j_* = \rho_* c_*$ (the subscript $*$ labels quantities at the location of the maximum in j)

Now consider flow of gas out of a large vessel through a tube of variable cross-section — a nozzle



if the tube is narrow and S varies slowly enough, the flow is 1D.

The continuity equation $\Rightarrow$ $jS =$ constant along the tube.

This tells us that if j reaches its maximum value j_* anywhere in the tube, it must be where S is minimum, ie at the narrowest point of the tube. (Otherwise j would exceed j_* somewhere else in the tube).

Therefore the maximum mass loss rate through the nozzle is

$$Q_{max} = j_* S_{min}$$

We can calculate Q_{max} in terms of the conditions in the reservoir. Bernoulli's principle tells us that $\dfrac{u^2}{2} + \dfrac{c^2}{\gamma-1} =$ constant in the flow. Let c_0 be the sound speed where $u=0$

then $$\underbrace{\frac{u_*^2}{2} + \frac{c_*^2}{\gamma-1}} = \frac{c_0^2}{\gamma-1}$$

$$c_*^2\left(1+\frac{2}{\gamma-1}\right) = \frac{2c_0^2}{\gamma-1} \Rightarrow \boxed{c_*^2 = \frac{2c_0^2}{\gamma+1}}$$

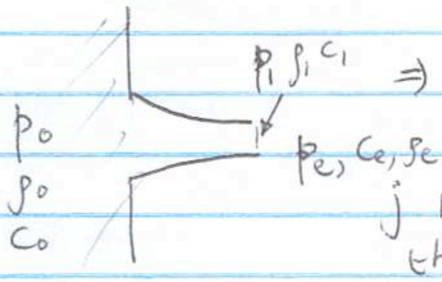
but $c^2 \propto \dfrac{P}{\rho} \propto \rho^{\gamma-1}$

$$\Rightarrow \quad \rho_* = \rho_0\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \qquad \left(\text{similarly } P_* = P_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}\right)$$

$$\Rightarrow \quad j_* S_{min} = \rho_* c_* S_{min} = \rho_0 c_0 \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}+\frac{1}{2}} S_{min}$$

$$\Rightarrow \quad \boxed{Q_{max} = S_{min}\sqrt{\gamma P_0 \rho_0}\left(\frac{2}{\gamma+1}\right)^{\left(\frac{\gamma+1}{2(\gamma-1)}\right)}}$$

Consider a nozzle with monotonically decreasing area



$\Rightarrow$ j increases monotonically along the tube

j reaches a maximum value if $u = c$ at the end of the tube; then $p_1 = p_*$ $v_1 = c_1 = c_*$

Now consider what happens as the external pressure p_e is reduced from p_0 downwards.

if $p_e \geq p_*$ the pressure drop from p_0 to p_e occurs in the nozzle

Bernoulli $\Rightarrow$ $\frac{1}{2}u^2 + \frac{c^2}{\gamma-1} = \frac{1}{2}u^2 + \frac{\gamma p}{(\gamma-1)\rho} = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}$

$$\Rightarrow \boxed{u^2 = \frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} \quad (1)$$

gives the velocity as a function of pressure along the nozzle

The mass loss rate is

$$Q = S_{min}\,\rho_1 u_1$$
$$= S_{min}\,\rho_1 \left\{ \frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2}$$

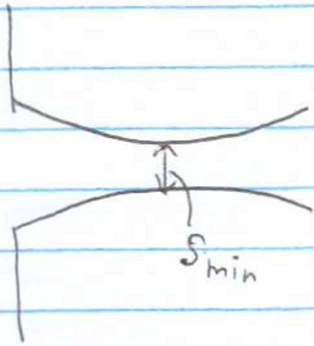
When

$p_e = p_*$ then $Q = Q_{max}$ and the velocity $u_1 = c_1$

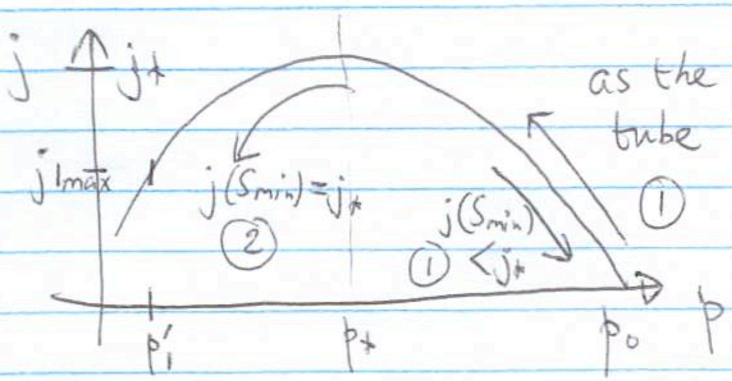
As the external pressure drops further, the pressure drop from p_* to p_e occurs outside the tube. The outflow rate remains the same. The gas cannot acquire a supersonic velocity in a nozzle of this kind!

Supersonic velocities can be achieved if the nozzle widens again

This is called a de Laval nozzle.



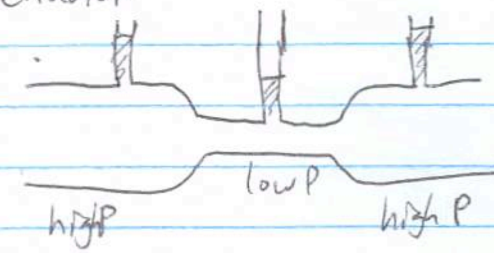
If the maximum flux is reached it must be reached at the narrowest point
$\Rightarrow$ maximum output is $Q_{max} = j_* S_{min}$



as the pressure drops along the tube j increases

① if $j < j_*$ at $S = S_{min}$ then the pressure increases again as the tube widens

this is the standard Venturi behavior



② if $j(S_{min}) = j_*$
then the pressure continues to decrease along the tube
the output is $S_1 j|_{max} = S_{min} j_*$

Now increase p_e from zero upwards

1) $p_e < p_1'$ the flow reaches j_* at S_{min}
the pressure drop from p_1' to p_e occurs outside the nozzle.

2) $p_e > p_1'$ a shock forms that compresses the gas from p_1' to p_e
As p_e increases, the shock moves into the tube, eventually reaching the location of S_{min}, at which point the flow becomes subsonic everywhere.

Applications

1) Blandford & Rees (1974) suggest that the collimated outflows observed from the centres of radio galaxies originate in a de Laval nozzle type flow. The idea is that the central engine produces hot relativistic plasma that flows outwards through a confining gas cloud.

The difference to the nozzle problem is that instead of the area S being specified in advance, the pressure p is specified along the flow → since it must match the pressure in the surrounding gas cloud at each radius.

∴ we can use equation (1) to write down the ~~corresponding Mach number~~ flow velocity

$$u^2 \cancel{M^2} = \frac{2}{\gamma-1}\left[1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\frac{\gamma p_0}{\rho_0}$$

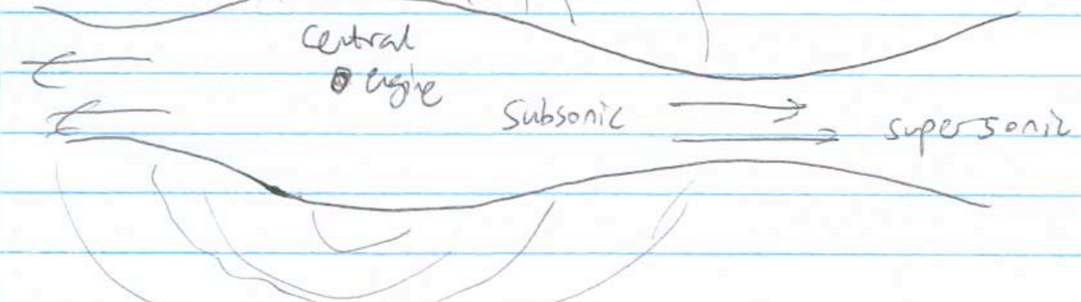
p_0 is called the "stagnation pressure" – just as in the previous example it is the pressure at a location where $u=0$.

or in terms of the local sound speed

$$M^2 = \frac{u^2}{c^2} = \frac{u^2}{\gamma P/\rho} = \frac{2}{\gamma-1}\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}-1\right]$$

Blandford & Rees argued that the flow will necessarily become supersonic since there is a substantial pressure drop through the cloud. The area of the outflow is set by the requirement $\rho u A = \text{constant}$.

In fact, Blandford & Rees obtain a different velocity profile in the outflow because they take into account the fact that the plasma is relativistic.



2) Spherical flow around a central point mass – accretion and winds

Bondi (1952) Parker (1958)

The momentum equation is

$$u\frac{du}{dr} = -\frac{c^2}{\rho}\frac{d\rho}{dr} - \frac{GM}{r^2}$$

Continuity demands that $\rho r^2 u =$ constant

or $$2 + \frac{d\ln\rho}{d\ln r} + \frac{d\ln u}{d\ln r} = 0$$

Eliminate $\frac{d\rho}{dr}$: $$u\frac{du}{dr} + \frac{GM}{r^2} = +\frac{c^2}{\rho}\left[\frac{2\rho}{r} + \frac{\rho}{r}\frac{d\ln u}{d\ln r}\right]$$

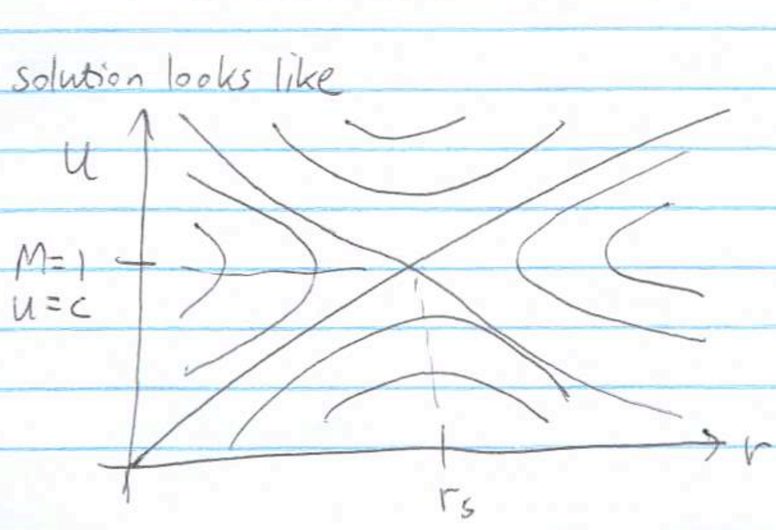
$$\Rightarrow \quad u\frac{du}{dr} - \frac{c^2}{u}\frac{du}{dr} = \frac{2c^2}{r} - \frac{GM}{r^2}$$

$$\Rightarrow \quad (1-M^2)\left(-\frac{d\ln u}{d\ln r}\right) = 2 - \frac{GM}{rc^2}$$

or define the characteristic radius $r_s = \frac{GM}{2c^2}$

$$\Rightarrow \quad \boxed{(1-M^2)\left(-\frac{d\ln u}{d\ln r}\right) = 2\left(1-\frac{r_s}{r}\right)}$$

The solution looks like



at $r=r_s$ $\frac{du}{dr}=0$ (for $M\neq 1$)

when $M=1$, $\frac{dr}{du}=0$ (unless $r=r_s$)

There are two solutions (one inflow, one outflow) that have a subsonic - supersonic transition where $M = 1$ at $r = r_s$, allowing $\frac{du}{dr}$ to remain finite.

As a concrete example, let's take the inward flowing solution, ie. accretion. We can calculate the accretion rate $\dot{M}$.
For isentropic flow, Bernoulli's equation gives

$$\frac{u^2}{2} + \frac{c^2}{\gamma-1} - \frac{GM}{r} = \text{constant} = \frac{c_\infty^2}{\gamma-1}$$

where

c_∞ = Sound speed at a large distance from the central object.

At the sonic point $u = c_s$ and $r = r_s = \frac{GM}{2c^2}$

$$\Rightarrow \quad \frac{c_s^2}{2} + \frac{c_s^2}{\gamma-1} - \underbrace{\frac{GM}{r_s}}_{2c_s^2} = \frac{c_\infty^2}{\gamma-1}$$

$$\Rightarrow \quad c_s^2 = \frac{2c_\infty^2}{5-3\gamma}$$

$$\Rightarrow \quad \rho_s = \rho_\infty \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}} \qquad \text{since } c^2 \propto \frac{P}{\rho} \propto \rho^{\gamma-1}$$

The accretion rate is

$$\dot{M} = 4\pi r_s^2 \rho_s c_s = \frac{4\pi (GM)^2}{4c_s^4} \rho_s c_s$$

$$= \frac{\pi (GM)^2 \rho_s}{c_s^3}$$

$$= \frac{\pi (GM)^2 \rho_\infty}{c_\infty^3} \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}} \left(\frac{5-3\gamma}{2}\right)^{3/2}$$

$$\boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_\infty^3} \left(\frac{2}{5-3\gamma}\right)^{\left(\frac{5-3\gamma}{2(\gamma-1)}\right)}}$$

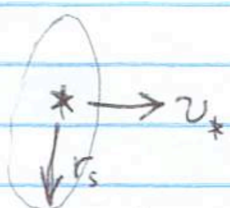
This is the Bondi accretion rate from Bondi (1952). For an isothermal flow $P = c^2\rho$ with $c^2 =$ constant, repeating this argument gives

$$\boxed{\dot{M} = \frac{\pi (GM)^2}{c^3}\rho_\infty\, e^{3/2}}$$

(which agrees with the previous expression in the limit $\gamma \to 1$)

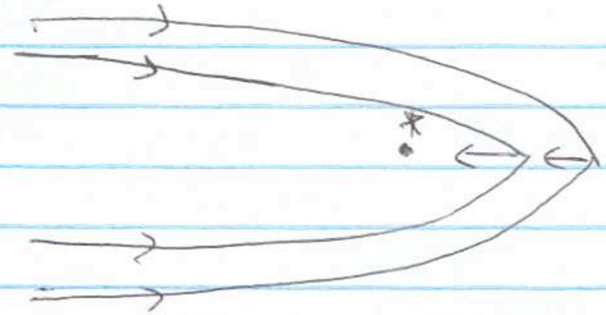
Hoyle & Lyttleton (1939) considered the related problem of a star moving through the ISM

[the idea was to power stellar luminosities by accretion — so that massive stars could have the same ages as the Sun!]



ρ_∞ $\qquad \dot{M} = \pi r_s^2 \rho_\infty v_*$

capture radius



capture radius from gravitational focussing

$$r_s \simeq \frac{GM}{v_*^2}$$

$$\Rightarrow \boxed{\dot{M} \approx \frac{\pi (GM)^2 \rho_\infty}{v_*^3}}$$

Bondi proposed an interpolation formula $\boxed{\dot{M} \simeq \frac{\pi (GM)^2 \rho_\infty}{(v_*^2 + c_\infty^2)^{3/2}}}$

"Bondi-Hoyle" accretion rate

Parker (1958) used the outgoing solution to model the solar wind. Properties of the solar wind had been inferred from observations of comet tails. Parker showed that they could be understood as an outflow originating in the $\simeq 3\times 10^6$K chromosphere of the Sun.

Relativistic hydrodynamics

(thermodynamic quantities are evaluated in proper frame)

stress-energy tensor $\quad T^{\mu\nu} = \underbrace{(e+p)}_{w} u^\mu u^\nu + p\,\eta^{\mu\nu}$

(metric $-1, +1, +1, +1$) $\quad u^\mu = \gamma(1, \underline{v}) \quad u^\mu u_\mu = -1$

in the fluid frame $\quad T^{\mu\nu} = \begin{pmatrix} e & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$

[e = energy per unit proper volume etc.]

equations of motion $\quad \partial_\mu T^{\mu\nu} = 0$

plus continuity $\quad \partial_\mu (n u^\mu) = 0$

Components of $T^{\mu\nu}$: $\quad T^{ij} = \dfrac{w\, v^i v^j}{c^2(1 - v^2/c^2)} + p\,\delta^{ij}$ } momentum flux density tensor

$$T^{00} = \frac{w}{1 - v^2/c^2} - p = \frac{e + p\,v^2/c^2}{1 - v^2/c^2}$$

$$T^{i0} = -\frac{w v^i}{c(1 - v^2/c^2)} = -w\,\beta^i \gamma^2$$ } energy flux density

– check the NR limit $\quad \cancel{T^{00} = e + p\frac{v^2}{c^2} + e\frac{v^2}{c^2} + O\left(\frac{v}{c}\right)^4}$

we should find

$$T^{00} = \rho c^2 + \rho\varepsilon + \tfrac{1}{2}\rho v^2$$

$$T^{ij} = \rho v^i v^j + p\,\delta^{ij}$$

$$\rho + \frac{U}{mc^2} - \frac{v^2}{2c^2}\left(\rho + \frac{U}{mc^2}\right)$$

in the rest frame, $\quad e = nmc^2 + nU$

write ρ = lab frame density $\quad c^2\rho = \gamma n m c^2$

$$\Rightarrow \quad e = \frac{\rho c^2}{\gamma} + \frac{U}{\ldots} = \rho c^2 (1 - \tfrac{v^2}{c^2})^{1/2} \simeq \rho c^2 - \frac{v^2}{2c^2}\rho c^2 + \frac{U}{\gamma m c^2}$$

November 1st, 2007

PHYS 643 lecture 15

Relativistic hydrodynamics

(see Choudhuri p 372
Landau & Lifshitz Chp XV)

There are many examples of flows that involve relativistic motions, eg. jets from black hole binaries, outflows from AGN, gamma-ray bursts.

Let's first look at the non-relativistic equations that we've dealt with so far, in conservative form.

First, momentum

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \partial_j u_i = -\partial_i P$$

$u_i \times$ (continuity) $\Rightarrow$ $$u_i \frac{\partial \rho}{\partial t} + u_i \partial_j (u_j \rho) = 0$$

add these $\Rightarrow$ $$\frac{\partial}{\partial t}(\rho u_i) + \partial_j(\rho u_i u_j) = -\partial_i P = -\partial_j(\delta_{ij} P)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho u_i) = -\partial_j \Pi_{ij}} \quad -(1)$$

where $\Pi_{ij} = \rho u_i u_j + P\delta_{ij}$

is the MOMENTUM FLUX DENSITY TENSOR

Similarly, the energy equation is

$$\boxed{\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho u^2 + \rho U\right)}_{\text{energy density}} = -\partial_j\Big(\underbrace{u_j\left[\frac{1}{2}\rho u^2 + \rho U + P\right]}_{\text{energy flux density}}\Big)} \quad -(2)$$

Once we introduce relativistic effects, we expect some mixing between energy and momentum, or in other words between eqns (1) + (2).

It turns out that indeed we can write the energy and momentum equations in terms of a single ENERGY-MOMENTUM TENSOR $T^{\mu\nu}$

$$T^{\mu\nu} = \left(\begin{array}{c|c} T^{00} & T^{i0} \\ \hline T^{0i} & T_{ij} \end{array}\right)$$

energy density

energy flux density

momentum flux density

[$T^{\mu\nu}$ is symmetric]

the energy equation is

$$\frac{\partial}{\partial t}\left(T^{00}\right) = -\partial_i T^{i0}$$

and momentum is $$\frac{\partial}{\partial t}\left(T^{i0}\right) = -\partial_j T^{ij}$$

or $$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

In terms of the 4-velocity of the fluid $u^\mu = \gamma(c, \underline{u})$,

$$\boxed{T^{\mu\nu} = \underbrace{(e+P)}_{w} \frac{u^\mu u^\nu}{c^2} + P\eta^{\mu\nu}}$$

where e = internal energy (includes rest mass)
P = pressure
$w = e + P$ = enthalpy

} all of these are evaluated in the proper frame of the fluid

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$$ is the metric

with this sign convention $u^\mu u_\mu = -c^2$

The components of $T^{\mu\nu}$ are

$$\boxed{\begin{aligned} T^{ij} &= \frac{w\,u^i u^j}{c^2(1-u^2/c^2)} + P\delta^{ij} \\ T^{00} &= \frac{w}{1-u^2/c^2} - P = \frac{e + P\,u^2/c^2}{1-u^2/c^2} \\ T^{i0} &= \frac{-w\,u^i}{c(1-u^2/c^2)} \end{aligned}}$$

(As I mentioned last time, these expressions have the expected non-relativistic limit, in which $e \to \frac{1}{2}\rho u^2 + \rho c^2 + \rho \mathcal{U}$)

To get the equations of motion in a coordinate independent way, we proceed as follows:

[But first, we need to talk about the continuity equation.

If the number density in the rest frame of a fluid element is n, the number density measured in a moving fluid element by a stationary observer is $n' = \gamma n$ because the volume is Lorentz contracted. Similarly, the number flux is $\gamma n \underline{u}$. The number flux 4-vector is

$$n^\mu = n u^\mu = \gamma n (c, \underline{u}).$$

The continuity equation is then $\boxed{\partial_\mu n^\mu = \partial_\mu (n u^\mu) = 0.}$]

1) First evaluate $\partial_\mu T^{\mu\nu} = \partial_\mu \left(w \frac{u^\mu u^\nu}{c^2} + P \eta^{\mu\nu} \right)$

$$= \frac{u^\nu}{c^2} \partial_\mu (w u^\mu) + w \frac{u^\mu}{c^2} \partial_\mu u^\nu + \partial^\nu P = 0$$

2) Now contract with u_ν

$$\Rightarrow \quad - \partial_\mu (w u^\mu) + u_\nu \partial^\nu P + \underbrace{w \frac{u_\nu u^\mu}{c^2} \partial_\mu u^\nu}_{\frac{w}{c^2} u^\mu \partial_\mu \cancel{\left(\frac{1}{2} u^\nu u_\nu\right)} \to 0} = 0$$

$$\Rightarrow \quad u_\nu \partial^\nu P - \partial_\mu (w u^\mu) = 0$$

now use continuity

$$n u_\nu \frac{\partial^\nu P}{n} - \partial_\mu \left(\frac{w}{n} n u^\mu \right) = 0$$

$$\Rightarrow \quad n u_\nu \left[\frac{\partial^\nu P}{n} - \partial_\nu \left(\frac{w}{n} \right) \right] = 0$$

But $\quad d\left(\frac{w}{n}\right) - \frac{dP}{n} = T\, d\left(\frac{s}{n}\right) \quad$ where s = entropy

$$\Rightarrow \quad n u_\nu T \partial^\nu \left(\frac{s}{n} \right) = 0$$

$\Rightarrow$ $\boxed{\frac{s}{n} \text{ is constant along particle trajectories} \quad \text{ie. } \underline{\text{adiabatic flow}}}$

3) Now project perpendicular to u_ν

$$u_\nu \cdot \left(\partial_\mu T^{\mu\nu} + \frac{u^\nu u_\sigma}{c^2} \partial_\mu T^{\mu\sigma} \right) = 0$$

evaluate the term in brackets

$$\Rightarrow \cancel{\frac{u^\nu}{c^2}\partial_\mu(wu^\mu)} + w\frac{u^\mu}{c^2}\partial_\mu u^\nu + \partial^\nu P$$

$$+ \frac{u^\nu u_\sigma}{c^2}\Big(\cancel{\frac{u^\sigma}{c^2}\partial_\mu(wu^\mu)} + \underbrace{wu^\mu\partial_\mu u^\sigma}_{\substack{0 \text{ as before} \\ \partial_\mu(u^\sigma)^2}} + \partial^\sigma P\Big)$$

$$\Rightarrow \frac{wu^\mu\partial_\mu u^\nu}{c^2} + \partial^\nu P + u^\nu u_\sigma \partial^\sigma P$$

$$\Rightarrow u_\nu \cdot \underbrace{\left(\frac{wu^\mu\partial_\mu u^\nu}{c^2} + \partial^\nu P + u^\nu u_\sigma \partial^\sigma P\right)}_{\text{constant along the flow}} = 0$$

set this constant to zero

$$\Rightarrow \boxed{wu^\mu\partial_\mu u^\nu/c^2 = -\partial^\nu P - u^\nu u_\mu \partial^\mu P}$$

* the relativistic generalization of Euler's equation. *

Now go through some examples:

① Sound waves

$$\delta T^{00} = \delta e \qquad [\ u=0 \text{ in the background}\ ] \Rightarrow \delta u^2 = 2u\,\delta u = 0$$

$$\delta T^{x0} = -w\frac{\delta u}{c}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\delta e - \frac{\partial}{\partial x}\left(w\frac{\delta u}{c}\right) = 0}$$

$$\delta T^{0x} = -w\,\delta u/c \qquad \delta T^{ix} = \delta P\,\delta^{ix}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(-w\frac{\delta u}{c}\right) + \frac{\partial}{\partial x}\delta P = 0}$$

Now write $\delta e = \delta p \left(\frac{\partial e}{\partial p}\right)_{ad} = \frac{c_s^2}{c^2}\,\delta p$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ↖ adiabatic

$$\Rightarrow \quad \delta\ddot{e} = \frac{w}{c}\frac{\partial}{\partial x}\delta\dot{u} = \boxed{\frac{\partial^2}{\partial x^2}\delta p = \frac{c_s^2}{c^2}\delta\ddot{p}}$$

wave speed is c_s

For an ultra-relativistic gas $p = e/3 \Rightarrow \boxed{c_s = \frac{c}{\sqrt{3}}}$

(2) Bernoulli's principle

Start with $\quad \frac{w u^\mu \partial_\mu u^\nu}{c^2} = -\partial^\nu p - u^\nu u_\mu \partial^\mu p$

for steady flow in some frame $\quad \frac{w\,\gamma\,\underline{v}\cdot\underline{\nabla}\,(\gamma\underline{v})}{c^2} = -\underline{\nabla}p - \gamma^2\underline{v}\,(\underline{v}\cdot\underline{\nabla})p$

Multiply by $\frac{v_j}{n}$: $\quad v_j\,\frac{w}{n}\,\frac{\gamma(\underline{v}\cdot\underline{\nabla})(\gamma v_j)}{c^2} = -[(\underline{v}\cdot\underline{\nabla})p]/n - \frac{\gamma^2 v^2(\underline{v}\cdot\underline{\nabla})p}{n}$

but $d\left(\frac{e}{n}\right) = -p\,d\left(\frac{1}{n}\right)$ if $ds = 0$

$$\Rightarrow \underline{\nabla}\left(\frac{p}{n}\right) = p\,\underline{\nabla}\left(\frac{1}{n}\right) + \frac{\underline{\nabla}p}{n} = \frac{\underline{\nabla}p}{n} - \underline{\nabla}\left(\frac{e}{n}\right)$$

$$\Rightarrow \boxed{\underline{\nabla}\left(\frac{w}{n}\right) = \frac{\underline{\nabla}p}{n}}$$

$\Rightarrow$ after a little more algebra and use of the identity $\boxed{\frac{\gamma^2 \underline{v}^2}{c^2} + 1 = \gamma^2}$

$$\Rightarrow \boxed{\underline{v}\cdot\underline{\nabla}\left(\frac{\gamma w}{n}\right) = 0}$$

$\Rightarrow$ Bernoulli's constant is $\boxed{\frac{\gamma w}{n} = \gamma\left(\frac{e+P}{n}\right)}$

(Compare the NR result $\frac{1}{2}u^2 + \frac{P}{\rho} + U$)

This is what Blandford and Rees (1974) applied to 1D flow from an AGN. They assume that the fluid is ultrarelativistic, so that

$$p = \tfrac{1}{3}e \qquad w = 4p \propto n^{4/3}$$

then $\frac{\gamma w}{n}$ = constant $\Rightarrow$ $\gamma n^{1/3}$ = constant

$\Rightarrow \gamma \propto n^{-1/3} \propto p^{-1/4}$ along the outflow

$\left[\text{or } p = p_0/\gamma^4 \quad \gamma = \left(\frac{p_0}{p}\right)^{1/4}\right]$

The energy flux is $L = w \underset{\text{velocity}}{\cancel{u}} \gamma^2 c A$ = constant
$\uparrow$ area of the jet

velocity $\frac{v}{c} = \frac{\sqrt{\gamma^2-1}}{\gamma}$ since $\frac{(\gamma v)^2}{c^2} = \gamma^2 - 1$

$$\Rightarrow \quad A = \frac{L}{w u \gamma^2}$$

$$= \frac{L}{4p_0}\left(\gamma^4\right) \cdot \frac{1}{\gamma^2} \cdot \frac{\gamma}{\sqrt{\gamma^2-1}} \frac{1}{c}$$

$$\boxed{A = \frac{L}{4p_0^{1/4}c}\, p^{-3/4}\left(\left(\frac{p_0}{p}\right)^{1/2} - 1\right)^{-1/2}}$$ area of the outflow

the minimum area occurs for $\boxed{p_* = \frac{4}{9}p_0}$ [check this by setting $\frac{\partial A}{\partial p} = 0$]

and is $A_{min} = \frac{3\sqrt{3}\,L}{8p_0 c}$

Finally one can show that $\gamma_* = \sqrt{\frac{3}{2}}$ or $v_* = \frac{c}{\sqrt{3}}$ = sound speed

→ So there is transsonic flow at the nozzle just as in the NR case.

③ Shock jump conditions (Blandford & McKee 1976 Phys. Fluids)

in the relativistic case, the jump conditions are (see components of $T^{\mu\nu}$ on page 3)

$$[n\beta\gamma] = 0 \qquad \text{continuity}$$
$$[(e+p)\gamma^2\beta] = 0 \qquad \text{energy flux}$$
$$[(e+p)\gamma^2\beta^2 + p] = 0 \qquad \text{momentum flux}$$

The simplest case to consider is a strong shock in the extreme relativistic limit: $p_1 = 0 \quad e_1 = n_1 mc^2$

$$p_2 = e_2/3 \Rightarrow w_2 = 4p_2$$

Work in the frame of the shock. Then the upstream fluid has $\beta_1 = 1$.

$$\Rightarrow \quad n_2\beta_2\gamma_2 = \gamma_1 n_1$$
$$4p_2\gamma_2^2\beta_2 = n_1 mc^2\gamma_1^2$$
$$4p_2\gamma_2^2\beta_2^2 + p_2 = \gamma_1^2 n_1 mc^2$$

$\Rightarrow$ (after some algebra)

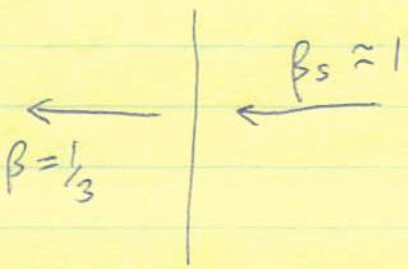
$$\boxed{\beta_2 = \frac{1}{3} \qquad \gamma_2 = \frac{3}{\sqrt{8}}}$$

$$\boxed{n_2 = \sqrt{8}\,\gamma_1 n_1}$$

$$\boxed{p_2 = \frac{2}{3}\gamma_1^2 e_1 \quad \text{or} \quad e_2 = 2\gamma_1^2 e_1}$$

Transform into the frame of the unshocked gas

Shock frame



If the shock has gamma factor $\Gamma \gg 1$ then $\beta_s \simeq 1 - \frac{1}{2\Gamma^2}$

BOOST this $\rightarrow \beta_s$

after applying the appropriate velocity transformation $\beta_2' = \frac{\beta_2 + \beta_s}{1 + \beta_2 \beta_s}$

one finds $\boxed{\gamma_2' = \frac{\Gamma}{\sqrt{2}}} + O\left(\frac{1}{\Gamma^2}\right)$

Lorentz factor of the shocked gas in the frame of the unshocked fluid

in this frame the jump conditions are: $\gamma_2' = \Gamma/\sqrt{2}$

$$n_2' = \gamma_2 n_2 = 2\Gamma^2 n_1$$

density of shocked gas as measured wrt unshocked gas

$$p_2 = \frac{2}{3}\Gamma^2 e_1 = \frac{1}{3} e_2$$

gas is compressed by factor Γ, energy density by Γ^2

Blandford & McKee (1976) give more general results for different degrees of relativity

Blandford & McKee apply these results to a

<u>Relativistic blast wave</u> (relativistic version of Sedov-Taylor)

The equations of motion are (see expressions for $T^{\mu\nu}$ given earlier)

$$\frac{\partial}{\partial t}\left(\frac{e+\beta^2 P}{1-\beta^2}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2(e+P)\beta}{1-\beta^2}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{(e+P)\beta}{1-\beta^2}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2(e+P)\beta^2}{1-\beta^2}\right) + \frac{\partial P}{\partial r} = 0$$

assume $p = \frac{1}{3}e$ behind the blast wave

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial r}\right)\left(p\gamma^4\right) = \gamma^2\frac{\partial p}{\partial t}$$

$$\left(\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial r}\right)\left(\ln p^3\gamma^4\right) = -\frac{4}{r^2}\frac{\partial}{\partial r}\left(r^2\beta\right)$$

and also we have $\quad \dfrac{\partial n'}{\partial t} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 n'\beta\right) = 0$

These equations imply that $\quad \left(\dfrac{\partial}{\partial t} + \beta\dfrac{\partial}{\partial r}\right)\left(\dfrac{p}{n^{4/3}}\right) = 0$

$(n' = \gamma n)$

Finally, there is a constraint on the total energy

$$\int 16\pi P\,\gamma^2 r^2\,dr$$

There is an <u>approximately</u> self-similar solution in which the shocked fluid lies in a shell of thickness R/Γ^2 (since $n_2' = 2\Gamma^2 n_1$)

Unlike the non-relativistic case, the energy is stored close to the shock front, also in the shocked material.

November 6, 2007.

PHYS 643 lecture 16

IV: Oscillations and Instabilities

Lagrangian vs. Eulerian perturbations

As we mentioned earlier, we can take an Eulerian or Lagrangian point of view when thinking about perturbations.

Let $\underline{x}_0$ label each fluid element (eg. $\underline{x}_0$ = initial position)

Then we define the displacement $\underline{\xi} = \underline{r}(\underline{x}_0, t) - \underline{r}_0(\underline{x}_0, t)$

position of the fluid element in the perturbed flow

position of the fluid element in the unperturbed flow.

The Eulerian perturbation in the quantity f is

$$\delta f(\underline{r}, t) = f(\underline{r}, t) - f_0(\underline{r}, t)$$

value in perturbed flow

value in unperturbed flow

The Lagrangian perturbation is

$$\Delta f(\underline{x}_0, t) = f(\underline{x}_0, t) - f_0(\underline{x}_0, t)$$

or $$\Delta f(\underline{r}, t) = f(\underline{r}, t) - f_0(\underline{r}_0, t)$$ where $\underline{r} = \underline{r}_0 + \underline{\xi}$

The relation between Δf and δf is therefore

$$\Delta f = \delta f + [\, f_0(\underline{r},t) - f_0(\underline{r}_0,t)\,]$$

To first order in $\underline{\xi}$, $\boxed{\Delta f = \delta f + \underline{\xi}\cdot\underline{\nabla} f_0}$.

Commutation relations: you can show that

$$\delta\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\delta f \qquad \delta(\underline{\nabla} f) = \underline{\nabla}\,\delta f$$

$$\frac{D}{Dt}(\Delta f) = \Delta\left(\frac{Df}{Dt}\right)$$

but
$$\delta\left(\frac{Df}{Dt}\right) \neq \frac{D}{Dt}\,\delta f \qquad \Delta\left(\frac{\partial f}{\partial t}\right) \neq \frac{\partial}{\partial t}(\Delta f)$$

$$\Delta(\underline{\nabla} f) \neq \underline{\nabla}(\Delta f)$$

Velocity perturbation

The fluid velocity in the perturbed and unperturbed flows is $\underline{u} = \dfrac{D}{Dt}\underline{r}$ and $\underline{u}_0 = \dfrac{D}{Dt}\underline{r}_0$

$$\Rightarrow \quad \Delta\underline{u} = \underline{u}(\underline{r},t) - \underline{u}_0(\underline{r}_0,t)$$

$$= \frac{D}{Dt}(\underline{r}-\underline{r}_0) = \boxed{\frac{\partial\underline{\xi}}{\partial t} + (\underline{u}_0\cdot\underline{\nabla})\underline{\xi} = \Delta\underline{u}}$$

The Eulerian perturbation is $\delta\underline{u} = \Delta\underline{u} - (\underline{\xi}\cdot\underline{\nabla})\underline{u}_0$

or
$$\boxed{\delta\underline{u} = \frac{\partial\underline{\xi}}{\partial t} + (\underline{u}_0\cdot\underline{\nabla})\underline{\xi} - (\underline{\xi}\cdot\underline{\nabla})\underline{u}_0}$$

if the background flow is stationary $\underline{u}_0 = 0$

then
$$\boxed{\delta\underline{u} = \Delta\underline{u} = \frac{D\underline{\xi}}{Dt} = \frac{\partial\underline{\xi}}{\partial t}}$$

eg. Continuity equation $\frac{\partial \rho}{\partial t} + \underline{\nabla}.(\rho \underline{u}) = 0$

when $\underline{u}_0 = 0$ $\frac{\partial}{\partial t} \delta\rho + \underline{\nabla}.(\rho \underbrace{\delta \underline{u}}_{\frac{\partial \underline{\xi}}{\partial t}}) = 0$

$$\Rightarrow \quad \boxed{\delta\rho + \underline{\nabla}.(\rho \underline{\xi}) = 0}$$

$$\text{or} \quad \boxed{\Delta\rho + \rho \underline{\nabla}.\underline{\xi} = 0}$$

You can show that this relation holds for $\underline{u}_0 \neq 0$ also.

Surface gravity waves

We've already seen that a uniform medium ~~support~~ supports compressible sound waves. Also, recall that when we introduced a magnetic field into the medium, the additional restoring force led to a set of incompressible Alfven waves.
(magnetic tension)

Now we'll discuss oscillation modes of stars. There is a set of incompressible modes restored by the bouyancy force — these are g-modes or gravity waves.

Look first at the simplest example: a plane-parallel layer of incompressible fluid, initially at rest, in hydrostatic balance with a vertical gravitational field — eg. the ocean

Continuity eqn

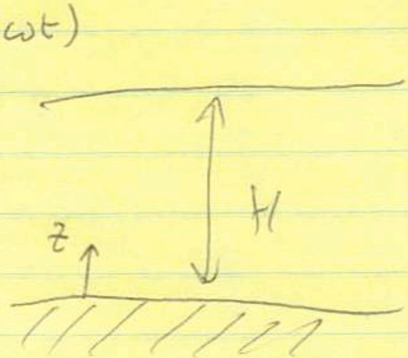
$$\Rightarrow \quad \Delta\rho = 0 = \underline{\nabla}.\underline{\xi} \qquad (\rho = \text{constant})$$

momentum $\Rightarrow$ $$\rho \frac{\partial^2 \underline{\xi}}{\partial t^2} = -\underline{\nabla}\,\delta P$$

(no gravity term $\delta\rho\, \underline{g}$ because $\delta\rho = 0$).

$$\Rightarrow \quad \boxed{\nabla^2 \delta p = 0}$$

Look for solutions $\quad \delta p = f(z)\, e^{i(k_\perp x - \omega t)}$



$$\Rightarrow \quad f'' - k_\perp^2 f = 0$$

$$\Rightarrow \quad \boxed{f = A e^{-k_\perp z} + B e^{+k_\perp z}}$$

What are the boundary conditions? At the floor of the ocean there is a hard surface $\Rightarrow \xi_z = 0$

the vertical momentum equation $\Rightarrow -\rho\omega^2 \xi_z = -\dfrac{d\delta p}{dz}$

$\Rightarrow$ we need $\quad \dfrac{df}{dz} = 0 \quad$ at $\quad z = 0$

$$\Rightarrow -k_\perp A + k_\perp B = 0 \quad \Rightarrow \quad \boxed{A = B}$$

At the top, the Lagrangian pressure change must vanish $\underline{\Delta p = 0}$

$$\Rightarrow \quad \delta p + \xi_z \frac{dp}{dz} = \delta p + \rho g \xi_z = 0 \quad \text{at } z = H$$

$$\Rightarrow \quad \underbrace{A e^{-k_\perp H} + B e^{k_\perp z}}_{\delta p} = -\rho g \cdot \underbrace{\frac{k_\perp}{\rho\omega^2}\left[-A e^{-k_\perp H} + B e^{k_\perp H}\right]}_{\xi_z}$$

$$\Rightarrow \quad \boxed{\tanh(k_\perp H) = \frac{\omega^2}{g k_\perp}}$$

This is the dispersion relation for the waves

Two limits 1) deep $k_\perp H \gg 1$ $\tanh(k_\perp H) \to 1$

$$\Rightarrow \boxed{\omega^2 \simeq g k_\perp}$$

2) shallow $k_\perp H \ll 1$ $\tanh(k_\perp H) \simeq k_\perp H$

$$\Rightarrow \boxed{\omega^2 = g k_\perp^2 H}$$

What do the eigenfunctions look like?

$$\delta p \propto \cosh(k_\perp z)$$
$$\xi_z \propto \sinh(k_\perp z)$$

in the deep limit $\delta p \propto \xi_z \propto e^{k_\perp z}$

" shallow " $\delta p = \text{constant}$
$\xi_z \propto z$

What about the horizontal displacement?
if we write $\delta p = \cosh(k_\perp z)$
then
$$\xi_z = \frac{k_\perp}{\rho\omega^2} \sinh(k_\perp z)$$
$$\xi_\perp = \frac{k_\perp}{\rho\omega^2} \cosh(k_\perp z)$$
(since the momentum eqns are $\rho\omega^2 \xi_z = \frac{d\delta p}{dz}$ and $\rho\omega^2 \xi_\perp = k_\perp \delta p$)

$$\Rightarrow \frac{\xi_z}{\xi_\perp} = \tanh(k_\perp z)$$

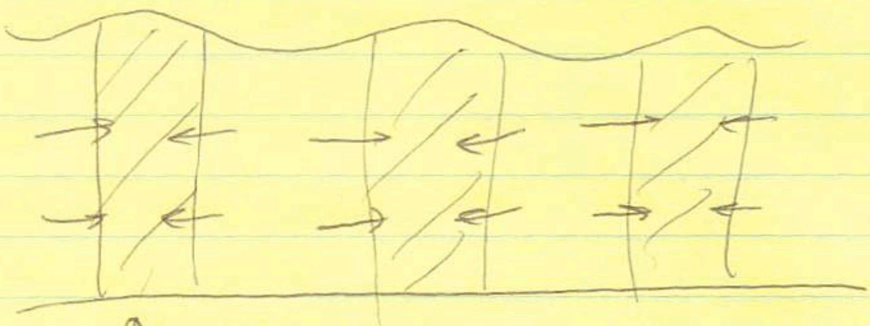
For deep waves $\xi_z \simeq \xi_\perp$ "circular motions"
shallow waves $\frac{\xi_z}{\xi_\perp} \simeq k_\perp H \ll 1$ eg. Tsunami

Estimate typical periods for water waves:

deep limit $\quad$ period $\simeq \frac{2\pi\lambda}{g} \simeq$ 1 second for $\lambda = 1\text{m}$

shallow $\quad$ period $\simeq \frac{\lambda}{\sqrt{gH}} \simeq$ 1 hour for $\lambda = 1000$ km, $H = 10$ km (Tsunami case)

Physics of the wave:



material builds up in the crests $\Rightarrow$ pressure gradient horizontally
$\Rightarrow$ restoring force

November 8, 2…

$\boxed{\text{PHYS 643 lecture 17}}$

Waves in a plane-parallel atmosphere

The background is in hydrostatic balance $\frac{dP}{dz} = -\rho g$

and static.

Continuity: $\boxed{\frac{\Delta\rho}{\rho} = -\nabla\cdot\underline{\xi}}$

or $\boxed{\frac{\Delta\rho}{\rho} = -\rho\frac{d\xi_z}{dz} - ik_\perp\xi_\perp}$ — (1)

Assume the perturbations are <u>adiabatic</u> $\boxed{\frac{\Delta p}{P} = \gamma\frac{\Delta\rho}{\rho}}$ — (1a)

ie. $$\frac{\delta p}{P} + \frac{\xi_z}{P}\frac{dP}{dz} = \gamma\left(\frac{\delta\rho}{\rho} + \frac{\xi_z}{\rho}\frac{d\rho}{dz}\right)$$

$$\frac{\delta p}{P} - \gamma\frac{\delta\rho}{\rho} = \gamma\xi_z\underbrace{\left(\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz}\right)}_{\text{this quantity is known as the convective discriminant } A}$$

$$A = \left(\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz}\right)$$

We also define the Brunt-Väisälä frequency or ~~by~~ buoyancy frequency N by $N^2 = -gA$

$$\Rightarrow \boxed{\frac{1}{\gamma}\frac{\delta p}{P} - \frac{\delta\rho}{\rho} = -\frac{\xi_z N^2}{g}} \text{ — (2)}$$

or

$$\frac{\delta p}{c_s^2} - \delta\rho = -\frac{\rho\,\xi_z N^2}{g}$$

The momentum equation is

$$\boxed{\begin{aligned} -\rho\omega^2 \xi_z &= -\frac{d}{dz}\delta p - g\,\delta\rho \quad &(3)\\ -\rho\omega^2 \xi_\perp &= -ik_\perp \delta p \quad &(4)\end{aligned}}$$

First substitute (2) for $\delta\rho$ in (3)

$$\Rightarrow \quad \boxed{\frac{d}{dz}\delta P + \frac{g}{c_s^2}\delta P = \rho(\omega^2 - N^2)\,\xi_z} \quad (*)$$

Next substite for $\xi_\perp$ and $\frac{\Delta\rho}{\rho}$ in (1) using (4) and (1a)

$$\Rightarrow \quad \boxed{\frac{d\xi_z}{dz} - \frac{g}{c_s^2}\xi_z = \frac{1}{\gamma}\frac{\delta P}{P}\left(\frac{c_s^2 k_\perp^2}{\omega^2} - 1\right)} \quad (**)$$

These are two coupled equations for ξ_z and δP. With boundary conditions, they form an eigenvalue problem for ω^2.

To get a sense of the solutions, try a WKB solution

$$\xi_z,\ \delta p \ \propto\ e^{ik_z z}$$

where we assume that $k_z H \gg 1$ ie short vertical wavelengths

(H is the pressure scale height $\quad H = -\frac{P}{dP/dz} = \frac{P}{\rho g} = \frac{c_s^2}{\gamma g}$)

This means that we can ignore the second terms in (*) and (**).
Now multiply (*) and (**) :

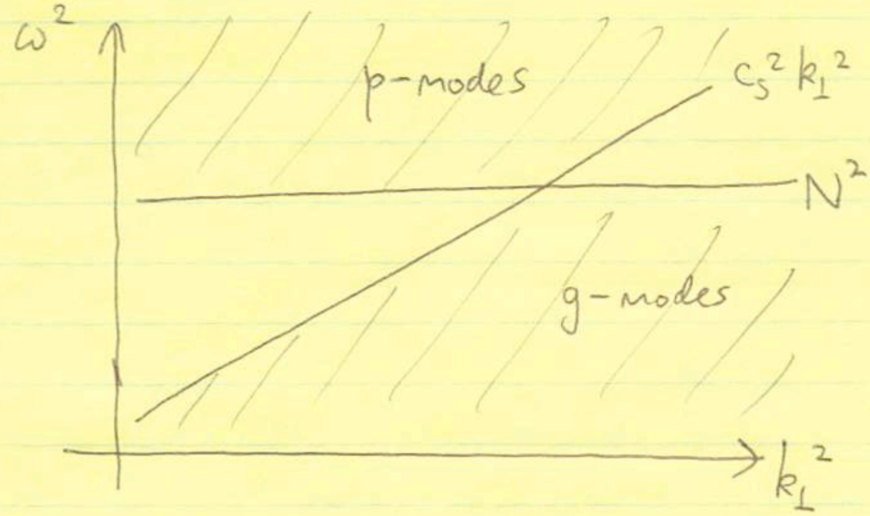
$$\Rightarrow \quad -k_z^2 = \frac{1}{c_s^2}(\omega^2 - N^2)\left(\frac{c_s^2 k_\perp^2}{\omega^2} - 1\right)$$

or

$$\boxed{c_s^2 k_z^2 = (\omega^2 - N^2)\left(1 - \frac{k_\perp^2 c_s^2}{\omega^2}\right)}$$

DISPERSION RELATION

Propagation diagram:



vertically propagating waves ($k_z^2 > 0$) only exist in the shaded regions

p-modes $\quad \omega^2 \gg N^2$ limit $\quad \omega^2 = c_s^2 (k_\perp^2 + k_z^2)$

$$\boxed{\omega^2 = c_s^2 k^2}.$$

compressible sound waves

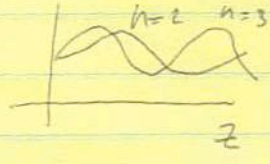
g-modes $\quad \left\{ \begin{matrix} \omega^2 \ll N^2 \\ \omega^2 \ll c_s^2 k^2 \end{matrix} \right\}$ limit $\quad \boxed{\omega^2 = \frac{N^2 k_\perp^2}{k^2}}$

these waves are incompressible.
They also have the property that the group velocity is perpendicular to the phase velocity.

If the atmosphere extends over a vertical scale L, there is a ~~finite~~ discrete set of mode frequencies that satisfy the boundary conditions, ie. an integer number of wavelengths must fit into L

We can write this as $\int_0^L k_z \, dz = n\pi$ in WKB.

Then $k_z = \frac{2\pi}{\lambda_z} \simeq 2\pi \frac{n}{L}$ ← number of nodes



when the vertical wavelength is much shorter than the horizontal one, $k_z \gg k_\perp$, we see that $\omega \propto k_z \propto n$ for p-modes

and $\omega \propto k_z \propto \frac{1}{n^2}$ for g-modes

In spherical geometry, the horizontal eigenfunctions are no longer $e^{ik_x x}$ but are $Y_{\ell m}$'s. The horizontal and radial solutions seperate, and the radial equations are the same as derived here but with $k_\perp^2 = \frac{\ell(\ell+1)}{r^2}$.

← ~~stellar~~ radial coordinate

The next pages show plots of N^2 and $c_s^2 k_\perp^2$ for the Sun, the range of predicted mode frequencies and some example eigenfunctions.

You can see the phenomenon of "mode trapping" in these figures. For example, take the g-modes: they can only propagate where $\omega^2 < N^2$ — so in the region $r/R \simeq 0-0.7$. Outside this region, the waves are evanescent. Similarly, the p-modes can only propagate where $\omega^2 > c_s^2 k^2 \propto \ell(\ell+1)$. The p-modes are trapped more and more towards the surface as ℓ increases.

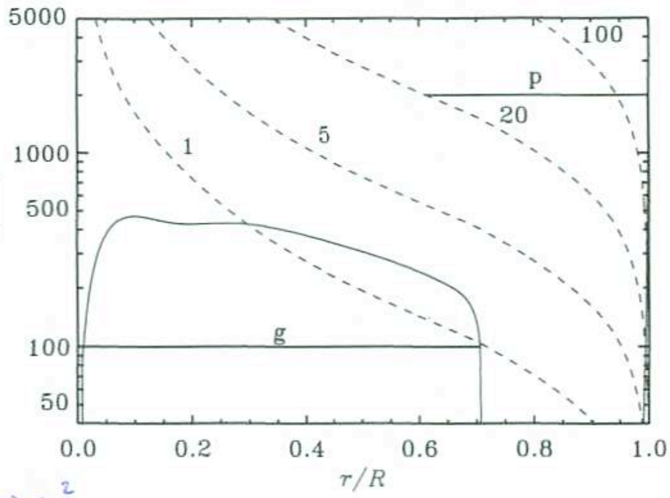
4a

These two equations can be combined into a single second-order differential equation for ξ_r; neglecting again derivatives of equilibrium quantities, the result is

$$\frac{\mathrm{d}^2\xi_r}{\mathrm{d}r^2} = \frac{\omega^2}{c^2}\left(1 - \frac{N^2}{\omega^2}\right)\left(\frac{S_l^2}{\omega^2} - 1\right)\xi_r \, . \tag{5.17}$$

These pages are from "Lecture Notes on Stellar Oscillations" by Jorgen Christensen-Dalsgaard

Brunt $N^2 = g\left(\frac{1}{\Gamma}\frac{d\ln P}{dr} - \frac{d\ln \rho}{dr}\right)$



acoustic frequency $S_\ell^2 = \frac{\ell(\ell+1)c_s^2}{r^2}$

Figure 5.2: Buoyancy frequency N [*cf.* equation (4.63); continuous line] and characteristic acoustic frequency S_l [*cf.* equation (4.60); dashed lines, labelled by the values of l], shown in terms of the corresponding cyclic frequencies, against fractional radius r/R for a model of the present Sun. The heavy horizontal lines indicate the trapping regions for a g mode with frequency $\nu = 100\,\mu$Hz, and for a p mode with degree 20 and $\nu = 2000\,\mu$Hz.

This equation represents the crudest possible approximation to the equations of non-radial oscillations. In fact the assumptions going into the derivations are questionable. In particular, the pressure scale height becomes small near the stellar surface (notice that $H_p = p/(\rho g)$ is proportional to temperature), and so derivatives of pressure and density cannot be neglected. I return to this question in Chapter 7. Similarly, the term in $2/r$ neglected in equation (5.12) is large very near the centre. Nevertheless, the equation is adequate to describe the overall properties of the modes of oscillation, and in fact gives a reasonably accurate determination of their frequencies.

From equation (5.17) it is evident that the characteristic frequencies S_l and N, defined in equations (4.60) and (4.63), play a very important role in determining the behaviour of the oscillations. They are illustrated in Figure 5.2 for a "standard" solar model. S_l tends to infinity as r tends to zero and decreases monotonically towards the surface, due to the decrease in c and the increase in r. As discussed in Section 3.3, N^2 is negative in convection zones (although generally of small absolute value), and positive in convectively stable regions. All normal solar models have a convection zone in the outer about 30 per

4b

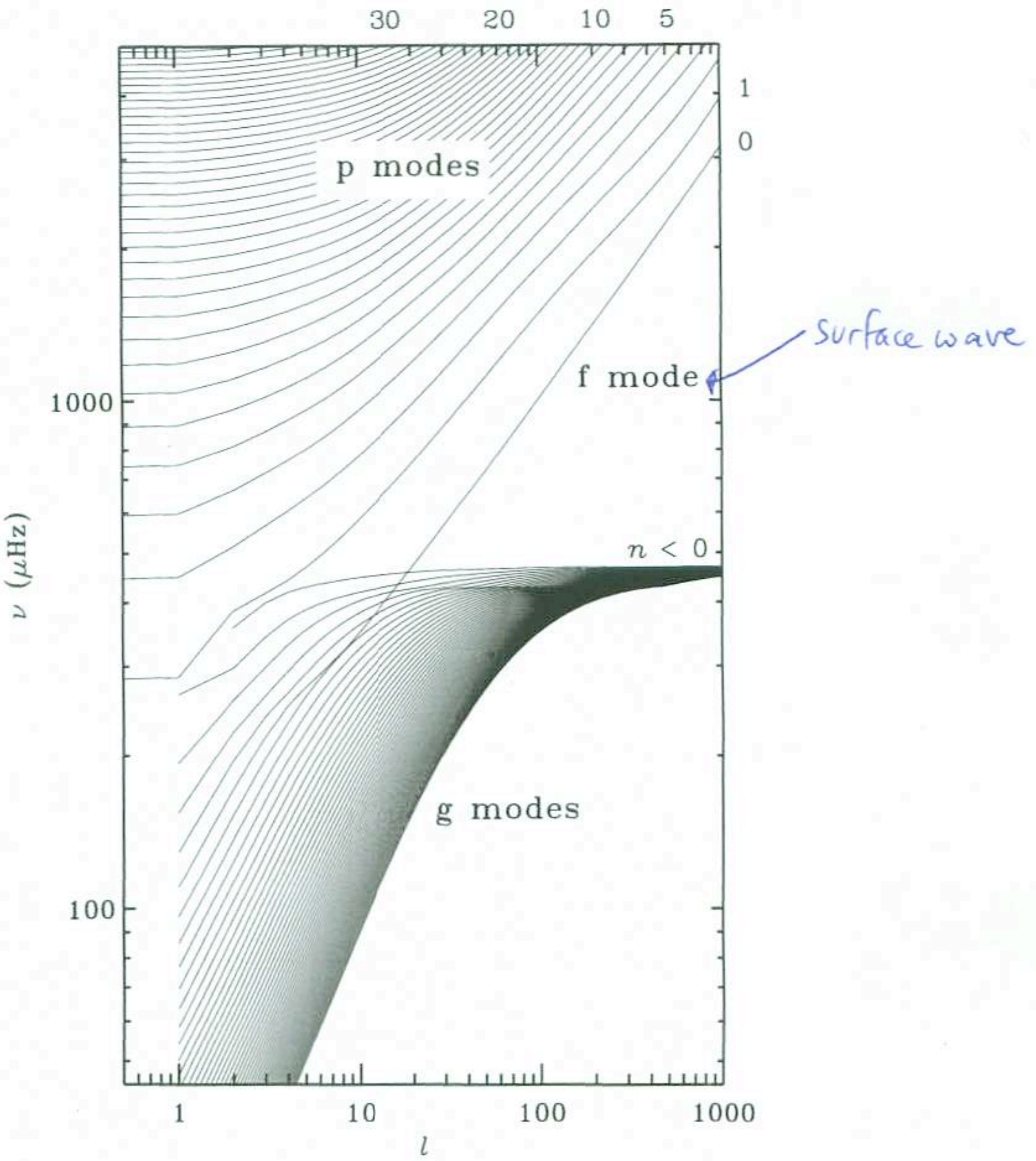


Figure 5.6: Cyclic frequencies $\nu = \omega/2\pi$, as functions of degree l, computed for a normal solar model. Selected values of the radial order n have been indicated.

The precise classification of the modes, *i.e.*, the assignment of radial orders to them, presents some interesting and so far unsolved problems. It appears that at each l it is possible to assign to each mode an integral order n, which ranges from minus to plus infinity, such that, at least for reasonably simple stellar models[3] $|n|$ gives the number of

[3]The definition of a 'simple' model in this context is not straightforward; examples might be zero-age main sequence models or, *e.g.*, polytropes of index between 1.5 and 3.

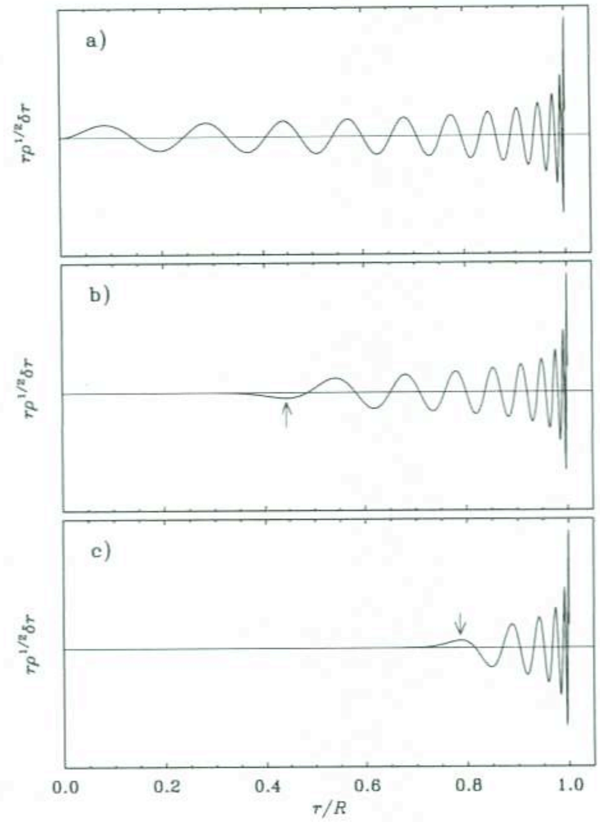


Figure 5.8: Scaled radial displacement eigenfunctions for selected p modes in a normal solar model, with a) $l = 0$, $n = 23$, $\nu = 3310\,\mu$Hz; b) $l = 20$, $n = 17$, $\nu = 3375\,\mu$Hz; c) $l = 60$, $n = 10$, $\nu = 3234\,\mu$Hz. The arrows mark the asymptotic location of the turning points r_t [*cf.* equation (5.28)].

Exercise 5.2:

Verify this statement.

It is interesting that this f mode with $l = 1$ behaves very differently in the Cowling approximation and for the full set of equations. In the Cowling approximation there is a mode with $l = 1$ having no nodes in the radial displacement, intermediate in frequency between the p and the g modes, which must be identified with the f mode. From a physical point of view it can be thought of roughly as an oscillation of the whole star in the gravitational potential defined by the equilibrium model. The connection between this mode and the zero-frequency mode for the full problem can be investigated by making a continuous transition from the Cowling approximation to the full set of equations; this can be accomplished formally by introducing a factor λ on the right-hand side of equation (4.21), 4d

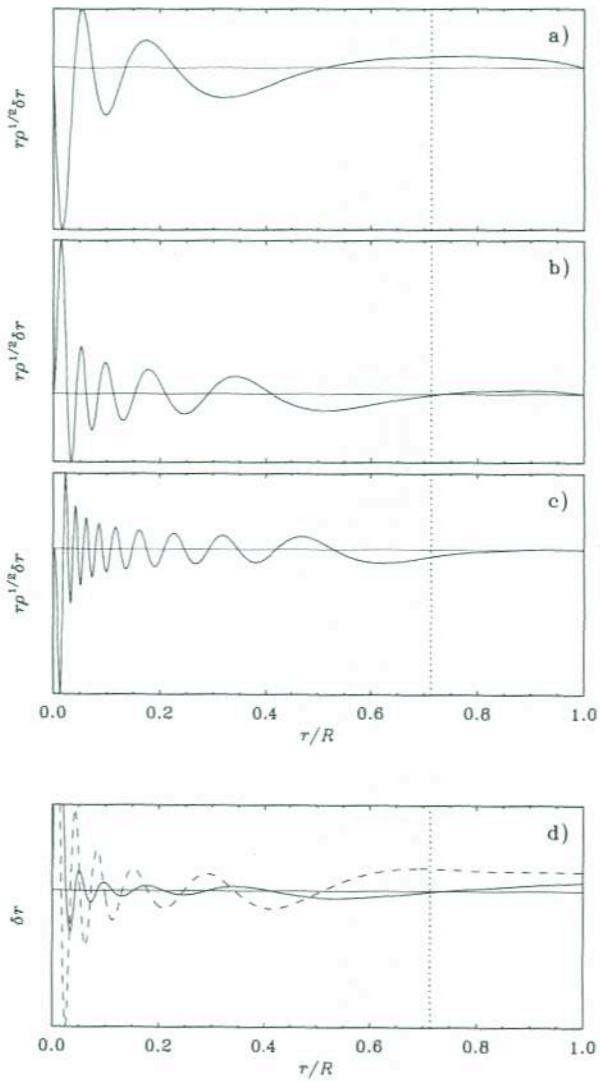


Figure 5.10: Eigenfunctions for selected g modes in a normal solar model. Panels a) to c) show scaled radial displacement eigenfunctions with a) $l = 1$, $n = -5$, $\nu = 110\,\mu$Hz; b) $l = 2$, $n = -10$, $\nu = 103\,\mu$Hz; c) $l = 4$, $n = -19$, $\nu = 100\,\mu$Hz. In panel d) the solid and dashed curves show unscaled radial (ξ_r) and horizontal displacement ($L\xi_{\rm h}$) eigenfunctions, for the $l = 2$, $n = -10$ mode. For clarity, the curve for ξ_r has been truncated: the maximum value is about 2.7 times higher than the largest value shown. The vertical dotted line marks the base of the convective envelope.

Figure 5.9 shows the eigenfunctions in the outer few per cent of the radius of a solar model, for modes of degree $l = 1$ with different frequencies. It is evident that the mode energy decreases in the atmosphere; this can be understood from the discussion in Sec-

Convective instability

We've been implicitly assuming that $N^2 > 0$, but that need not be the case for a particular stellar model. One way to see this is to rewrite N^2 in terms of the temperature gradient.

We write $\frac{d\ln p}{dz} = \chi_\rho \frac{d\ln\rho}{dz} + \chi_T \frac{d\ln T}{dz}$

where $\chi_T = \left.\frac{\partial \ln p}{\partial \ln T}\right|_\rho$ and $\chi_\rho = \left.\frac{\partial \ln p}{\partial \ln \rho}\right|_T$

then
$$A = \frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln p}{dz}$$
$$= \frac{d\ln p}{dz}\left(-\frac{1}{\gamma} + \frac{1}{\chi_\rho}\right) - \frac{\chi_T}{\chi_\rho}\frac{d\ln T}{dz}$$

Then use the thermodynamic identity $\gamma - \chi_\rho = \gamma \nabla_{ad} \chi_T$

where $\nabla_{ad} = \left.\frac{\partial \ln T}{\partial \ln P}\right|_{ad}$

$$\Rightarrow \boxed{A = -\frac{1}{H}\frac{\chi_T}{\chi_\rho}\left(\nabla_{ad} - \left.\frac{d\ln T}{d\ln P}\right|_*\right)}$$

→ temperature gradient with respect to pressure in the star

where $H = -\frac{dz}{d\ln P}$ is the pressure scale height

For $\left.\frac{d\ln T}{d\ln P}\right|_* < \nabla_{ad}$ then $A < 0$ and $N^2 = -gA > 0$

But if the temperature gradient is too steep $\left.\frac{d\ln T}{d\ln P}\right|_* > \nabla_{ad}$

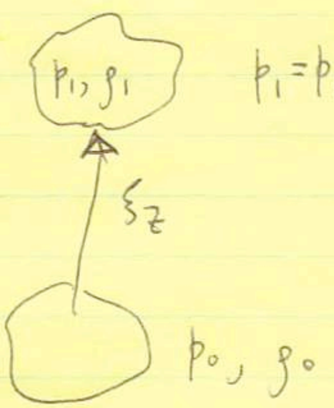
then $A > 0$ and $N^2 < 0$. In that case the perturbations grow exponentially in time $\omega^2 < 0 \Rightarrow \delta p, \xi_z \propto e^{\gamma t}$

with $\gamma = i\omega = $ real

There is an <u>instability</u> that leads to <u>convection</u>. If we make a stellar model and the temperature gradient (set by the opacity $F = \frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$) is $\left.\frac{d\ln T}{d\ln p}\right|_* > \nabla_{ad}$ then we need to include heat transport by convection.

The criterion $A > 0$ for instability is known as the <u>Schwarzschild criterion</u> for convection. There is a simple argument to see where it comes from that is given in stellar structure books.

Displace a fluid element upwards, allowing it to come into pressure balance with its surroundings, but its entropy doesn't change (adiabatic)



$$p_1 = p_{1*} = p_0 + \xi_z \left.\frac{dP}{dz}\right|_*$$

$$\rho_1 = \rho_0 \left(\frac{p_1}{p_0}\right)^{1/\gamma}$$

(adiabatic)

this is the new density of the fluid element

$$\rho_1 = \rho_0 \left(1 + \frac{\xi_z}{p_0}\frac{dp}{dz}\right)^{1/\gamma}$$

$$\simeq \rho_0 \left(1 + \frac{1}{\gamma}\, \xi_z \left.\frac{d\ln p}{dz}\right|_* \right)$$

the density of the stellar background at the new location is

$$\rho_* = \rho_0 + \xi_z \left.\frac{d\rho}{dz}\right|_*$$

The argument is then if $\rho_1 > \rho_*$ (fluid element heavier than surroundings)

it will fall back

but if $\rho_1 < \rho_*$ the fluid element is less dense than its surroundings, it will keep going — instability.

The condition for instability is therefore that

$$\rho_1 < \rho_*$$

$$\rho_0\left(1 + \frac{1}{\gamma}\xi_z \frac{d\ln p}{dz}\Big|_*\right) < \rho_0\left[1 + \xi_z \frac{d\ln \rho}{dz}\Big|_*\right]$$

$$\Rightarrow \quad \frac{1}{\gamma}\frac{d\ln p}{dz} < \frac{d\ln \rho}{dz}$$

or $\boxed{A = \frac{d\ln \rho}{dz} - \frac{1}{\gamma}\frac{d\ln p}{dz} > 0}$ <u>is unstable</u>

Since γ is defined as $\gamma = \frac{d\ln p}{d\ln \rho}\Big|_{ad}$ $\left[\text{then } dS \propto \frac{d\ln p}{\gamma} - d\ln \rho\right]$

then A can be rewritten as $\boxed{A \propto -\frac{dS}{dz}}$

the <u>unstable</u> situation is

$$\frac{dS}{dz} < 0$$

ENTROPY DECREASING OUTWARDS IS UNSTABLE

	Stable		unstable
$g\downarrow$	high entropy	$\uparrow z$	low entropy
	low entropy		high entropy

November 13, 2007.

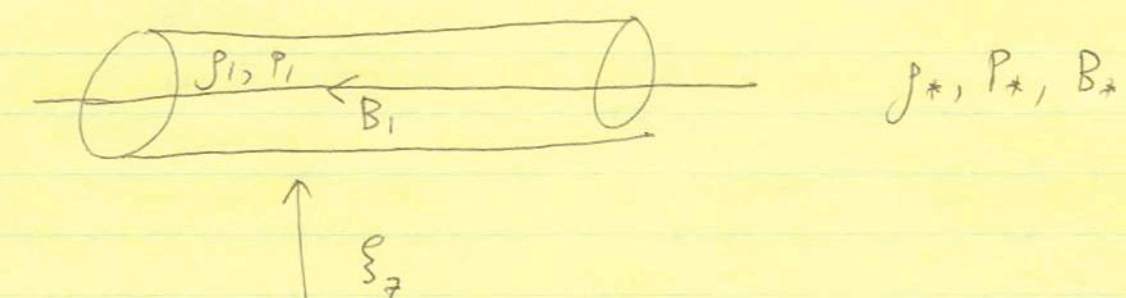
PHYS 643 lecture 18

Interchange and Parker instabilities

(see Newcomb 1961)
Parker 1966

What happens if we add a magnetic field to the atmosphere we considered last time? A vertical magnetic field will presumably not change the stability criterion since the pressure gradient vanishes along the field. But what about a horizontal field?

Let's repeat the simple argument from last time, but now we displace a field line and associated cylinder of fluid vertically. We consider perturbations in which the field line does not bend ($k=0$ along the field) — these perturbations are known as <u>interchanges</u>.



Pressure balance $\Rightarrow$ $$P_1 + \frac{B_1^2}{8\pi} = P_* + \frac{B_*^2}{8\pi}$$

for an adiabatic perturbation
$$P_1 = \left(\frac{\rho_1}{\rho_0}\right)^\gamma P_0$$

and flux conservation $\Rightarrow$ $$B \propto \frac{1}{(\text{area})} \propto \rho$$

[since the mass per unit length $= \rho A$ in the cylinder is conserved]

$$\Rightarrow \quad B_1 = B_0\left(\frac{\rho_1}{\rho_0}\right)$$

$$\Rightarrow \quad \left(\frac{\rho_1}{\rho_0}\right)^\gamma p_0 + \frac{B_0^2}{8\pi}\left(\frac{\rho_1}{\rho_0}\right)^2 = p_0 + \xi_z \frac{dP}{dz}\bigg|_* + \frac{1}{8\pi}\left(B_0 + \xi_z \frac{dB}{dz}\bigg|_*\right)^2$$

write $\rho_1 = \rho_0 + \delta\rho$

to first order in $\frac{\delta\rho}{\rho}$:

$$\gamma p_0 \frac{\delta\rho}{\rho_0} + 2\frac{B_0^2}{8\pi}\frac{\delta\rho}{\rho_0} = \underbrace{\xi_z\left(\frac{dP}{dz}\bigg|_* + \frac{d(B_0^2/8\pi)}{dz}\bigg|_*\right)}_{\frac{dP_{tot}}{dz}\big|_*}$$

where $P_{tot} = P + B^2/8\pi$ is the total pressure

$$\Rightarrow \quad \frac{\delta\rho}{\rho} = \frac{\xi_z \frac{dP_{tot}}{dz}\big|_*}{\gamma P + 2\left(B_0^2/8\pi\right)}$$

the condition for STABILITY is that this be larger than the change in the background density, $\xi_z \frac{d\rho}{dz}$

$$\Rightarrow \left\{\begin{array}{c}\text{STABLE}\\ \text{against}\\ \text{interchanges}\end{array}\right\} \text{if} \quad \boxed{\frac{d\ln\rho}{dz}\bigg|_* < \frac{\frac{dP_{tot}}{dz}}{\gamma P + 2B^2/8\pi}}$$

How do we obtain this condition from the fluid equations? A useful approach to analyse the stability of fluid–magnetic field configurations is an <u>energy principle</u>.

Start with the momentum equation

$$\rho \frac{\partial^2 \underline{\xi}}{\partial t^2} = -\underline{\nabla}\delta p + \delta\rho\, \underline{g} + \frac{1}{c}\delta(\underline{J}\times\underline{B})$$

Take $\int \underline{\xi}^* \cdot (\text{momentum eqn})\, dV$

1) First consider the terms $-\underline{\xi}^*\cdot\underline{\nabla}\delta p + \underline{\xi}^*\cdot\underline{g}\,\delta\rho$

and use $\delta\rho = -\underline{\nabla}\cdot(\rho\underline{\xi})$ (continuity)

$$\frac{1}{\gamma}\frac{\delta p}{p} = -\frac{1}{\gamma}\frac{\underline{\xi}\cdot\underline{\nabla}P}{P} + \frac{\delta\rho}{\rho} + \frac{\underline{\xi}\cdot\underline{\nabla}\rho}{\rho}$$ (adiabatic perturbation)

or $\delta p = -\underline{\xi}\cdot\underline{\nabla}P + \gamma P\left(\frac{\Delta\rho}{\rho}\right)$

$\therefore$ we have $-\underline{\xi}^*\cdot\underline{\nabla}\delta P + \underline{\xi}^*\cdot g\,\delta\rho$

$$= +\underbrace{(\underline{\xi}^*\cdot\underline{\nabla})(\underline{\xi}\cdot\underline{\nabla})P} + \underbrace{\underline{\xi}^*\cdot\underline{\nabla}\left[\gamma P\,\underline{\nabla}\cdot\underline{\xi}\right]} - (\underline{\xi}^*\cdot\underline{g})\,\underline{\nabla}\cdot(\rho\underline{\xi})$$

$$\underbrace{\underline{\nabla}\cdot\left[\cancel{\underline{\xi}^*(\underline{\xi}\cdot\underline{\nabla})P}\right]}_{\text{surface term when we integrate over volume — assume vanishes}} - (\underline{\xi}\cdot\underline{\nabla}P)(\underline{\nabla}\cdot\underline{\xi}^*) \qquad \overbrace{\gamma\underline{\nabla}\cdot\left[\cancel{\underline{\xi}^* P\underline{\nabla}\cdot\underline{\xi}}\right]}^{\text{Surface term}} - \gamma P(\underline{\nabla}\cdot\underline{\xi})(\underline{\nabla}\cdot\underline{\xi}^*)$$

0 $\quad$ 0

$$\boxed{= -(\underline{\xi}\cdot\underline{\nabla}P)(\underline{\nabla}\cdot\underline{\xi}^*) - \gamma P|\underline{\nabla}\cdot\underline{\xi}|^2 - \underline{\nabla}\cdot(\rho\underline{\xi})(\underline{\xi}^*\cdot\underline{g})}$$

2) Now $\underline{\xi}^*\cdot\left(\frac{1}{c}(\delta\underline{J}\times\underline{B}) + \frac{1}{c}\underline{J}\times\delta\underline{B}\right)$

$$\left(\underline{\nabla}\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot\underline{\nabla}\times\underline{A} - \underline{A}\cdot\underline{\nabla}\times\underline{B}\right)$$

$$\frac{\underline{\xi}^*\cdot\delta\underline{J}\times\underline{B}}{c} = -\frac{1}{4\pi}(\underline{\nabla}\times\delta\underline{B})\cdot(\underline{\xi}^*\times\underline{B})$$

$$= -\underline{\nabla}\cdot\left[\delta\underline{B}\times(\underline{\xi}^*\times\underline{B})\right]/4\pi - \delta\underline{B}\cdot\underline{\nabla}\times(\underline{\xi}^*\times\underline{B})/4\pi$$

but from the induction equation $\quad \delta\underline{B} = \underline{\nabla} \times (\underline{\xi} \times \underline{B})$

$$\Rightarrow \quad \boxed{\frac{\underline{\xi}^* \cdot \delta\underline{J} \times \underline{B}}{c} = -\frac{|\delta\underline{B}|^2}{4\pi}} + (\text{surface term})$$

the other term is

$$\frac{\underline{\xi}^* \cdot \underline{J} \times \delta\underline{B}}{c} = \boxed{-\frac{1}{4\pi}(\underline{\nabla} \times \underline{B}) \cdot (\underline{\xi}^* \times \delta\underline{B})}$$

3) the LHS is $\boxed{-\omega^2 \int dV\, \rho\, |\underline{\xi}|^2}$

now put it all together

$\Rightarrow$

$$\omega^2 \int \rho |\xi|^2 dV = 2W$$

energy associated with the perturbation

$$\boxed{2W = \int dV \left[\frac{|\delta\underline{B}|^2}{4\pi} + \frac{(\underline{\nabla} \times \underline{B}) \cdot (\underline{\xi}^* \times \delta\underline{B})}{4\pi} + \gamma P |\underline{\nabla} \cdot \underline{\xi}|^2 + (\underline{\xi} \cdot \underline{\nabla} P)(\underline{\nabla} \cdot \underline{\xi}^*) + (\underline{\xi}^* \cdot \underline{g})\, \underline{\nabla} \cdot (\rho \underline{\xi}) \right]}$$

The idea is that if $W > 0$ for a particular perturbation then the configuration is stable to that perturbation.
To address stability the idea is to find out whether there is any perturbation for which $W < 0 \Rightarrow$ unstable.

The energy principle can be written in different ways – we follow Newcomb (1961) here. All the different MHD instabilities – interchange, pinch, kink etc. can be understood as different terms in the energy integral – see Greene & Johnson (1968 Plasma Phys 10, 729) and the original Bernstein et al. (1958).

Now apply this to our atmosphere – this is the argument from Newcomb (1961).
The background has $\frac{d}{dz}\left(P + \frac{B^2}{8\pi}\right) = -\rho g$

with $\underline{B} = \hat{\underline{x}}\, B(z)$, $\underline{J} = \frac{c}{4\pi} \nabla \times \underline{B} = \frac{c}{4\pi}\frac{dB}{dz}\hat{\underline{y}}$

after quite a lot of algebra, which is given on pages 5a–5c, we find:

$$\boxed{\begin{aligned} 2W = \int dz \Big[& \frac{B^2}{4\pi} \left| \xi_z' + i k_y \xi_y \right|^2 + k_x^2 \frac{B^2}{4\pi}\left(|\xi_y|^2 + |\xi_z|^2\right) \\ & + \gamma P \left|\nabla\cdot\underline{\xi}\right|^2 - \rho' g |\xi_z|^2 - \rho g \left[\xi_z \nabla\cdot\underline{\xi}^* + \xi_z^* \nabla\cdot\underline{\xi} \right] \Big] \end{aligned}}$$

Now the question is – for arbitrary displacements, is there a necessary condition for stability?

1) first consider the case $k_x = 0$. This is what we assumed above when we displaced the field lines without bending them.

Then you can show that

$$2W = \int dz \left[-|\xi_z|^2 \left(g\rho' + \frac{\rho^2 g^2}{\gamma P + B^2/4\pi} \right) + \left(\gamma P + B^2/4\pi\right) \left| \xi_z' + i k_y \xi_y - \frac{\rho g\, \xi_z}{\gamma P + B^2/4\pi} \right|^2 \right]$$

so if $\left(-\frac{d\rho}{dz}\right) > \frac{\rho^2 g}{\gamma P + B^2/4\pi}$ everywhere then $W > 0$
STABLE

This is the same condition we had earlier.
If this condition is not satisfied, then we can choose a displacement ξ_z which makes the first term negative, and then choose ξ_y so that the 2nd term vanishes $\Rightarrow$ $W < 0$ $\Rightarrow$ UNSTABLE.

Derivation of Newcomb's eq. (12)

$$2W = \int dV \Big[\frac{|\delta \underline{B}|^2}{4\pi} + \frac{\underline{\nabla} \times \underline{B}}{4\pi} \cdot (\underline{\xi}^* \times \delta \underline{B}) + \gamma P |\underline{\nabla} \cdot \underline{\xi}|^2$$
$$+ (\underline{\xi} \cdot \underline{\nabla} P)(\underline{\nabla} \cdot \underline{\xi}^*) + (\underline{\xi}^* \cdot \underline{g}) \underline{\nabla} \cdot (\rho \underline{\xi}) \Big]$$

this is Newcomb eq. (10)

$$\gamma P (\underline{\nabla} \cdot \underline{\xi})^2 = \gamma P | \xi_z' + i k_x \xi_x + i k_y \xi_y |^2$$

$$(\underline{\xi}^* \cdot \underline{g}) \underline{\nabla} \cdot (\rho \underline{\xi}) = (\underline{\xi}^* \cdot \underline{g}) \underline{\xi} \cdot \underline{\nabla} \rho + \rho \underline{\xi}^* \cdot \underline{g} \, \underline{\nabla} \cdot \underline{\xi}$$
$$= -|\xi_z|^2 g \rho' + - \rho \xi_z^* g (\xi_z' + i k_x \xi_x + i k_y \xi_y)$$

$$(\underline{\xi} \cdot \underline{\nabla} P)(\underline{\nabla} \cdot \underline{\xi}^*) = (\underline{\nabla} \cdot \underline{\xi}^*) \xi_z \left(-\rho g - \left(\frac{B^2}{8\pi} \right)' \right)$$
$$= -\rho g \xi_z (\underline{\nabla} \cdot \underline{\xi}^*) - \xi_z \frac{B B'}{4\pi} (\underline{\nabla} \cdot \underline{\xi}^*)$$

add these:

$$\boxed{\gamma P |\underline{\nabla} \cdot \underline{\xi}|^2 - |\xi_z^2| g \rho' - \rho g [\xi_z \underline{\nabla} \cdot \underline{\xi}^* + \xi_z^* \underline{\nabla} \cdot \underline{\xi}]}$$
$$- \frac{\xi_z B B'}{4\pi} (\underline{\nabla} \cdot \underline{\xi}^*)$$

← (**) we'll add this term to the magnetic terms below.

Now what about the magnetic terms?

$$\delta \underline{B} = \underline{\nabla} \times (\underline{\xi} \times \underline{B}) \qquad \underline{B} = B(z) \hat{\underline{x}}$$

$$\underline{\xi} \times \underline{B} = \hat{\underline{y}} \xi_z B - \hat{\underline{z}} \xi_y B$$

$$\delta \underline{B} = \underline{\nabla} \times (\underline{\xi} \times \underline{B}) = \hat{\underline{x}} \left(-i k_y \xi_y B - \frac{d}{dz} (\xi_z B) \right) + \hat{\underline{y}} \, i k_x \xi_y B$$
$$+ \hat{\underline{z}} \, i k_x \xi_z B$$

$$\Rightarrow |\delta B|^2 = \left((\xi_z B)' + i k_y \xi_y B\right)\left((\xi_z^* B)' - i k_y \xi_y^* B\right) + k_x^2 |\xi_y|^2 B^2 + k_x^2 |\xi_z|^2 B^2$$

$$= |(\xi_z B)'|^2 + k_y^2 |\xi_y|^2 B^2 + k_x^2 B^2 (|\xi_y|^2 + |\xi_z|^2)$$

$$+ i k_y B \xi_y (\xi_z^* B^*)' - i k_y B \xi_y^* (\xi_z B)'$$

$$(\underline{\xi}^* \times \delta \underline{B}) = \underline{\hat{y}} \left(-\xi_z^* \left(i k_y \xi_y B + (\xi_z B)' \right) - \xi_x^* i k_x \xi_z B \right)$$

(we only need the y cpt)

$$\Rightarrow \frac{\underline{\nabla \times B}}{4\pi} \cdot (\xi^* \times \delta \underline{B}) = -\frac{B'}{4\pi} \left[\xi_z^* \left(i k_y \xi_y B + (\xi_z B)' \right) + \xi_x^* i k_x \xi_z B \right]$$

add these: $|(\xi_z B)'|^2 + k_y^2 |\xi_y|^2 B^2 + k_x^2 B^2 (|\xi_y|^2 + |\xi_z|^2)$

$$+ i k_y \cancel{B B' \xi_z^* \xi_y} - i k_y \cancel{B B' \xi_y^* \xi_z} - i k_y B^2 \xi_y^* \xi_z' + i k_y B^2 \xi_z^{*\prime} \xi_y$$

$$- \cancel{i k_y \xi_y B B' \xi_z^*} - \cancel{i k_x \xi_z \xi_x^* B B'} - \xi_z^* B' (\xi_z B)'$$

and include the term (**) from previous page $\quad - \xi_z B B' \left(\cancel{-i k_x \xi_x^* \pm i k_y \xi_y^*} + \xi_z^{*\prime} \right)$

$$\rightarrow k_y^2 |\xi_y|^2 B^2 + k_x^2 B^2 (|\xi_y|^2 + |\xi_z|^2) - i k_y B^2 \xi_y^* \xi_z' + i k_y B^2 \xi_z^{*\prime} \xi_y$$

$$+ \cancel{|\xi_z|^2 B'^2} + \cancel{\xi_z^* \xi_z' B B'} + \cancel{\xi_z^{*\prime} \xi_z B B'} + B^2 |\xi_z'|^2$$

$$- \cancel{\xi_z^* B' (\xi_z B' + B \xi_z')} - \cancel{\xi_z \xi_z^{*\prime} B B'}$$

$$\rightarrow \boxed{B^2 |\xi_z' + i k_y \xi_y|^2 + k_x^2 B^2 (|\xi_y|^2 + |\xi_z|^2)} \quad \checkmark$$

put it all together:

$$2W = \int dz \Big[B^2\left(|\xi_z' + ik_y \xi_y|^2 + k_x^2 (|\xi_y|^2 + |\xi_z|^2) \right)$$
$$+ \gamma P |\underline{\nabla}.\underline{\xi}|^2 - \rho' g |\xi_z|^2 - \rho g \left[\xi_z \underline{\nabla}.\underline{\xi}^* + \xi_z^* \underline{\nabla}.\underline{\xi} \right] \Big]$$

which agrees with Newcomb eq. (12).

2) We've reproduced our previous result. But what if we keep $k_x \neq 0$?

In this case, Newcomb shows that

$$2W = \int dz \Big[|\xi_z|^2 \left(k_x^2 B^2 - \frac{\rho^2 g^2}{\gamma P} - g\rho' \right) + |\xi_z'|^2 \frac{k_x^2 B^2}{k_x^2 + k_y^2}$$
$$+ \gamma P \left| \xi_z' + i k_x \xi_x + i k_y \xi_y - \frac{\rho g \xi_z}{\gamma P} \right|^2$$
$$+ \left(k_x^2 + k_y^2 \right) B^2 \left| \xi_y + \frac{i k_y \xi_z'}{k_x^2 + k_y^2} \right|^2 \Big]$$

Now, for any given $\xi_z(z)$ we can choose ξ_y and ξ_x so that the 3rd and 4th terms vanish.

Then the smallest value of W occurs in the $k_x \to 0$ limit:

$$2W = -\int dz \, |\xi_z|^2 \left(g\rho' + \frac{\rho^2 g^2}{\gamma P} \right)$$

So the condition for stability is now

$$\left(-\frac{d\rho}{dz} \right) > \frac{\rho^2 g}{\gamma P}$$

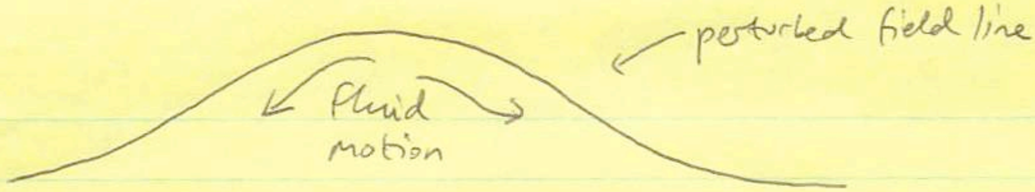
This is a more stringent condition for stability!

different from the previous result!

We might have argued that perturbations with $k_x \neq 0$ would always be stable if perturbations with $k_x = 0$ are stable, since bending the field lines generates a restoring force (tension).

But by setting $k_x = 0$ initially, we inadvertently exclude a different kind of perturbation in which the fluid elements slide along the magnetic field lines. → [for $k_x \equiv 0$ these kind of perturbations are neutrally stable, ie. $W = 0$]

This is the PARKER INSTABILITY.



Parker (1966) "The dynamical state of the interstellar gas and field"
– role of this instability in determining the configuration of Galactic fields

For ~~large~~ small enough k_x, the energy released as the fluid elements fall in the gravitational field is more than the work done against the magnetic tension.

If we write the instability criterion as

$$\left(-\frac{d\rho}{dz}\right) < \left(\frac{-\rho}{\gamma P}\right)(-\rho g)$$

$$\frac{dP_g}{dz} + \frac{d}{dz}\left(\frac{B^2}{8\pi}\right)$$

then we see that instability requires

$$\frac{1}{\gamma P}\left[-\frac{d}{dz}\left(\frac{B^2}{8\pi}\right)\right] > -\frac{1}{\rho}\frac{d\rho}{dz} + \frac{1}{\gamma P}\frac{dP_g}{dz} = -A = \frac{N^2}{g}$$

or
$$\boxed{\left(\frac{B^2}{8\pi}\frac{1}{\gamma P}\right)\left(-\frac{d\ln B^2}{dz}\right) > \left(\frac{N^2}{g}\right)}$$

if we add a magnetic field to a stably stratified atmosphere, it will be unstable if $\left(\frac{\text{magnetic pressure}}{\text{gas pressure}}\right)$ is large enough or if the gradient $\left(\frac{-d\ln B^2}{dz}\right)$ is large enough.

[e.g. a discontinuous field is unstable for any magnitude of B if B drops with height across the discontinuity.

small B $\qquad \rho_2 > \rho_1$

large B $\qquad \rho_1$]

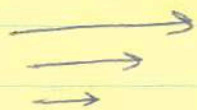
November 15, 2007

PHYS 643 lecture 19

Shear instabilities

Shearing flows are notoriously unstable. They are common in astrophysics - for example an outflow penetrating into the surrounding ISM, or an accretion disk in which neighbouring Keplerian orbits have different angular velocities. We'll deal with the rotating flow case in the final section of the course, here we talk about linear shear flows.

For example $\underline{u} = U(z)\,\hat{\underline{x}}$



The basic reason for instability is that the kinetic energy of the shear can be released if the fluid mixes.

eg. mix two fluid elements $\rightarrow u_1$, $\rightarrow u_2 = u_1 + \Delta u$ $\Big\}$ final velocity is $U = \dfrac{u_2+u_1}{2}$ (to conserve momentum)

The final K.E. is $2\times\frac{1}{2}U^2 = \frac{1}{4}u_1^2 + \cancel{\frac{1}{2}u_1\Delta u} + \cancel{\frac{(\Delta u)^2}{8}}$
$$+\frac{1}{4}u_2^2 + \frac{u_1u_2}{2}$$

$$\Rightarrow \Delta(KE) = 2\times\frac{1}{2}U^2 - \frac{1}{2}u_1^2 - \frac{1}{2}u_2^2$$

$$= -\frac{1}{4}u_1^2 - \frac{1}{4}u_2^2 + \frac{u_1u_2}{2} \qquad = -\left(\frac{u_1-u_2}{2}\right)^2 < 0$$

energy is released

Shear is a source of free energy.

In astrophysical examples you often don't get to mix the fluid for free because you have to do work against gravity. If the fluid elements have densities ρ_1 and ρ_2 initially, this is

$$\Delta\rho g \xi_z = (\rho_2-\rho_1) g \xi_z$$

for incompressible fluid, or for adiabatic changes of a compressible gas (assuming pressure equilibrium) from last time we have

$$\text{work} = -\rho g \xi_z^2 A = +\rho N^2 \xi_z^2$$

We expect instability if $\rho \frac{(U_1-U_2)^2}{4} > \rho N^2 \xi_z^2$

but $U_1 - U_2 = \left(\frac{dU}{dz}\right)\xi_z$

$$\Rightarrow \quad \text{instability if } \boxed{\frac{N^2}{(dU/dz)^2} < \frac{1}{4}}$$

this quantity is known as the Richardson number Ri

Note that we~~'ve been a little vague here~~ haven't quite done this correctly because if the two fluid elements are different densities initially, their final velocity after mixing will be different from their average velocity (conserving momentum).

The assumption is that the density variation plays its main role in the buoyancy of the fluid elements, and its effect on the inertia is small. (~~We'll see this come out of the mathematics later~~).

Those are the basic ideas. Now look at how these results emerge from the fluid equations.

Results for incompressible flow

Take the background flow to be $\underline{u} = U(z)\,\hat{\underline{x}}$
the fluid is incompressible ρ = constant.

The momentum and continuity equations are

$$\underline{\nabla}.\,\delta\underline{u} = 0$$

$$\frac{\partial\,\delta\underline{u}}{\partial t} + U\frac{\partial}{\partial x}\delta\underline{u} + \delta u_z\frac{dU}{dz}\,\hat{\underline{x}} = -\underline{\nabla}\,\delta p$$

We'll look for solutions which are $\propto e^{ik_x x}\,e^{-i\cancel{\omega t}k_x ct}$
$\propto e^{ik_x(x-ct)}$

and assume $k_y = 0$. One can show in fact that any 3D ($k_y \neq 0$) disturbance can be transformed into an equivalent 2D problem by a "Squires transformation", and furthermore for every 3D disturbance there is a more unstable 2D disturbance (Squires' theorem), so we only have to solve the 2D problem ($k_y = 0$).

$$\Rightarrow \boxed{\begin{array}{l} ik_x\,\delta u_x = -\dfrac{d}{dz}\,\delta u_z\ ; \quad ik_x(U-c)\,\delta u_z = -\dfrac{d\,\delta p}{dz} \\ ik_x(U-c)\,\delta u_x + \delta u_z\,U' = -ik_x\,\delta p \end{array}}$$

Now define a stream function

$$\delta u_x = \frac{\partial\psi}{\partial z} \qquad \delta u_z = -\frac{\partial\psi}{\partial x}$$

[ie. $\delta\underline{u} = -\underline{\nabla}\times(\psi(x,y)\,\hat{\underline{y}})$ which means that $\underline{\nabla}.\,\delta\underline{u} = 0$ is automatically satisfied.]

and write $\psi = \phi\, e^{ik_x(\cancel{\ }x-ct)}$ $\Rightarrow$ $\boxed{\begin{array}{l}\delta u_x = \dfrac{d\phi}{dz} \\ \delta u_z = -ik_x\phi\end{array}}$

$\therefore$ the momentum equations are

$$ik_x (U-c)(-ik_x)\phi = -\frac{d}{dz}\delta p = k_x^2 (U-c)\phi$$

$$ik_x (U-c)\phi' - ik_x \phi U' = -ik_x \delta p$$
$$\Rightarrow \quad -\delta p = (U-c)\phi' - \phi U'$$

$$\therefore \quad k_x^2 (U-c)\phi = (U-c)\phi'' - \phi U''$$

$$\Rightarrow \quad \boxed{(U-c)(\phi'' - k_x^2\phi) - U''\phi = 0} \quad (*)$$

Rayleigh's Stability equation ~~Criterion~~

Notice that if ϕ is an eigenfunction with eigenvalue c then
ϕ^* " " c^*
so for each unstable mode $c_I > 0$ there is a decaying mode with $c_I < 0$.

<u>Rayleigh's inflexion point theorem</u> (1880)

Start with $$\phi'' - k_x^2\phi - \frac{U''}{U-c}\phi = 0$$

(assume $c_I > 0$ so that the last term is non-singular).

Now $\int dz\, \phi^* (\;)$ and integrate by parts

$$\Rightarrow \quad \boxed{\int_{z_1}^{z_2} dz \left(|\phi'|^2 + k_x^2 |\phi|^2 \right) = -\int_{z_1}^{z_2} \frac{U''}{U-c} |\phi|^2 \, dz} \quad (1)$$

The imaginary part of this equation is

$$c_I \int_{z_1}^{z_2} dz \, \frac{U''}{|U-c|^2} \, |\phi|^2 = 0$$

$\Rightarrow$ $\boxed{\text{$U''$ must change sign somewhere in the flow, otherwise $c_I = 0$ ie. $U(z)$ must have an inflexion point for instability to occur}}$

Fjørtoft's theorem

Now take the real part of equation (1).

$$\int dz \, \frac{U''(U-c_r)|\phi|^2}{|U-c|^2} = -\int dz \left(|\phi'|^2 + k_x^2|\phi|^2 \right)$$

if $c_I \neq 0$ we can add to each side the vanishing quantity

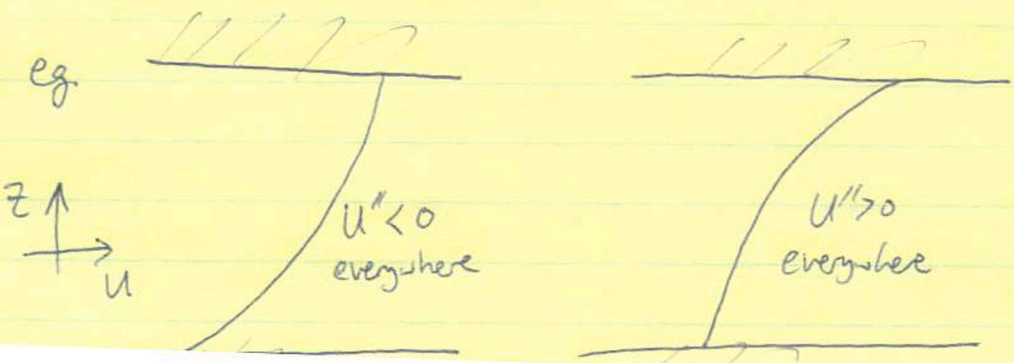
$$(c_r - U_s) \int \frac{U'' \, |\phi|^2}{|U-c|^2} \, dz \qquad \text{where } U_s = \left(\begin{array}{c}\text{velocity where} \\ U''=0\end{array}\right)$$

$$\Rightarrow \quad \int dz \, \frac{U''(U-U_s)}{|U-c|^2} \, |\phi|^2 = -\int dz \left(\left|\frac{d\phi}{dz}\right|^2 + k_x^2 |\phi|^2 \right) < 0$$

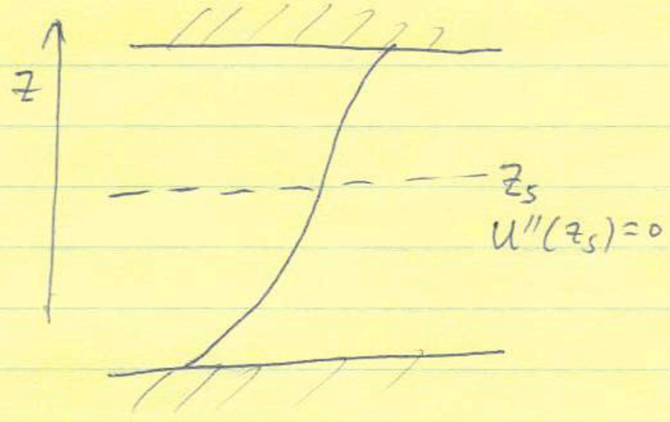
$\Rightarrow$ $\boxed{\text{a necessary condition for instability is that $U''(U-U_s) < 0$ somewhere in the flow.}}$

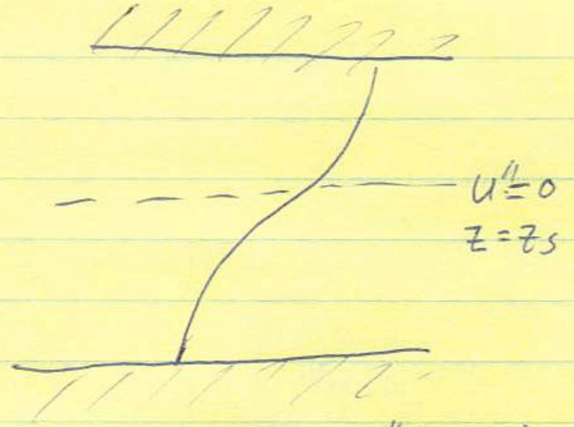
[Note that neither Rayleigh's or Fjørtoft's theorems are sufficient conditions, only necessary ones. For example $U = \sin z$ (Tollmien 1935) is stable.]

eg.



BOTH STABLE.





stable because $U''(U-U_s) \geq 0$ <u>unstable</u> $U''(U-U_s) < 0$

<u>Vorticity</u>

It's helpful to think of these results in terms of <u>vorticity</u>. In fact, Rayleigh's stability equation for the fluid is simply $\delta\left(\frac{D\omega}{Dt}\right) = 0$

The background flow has $\underline{\omega} = \underline{\nabla} \times \underline{u} = \hat{\underline{y}}\, \frac{dU}{dz}(z)$

the perturbed vorticity is $\delta\underline{\omega} = \hat{\underline{y}}\left(\frac{\partial \delta u_x}{\partial z} - i k_x\, \delta u_z\right)$

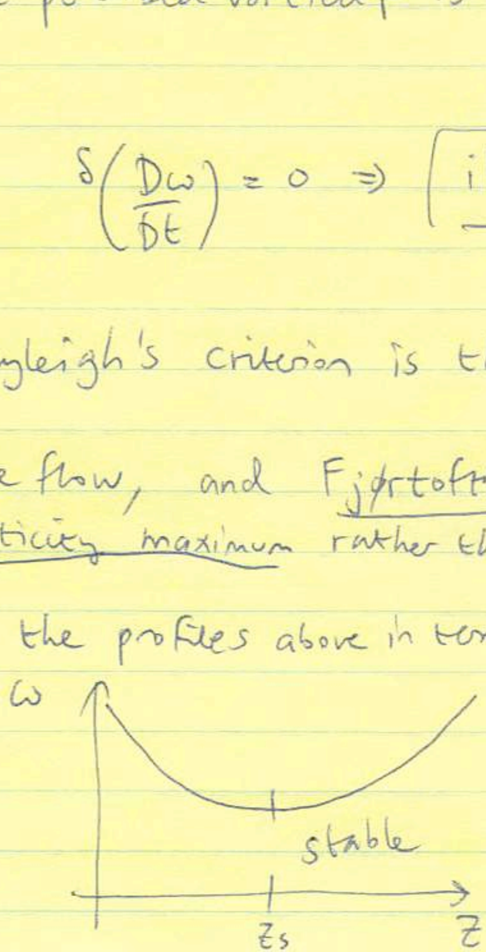
$$= \hat{\underline{y}}\left(\phi'' - k_x^2 \phi\right)$$

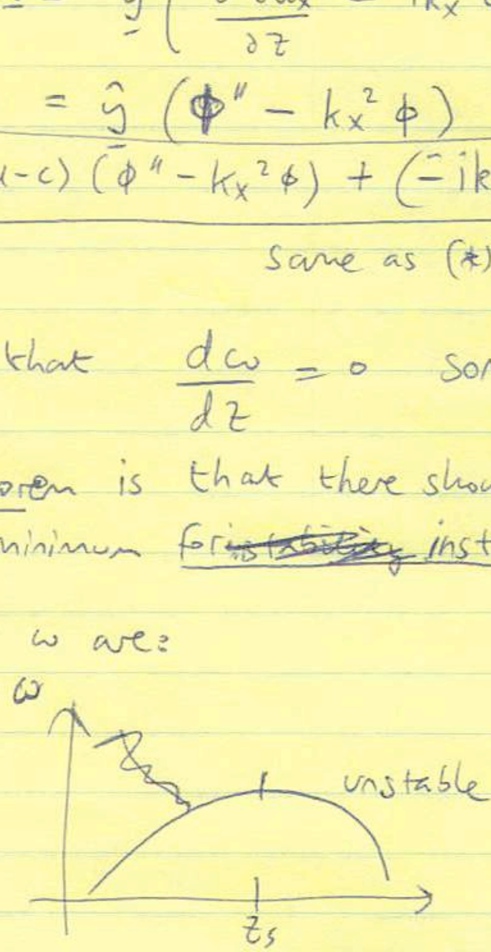
$$\Rightarrow \quad \delta\left(\frac{D\omega}{Dt}\right) = 0 \quad \Rightarrow \quad \boxed{i k_x (U-c)(\phi'' - k_x^2\phi) + (-i k_x)\phi U'' = 0}$$

same as (*) ✓

Rayleigh's criterion is then that $\frac{d\omega}{dz} = 0$ somewhere in the flow, and <u>Fjørtoft's theorem</u> is that there should be a <u>vorticity maximum</u> rather than minimum <u>for ~~stability~~ instability</u> to occur.

eg. the profiles above in terms of ω are:





Howard's semicircle theorem (Howard 1961)

Rayleigh's equation can be written as

$$\frac{d}{dz}\left[(U-c)^2 \frac{d\psi}{dz} \right] - k_x^2 (U-c)^2 \psi = 0$$

where $\psi = \dfrac{\phi}{U-c}$. [Assume $c_I \neq 0$ so non-singular.]

Now multiply by ψ^* and $\int dz$, integrating the first term by parts:

$$\int dz\, (U-c)^2 \left[\left|\frac{d\psi}{dz}\right|^2 + k_x^2 |\psi|^2 \right] = 0 .$$

The imaginary part of this eqn is

$$2 c_i \int dz\, (U - c_R) \left[\left|\frac{d\psi}{dz}\right|^2 + k_x^2 |\psi|^2 \right] = 0$$

$\Rightarrow$ $\boxed{c_R \text{ must lie within the range of the fluid velocity } U}$

ie the unstable mode has a wave speed that matches the fluid velocity at some location — the critical level.

The real part is $\int \left[(U-c_R)^2 - c_I^2 \right] Q \, dz = 0$
where $Q = |\psi'|^2 + k_x^2 |\psi|^2 > 0$

or $\int U^2 Q \, dz = (c_R^2 + c_I^2) \int Q \, dz$

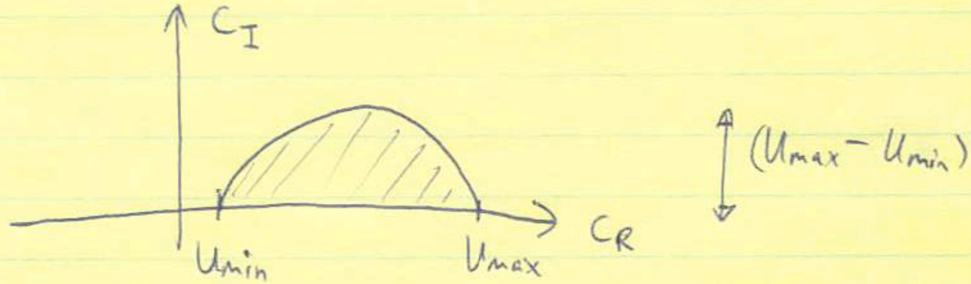
Now consider $\int dz\, (U - U_{min})(U - U_{max}) Q \leq 0$

$$\Rightarrow \int dz \left[c_R^2 + c_I^2 - (U_{min} + U_{max}) c_R + U_{min} U_{max} \right] Q \leq 0$$

$$\Rightarrow \int dz \left[\left[c_R - \tfrac{1}{2}(U_{min} + U_{max}) \right]^2 + c_I^2 - \left(\frac{U_{min} - U_{max}}{2} \right)^2 \right] Q \leq 0$$

$\Rightarrow$ We must have $\boxed{\left(c_R - \dfrac{U_{min} + U_{max}}{2} \right)^2 + c_I^2 \leq \left(\dfrac{U_{min} - U_{max}}{2} \right)^2}$

c must lie within the semi-circle:



Instability in a stratified flow

Miles & Howard (1961) in two back-to-back papers.

Assume incompressible perturbations. Then we add the term

$$g\,\delta\rho = -g\frac{d\rho}{dz}\frac{\delta u_z}{ik_x(U-c)} \qquad \left[\text{here we used } \delta\left(\frac{D\rho}{Dt}\right) = 0\right]$$

to the z-momentum equation.

Squires theorem applies also in this case.

We obtain the Taylor-Goldstein equation

$$(U-c)(\phi'' - k_x\phi) - U''\phi + \frac{N^2\phi}{U-c} = 0$$

where $N^2 = -\frac{g}{\rho}\frac{d\rho}{dz}$ is the Brunt frequency.

To show that a necessary condition for instability is that

$$R_i = \frac{N^2}{(dU/dz)^2} < \frac{1}{4}$$

first define

$$H = \frac{\phi}{\sqrt{U-c}}$$

then

$$\frac{d}{dz}\left[(U-c)\frac{dH}{dz}\right] - \left\{k_x^2(U-c) + \frac{U''}{2} + \frac{\frac{1}{4}U'^2 - N^2}{U-c}\right\}H = 0$$

Now $\int dz\, H^*(\ )$ and take the imaginary part

$$\Rightarrow \quad c_I \int_{z_1}^{z_2} \left[|H'|^2 + k_x^2 |H|^2 + \frac{\{N^2 - U'^2/4\}}{|U-c|^2} \right] dz = 0$$

$\Rightarrow$ necessary condition for instability is $N^2 < \frac{1}{4} U'^2$

or $\underline{Ri < \frac{1}{4}}$.

Howard (1961) also derived a bound on the growth rate

$$k_x^2 c_I^2 \leq \max_{z_1 < z < z_2} \left(\frac{1}{4} U'^2 - N^2 \right)$$

and derived the semicircle theorem in this case (it looks the same as we showed previously).

November 20, 2007.

PHYS 643 lecture 20

We'll return to the topic of instabilities next week - but now look at

V. Numerical techniques

see Numerical Recipes Chp 19 or Thompson Chp 6.

We'll start by looking at how to solve the 1D advection-diffusion equation by finite differencing,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$

Represent f on a grid f_j $j=1,N$
assume equal grid spacing Δx for simplicity

Now use Taylor expansion

$$f_{j+1} = f_j + \Delta x\, f_j' + \frac{\Delta x^2}{2} f_j'' + O(\Delta x^3)$$

$$f_{j-1} = f_j - \Delta x\, f_j' + \frac{\Delta x^2}{2} f_j'' + O(\Delta x^3)$$

add and subtract these:

$$\Rightarrow \quad f_j' = \frac{f_{j+1} - f_{j-1}}{2\Delta x} + O(\Delta x^2)$$

$$\left[\text{or} \quad f_j' = \frac{f_{j+1} - f_j}{\Delta x} + O(\Delta x) = \frac{f_j - f_{j-1}}{\Delta x} + O(\Delta x) \right]$$

and $$f''_j = \frac{f_{j+1} - 2f_j + f_{j-1}}{(\Delta x)^2} + O(\Delta x^2)$$

The idea is to use expressions like these to write a finite difference representation of the differential equation we are trying to solve.

Focus first on the advection part, $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$.

The first thing we might try is the FTCS scheme:

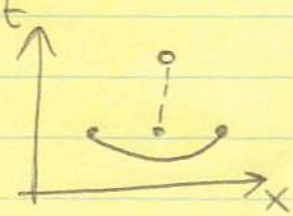
$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v\,\frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$$

n labels the time step

or $$f_j^{n+1} = f_j^n - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right)$$

which gives the values of the function at timestep $n+1$ in terms of the values at the previous timestep n.

We can represent this scheme in a diagram:



We say that this scheme is <u>explicit</u>.

<u>Von Neumann stability analysis</u>

Unfortunately the previous scheme is numerically unstable!

To see why look for solutions $f_j^n = \xi^n e^{ik(j\Delta x)}$ and assume coefficients are constant (cf. WKB)

ξ is complex — if $|\xi|>1$ for any value of k then there are exponentially growing modes in time (n).

For the FTCS scheme:

$$e^{ijk\Delta x}\,\xi^{n+1} = e^{ikj\Delta x}\,\xi^{n} - v\Delta t\left(\frac{\xi^n e^{i(j+1)k\Delta x} - \xi^n e^{i(j-1)k\Delta x}}{2\Delta x}\right)$$

$$\Rightarrow \quad \xi = 1 - \frac{v\Delta t}{2\Delta x}\, 2i \sin k\Delta x$$

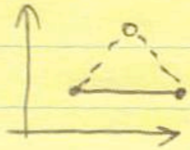
$$= 1 - i\,\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

$$\Rightarrow \quad |\xi| = 1 + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)$$

>1 for all k! FTCS is unstable

Lax method

$$f_j^{n+1} = \tfrac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right) \quad -(*)$$



for this scheme $\quad \xi = \cos k\Delta x - i\,\frac{v\Delta t}{\Delta x}\sin k\Delta x \quad -(**)$

$$\text{or} \quad |\xi|^2 = \cos^2 k\Delta x + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x$$

$$= 1 + \sin^2(k\Delta x)\left[\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1\right]$$

$\Rightarrow$ this scheme is stable if $\boxed{\dfrac{v\Delta t}{\Delta x} \leq 1}$

this is the Courant-Friedrichs-Levy criterion
"Courant condition"
Can understand this in terms of causality.

How to understand why this method is stable?

(*) is $\dfrac{f_j^{n+1} - f_j^n}{\Delta t} = -v\left(\dfrac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}\right) + \dfrac{1}{2}\left(\dfrac{f_{j+1}^n - 2f_j^n + f_{j-1}^n}{\Delta t}\right)$

which is the FTCS representation of
$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \underbrace{\frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 f}{\partial x^2}}_{\text{diffusion term}}$$

This scheme has "numerical dissipation"
For $|v|\Delta t < \Delta x$ then $|\xi| < 1$ and the amplitude decreases.

The damping is small for long wavelength features $k\Delta x \ll 1$.
Short scales with $k\Delta x \sim 1$ damp away quickly.

<u>Different kinds of errors</u>

1) amplitude errors $|\xi| \neq 1$
2) phase errors:
(**) is $\xi = e^{-ik\Delta x} + \underbrace{i\left(1 - \frac{v\Delta t}{\Delta x}\right)\sin k\Delta x}_{\text{dispersion when } v\Delta t \neq \Delta x}$
3) transport errors: in the Lax scheme, information at gridcell j propagates to $j-1$ and $j+1$ on the next timestep.

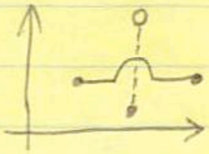
"Upwind differencing"

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v_j^n \begin{cases} \dfrac{f_j^n - f_{j-1}^n}{\Delta x} & v_j^n > 0 \\ \dfrac{f_{j+1}^n - f_j^n}{\Delta x} & v_j^n < 0 \end{cases}$$

The stability condition is again the Courant condition.

<u>2nd order accuracy in time</u>

1) staggered leapfrog $\quad f_j^{n+1} - f_j^{n-1} = -\dfrac{v\Delta t}{\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right)$



can show that $\xi = -i\dfrac{v\Delta t}{\Delta x}\sin k\Delta x \pm \sqrt{1 - \left(\dfrac{v\Delta t}{\Delta x}\sin k\Delta x\right)^2}$

- Stable if $\dfrac{v\Delta t}{\Delta x} < 1$
- $|\xi| = 1$ for all k! No amplitude dissipation.
- subject to "mesh drifting"

2) Two step LAX-Wendroff

$$f_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(f_{j+1}^n + f_j^n\right) - \left(\frac{(\Delta t)v}{2\Delta x}\right)\left(f_{j+1}^n - f_j^n\right)$$

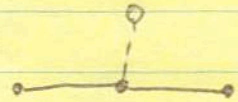
followed by $\quad f_j^{n+1} = f_j^n - \dfrac{\Delta t}{\Delta x}\left(f_{j+\frac{1}{2}}^{n+\frac{1}{2}} - f_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)$

[the $f^{n+1/2}$ values are discarded]

This scheme has $|\xi| < 1$ (for $v\Delta t < \Delta x$) but $|\xi|^2 = 1 + O(k\Delta x)^4$ [compared with $(k\Delta x)^2$ for Lax]

Now look at the $\boxed{\text{diffusion}}$ part $\quad \dfrac{\partial f}{\partial t} = D \dfrac{\partial^2 f}{\partial x^2}$

try $\quad \dfrac{f_j^{n+1} - f_j^n}{\Delta t} = \dfrac{D}{\cancel{2}(\Delta x^2)} \left(f_{j+1}^n - 2 f_j^n + f_{j-1}^n \right)$



Is this stable? $\quad f_j^n = \xi^n e^{ikj\Delta x}$

$$\Rightarrow \quad \xi^{n+1} - \xi^n = \xi^n \frac{\Delta t D}{(\Delta x)^2} \underbrace{\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x} \right)}_{2(-1 + \cos k\Delta x)}$$

$$\Rightarrow \boxed{\xi = 1 - \frac{4 D \Delta t}{(\Delta x)^2} \sin^2\left(\frac{k\Delta x}{2}\right)}$$

$\Rightarrow \quad |\xi| < 1 \quad$ if $\quad \boxed{\dfrac{2 D \Delta t}{(\Delta x)^2} \leq 1} \qquad \Delta t \lesssim$ diffusion time across a grid cell

The problem with this is that if we are interested in the evolution on a lengthscale $L \gg \Delta x$, this is slow

time-needed steps $\simeq \dfrac{L^2}{D} \dfrac{1}{\Delta t} \simeq \left(\dfrac{L}{\Delta x}\right)^2 \simeq N_{grid}^2$

We need a scheme that allows larger timesteps (at the expense of accuracy on the smallest scales).

1) "implicit" scheme $\quad$ 1st order in time

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2} \left(f_{j+1}^{n+1} - 2 f_j^{n+1} + f_{j-1}^{n+1} \right)$$

or if we write $\beta = D\Delta t / (\Delta x)^2$ this is

$$-\beta f_{j+1}^{n+1} + (1+2\beta) f_j^{n+1} - \beta f_{j-1}^{n+1} = f_j^n$$

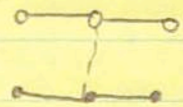
we need to invert a tridiagonal matrix $\underline{\underline{A}}\, \underline{f}^{n+1} = \underline{f}^n$

$$\rightarrow \underline{f}^{n+1} = \underline{\underline{A}}^{-1} \underline{f}^n$$

The stability is $\xi = \dfrac{1}{1 + 4\beta \sin^2\left(\frac{k\Delta x}{2}\right)} < 1$ for all Δt

For large timesteps, the solution goes to the equilibrium solution (which satisfies $f'' = 0$). So the short wavelengths are not followed accurately, but adopt their equilibrium solution.

2) <u>Crank-Nicholson</u>



2nd order in space and time

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left[\frac{1}{2}\left(f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}\right) + \frac{1}{2}\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right)\right]$$

This is also stable for all timesteps Δt.

Finally, a note about <u>boundary conditions</u>. Often it is useful to use a dummy grid cell.

eg. we want to enforce $f' = C$ at the boundary.

Write $\dfrac{f_2 - f_0}{2\Delta x} = C \Rightarrow f_0 = f_2 - C\,2\Delta x$ — (***)

Now in the equation for the evolution of f_1, wherever f_0 appears we can substitute for it using (***). This implements the boundary condition.

November 22, 2007

PHYS 643 lecture 21

Finish off last time by discussing how to implement boundary conditions.

Operator splitting

Suppose we have an equation of the form

$$\frac{\partial f}{\partial t} = Lf = (L_1 + L_2 + \dots)f$$

eg. L_1 = advection

L_2 = diffusion

then one way to proceed is to do the update for each L sequentially:

$$f^{n+\frac{1}{m}} = U_1(f^n, \Delta t)$$

$$f^{n+2/m} = U_2(f^{n+1/m}, \Delta t)$$

$$\dots$$

$$f^{n+1} = U_m(f^{n+\frac{m-1}{m}}, \Delta t)$$

in each step use the full timestep Δt

eg. advection-diffusion

$$f_j^{n+1/2} = \frac{1}{2}(f_{j+1}^n + f_{j-1}^n) - \frac{v\Delta t}{2\Delta x}(f_{j+1}^n - f_{j-1}^n)$$

$$f_j^{n+1} = f_j^{n+1/2} + \frac{D\Delta t}{(\Delta x)^2}(f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1})$$

an alternative is to use an update scheme for the entire operator L at each step, where the update at each step need only be stable for each piece L_1, L_2 and so on..

$$f^{n+1/m} = U_1(f^n, \Delta t/m)$$

$$\dots$$

$$f^{n+1} = U_m\left(f^{n+\frac{m-1}{m}}, \frac{\Delta t}{m}\right)$$

eg. 2D diffusion $\qquad \dfrac{\partial f}{\partial t} = D\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right)$

Alternating-direction implicit (ADI) scheme

$$f^{n+1/2}_{j,l} = f^n_{j,l} + \frac{\beta}{2}\Big(f^{n+1/2}_{j+1,l} - 2f^{n+1/2}_{j,l} + f^{n+1/2}_{j-1,l} + f^n_{j,l+1} - 2f^n_{j,l} + f^n_{j,l-1}\Big)$$

timestep $\Delta t/2$

$$f^n_{j,l} = f^{n+1/2}_{j,l} + \frac{\beta}{2}\Big(f^{n+1/2}_{j+1,l} - 2f^{n+1/2}_{j,l} + f^{n+1/2}_{j-1,l} + f^{n+1}_{j,l+1} - 2f^{n+1}_{j,l} + f^{n+1}_{j,l-1}\Big)$$

<u>Flux conserving formulation</u> [see Heidelberg lecture notes Chp 4,5]

The equations of hydrodynamics are of the form of conservation equations $\dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial x}\big(\underbrace{uf}\big) = 0$

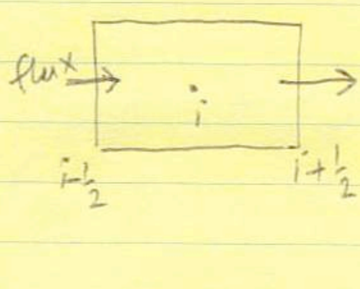
more generally, flux $J(f)$

if possible we should use a formulation that conserves the quantity f.

"Finite volume methods" [assume Δx = constant here for simplicity]

divide the volume into cells

- the grid points are now the cell centers x_i
- the cell boundaries ($N+1$ of these) are at $x_{i\pm\frac{1}{2}} = \frac{1}{2}(x_i + x_{i\pm 1})$



then write

$$\frac{d}{dt}(f_i \Delta x) = J_{i-1/2} - J_{i+1/2}$$

or

$$\boxed{\frac{f_i^{n+1} - f_i^n}{\Delta t} = \frac{J^{n+1/2}_{i-1/2} - J^{n+1/2}_{i+1/2}}{\Delta x}}$$

We write the fluxes as being evaluated at the half timestep, in fact they are averages over the timestep

$$J_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} J_{i+1/2}(t)\, dt$$

Note that if we sum over all cells (over the volume)

$$\sum_i \frac{d}{dt}\left(f_i \Delta x\right) = \underbrace{J_{-1/2} - J_{N+1/2}}_{\text{flow through the boundaries}}$$

— otherwise f_i is conserved.

The only question is how to choose the fluxes J.

eg. donor cell advection (1st order)

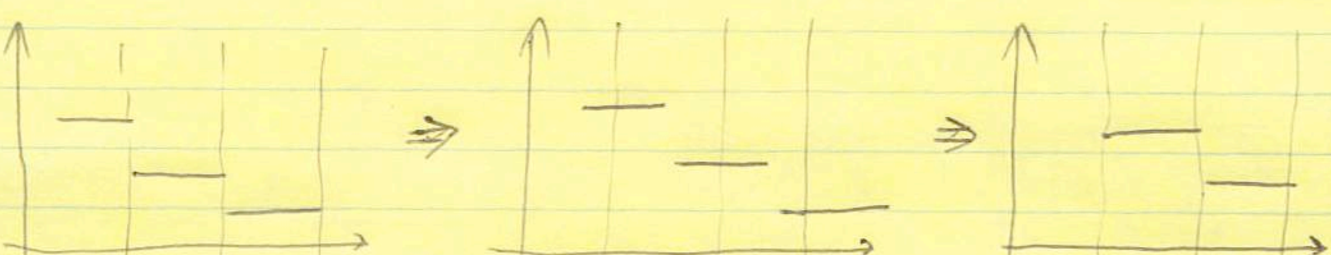
$$J_{i+1/2} = \begin{cases} v_{i+1/2} f_i^n & v_{i+1/2} > 0 \\ v_{i+1/2} f_{i+1}^n & v_{i+1/2} < 0 \end{cases}$$

$$J_{i-1/2} = \begin{cases} v_{i-1/2} f_{i-1} & v_{i-1/2} > 0 \\ v_{i-1/2} f_i & v_{i-1/2} < 0 \end{cases}$$

[cf upwind differencing technique from last time]

~~eg. piecewise linear (2nd order in time)~~

Graphically, this update scheme is (for flow to the right)



eg. piecewise linear (2nd order in time)

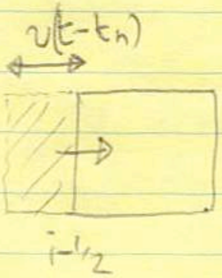
The idea here is to allow the quantity f to vary linearly across the cell rather than assume it is constant across the cell.

$$f(x, t=t_n) = f_i^n + \sigma_i^n (x - x_i)$$

$$x_{i-1/2} < x < x_{i+1/2}$$

[defined so ~~the~~ that f_i^n is the average value in the cell
i.e. $x_i = \frac{1}{2}(x_{i+1/2} + x_{i-1/2})$]

For simplicity here, assume v is constant and $v > 0$ then (flow to the right)



$$J_{i-1/2}(t) = v f(x = x_{i-1/2}, t)$$

$$= v f^n_{i-1} + v \sigma^n_{i-1} \left(x_{i-1/2} - v(t - t_n) - x_{i-1} \right)$$

$$= v f^n_{i-1} + v \sigma^n_{i-1} \left(\tfrac{1}{2}\Delta x - v(t - t_n) \right)$$

$$\Rightarrow J^{n+1/2}_{i-1/2} = \frac{1}{\Delta t} \int_{t_n}^{t_n + \Delta t} J_{i-1/2}(t)$$

$$= v f^n_{i-1} + \tfrac{1}{2} v \sigma^n_{i-1} (\Delta x - v \Delta t)$$

and

$$J^{n+1/2}_{i+1/2} - J^{n+1/2}_{i-1/2} = v(f_i^n - f^n_{i-1}) + \tfrac{1}{2} v (\sigma_i^n - \sigma^n_{i-1})(\Delta x - v\Delta t)$$

$\Rightarrow$ the update is

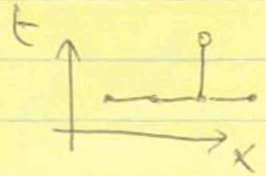
$$\boxed{f_i^{n+1} - f_i^n = -\frac{v\Delta t}{\Delta x}(f_i^n - f^n_{i-1}) - \frac{u\Delta t}{2\Delta x}(\sigma_i^n - \sigma^n_{i-1})(\Delta x - v\Delta t)}$$

There are different choices we can make for the slope:

centered	$\sigma_i^n = \dfrac{f_{i+1}^n - f_{i-1}^n}{2\Delta x}$	Fromm's
upwind	$\sigma_i^n = \dfrac{f_i^n - f_{i-1}^n}{\Delta x}$	Beam warming
downwind	$\sigma_i^n = \dfrac{f_{i+1}^n - f_i^n}{\Delta x}$	Lax-Wendroff

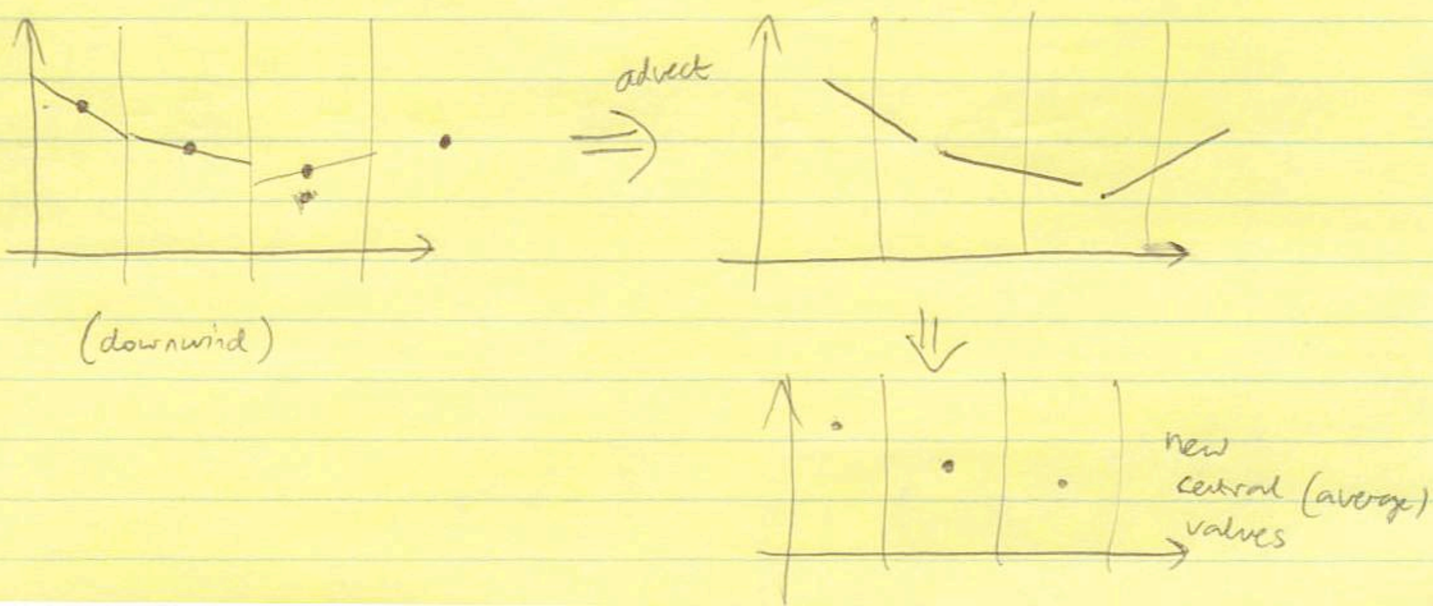
eg. if we choose the centered expression then we get Fromm's method

$$f_i^{n+1} = f_i^n - \frac{v\Delta t}{4\Delta x}\left(f_{i+1}^n + 3f_i^n - 5f_{i-1}^n + f_{i-2}^n\right) - \frac{v^2\Delta t^2}{4(\Delta x)^2}\left(f_{i+1}^n - f_i^n - f_{i-1}^n + f_{i-2}^n\right)$$

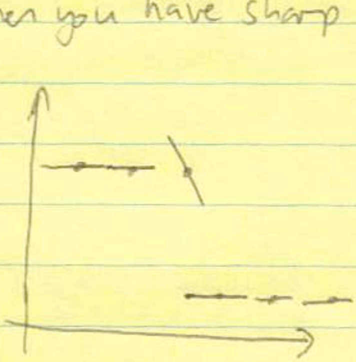


the downwind choice leads to the Lax Wendroff scheme that we saw last time.

~~one of the pr~~ Graphically,



One of the problems with this method is that it leads to overshoot when you have sharp discontinuities.



to avoid this, can use slope limiters or flux limiters

eg. total variation diminishing (TVD) method.

See Figure taken from the Heidelberg course which shows the performance of different schemes for calculating $J^{n+\frac{1}{2}}$

<u>1D hydro example</u> (taken from Heidelberg Chp 5)

$$f_1 = \rho \qquad P = \rho c_s^2 \qquad c_s^2 = \text{constant}$$
$$f_2 = \rho u$$

then
$$\frac{\partial}{\partial t} f_1 + \frac{\partial}{\partial x}(u f_1) = 0$$
$$\frac{\partial}{\partial t} f_2 + \frac{\partial}{\partial x}(u f_2) = -\frac{\partial P}{\partial x}$$

The algorithm is

1) $$f_{1,i}^{n+1/2} = f_{1,i}^{n} - \Delta t \left(\frac{J_{1,i+1/2} - J_{1,i-1/2}}{\Delta x} \right)$$

donor cell
$$J_{1,i+1/2} = \begin{cases} f_{1,i}^{n} \; u^n_{i+1/2} & \text{if } u^n_{i+1/2} > 0 \\ f_{1,i+1}^{n} \; u^n_{i+1/2} & \text{if } u^n_{i-1/2} < 0 \end{cases}$$

where $u^n_{i+1/2} = \frac{1}{2}\left(\frac{f_{2,i}}{f_{1,i}} + \frac{f_{2,i+1}}{f_{1,i+1}} \right)$ [and similarly for $J_{i-\frac{1}{2}}$]

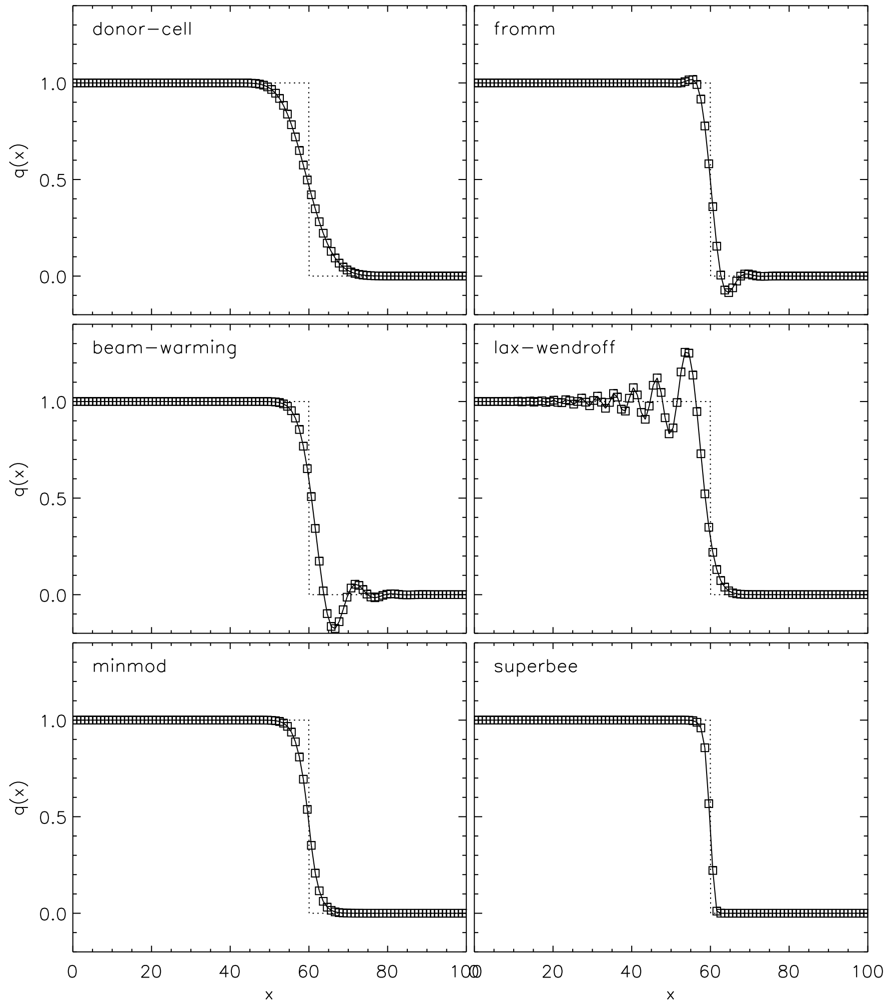


Figure 4.5. Advection with the piecewise linear advection algorithm with 6 different choices of the slope. Results are shown of the advection of a step function over a grid of 100 points with grid spacing $\Delta x = 1$, after 300 time steps with $\Delta t = 0.1$.

The second step is to add the source term

2) $$f_{1,i}^{n+1} = f_{1,i}^{n+1/2}$$

$$f_{2,i}^{n+1} = f_{2,i}^{n+1/2} \underbrace{- c_s^2 \left(\frac{f_{1,i+1}^{n+1/2} - f_{1,i-1}^{n+1/2}}{\Delta x} \right)}_{\text{pressure gradient term}}$$

The implementation of this method from the Heidelberg Chp 5 is on the website.

November 27, 2007

PHYS 643 lecture 22

Viscosity

In a viscous fluid, microscopic motions of molecules transport momentum.

eg. a plane-parallel shear flow $\underline{U} = U(z)\,\hat{\underline{x}}$

$\longrightarrow U + \lambda \dfrac{dU}{dz}$

$\longrightarrow U$ $\qquad \lambda$ = mean free path

the net flux of momentum here
(ie. the flux of x-momentum across the z-surface)

is $\quad -\dfrac{1}{3}\, n\, v_{th}\, m \left(\lambda \dfrac{dU}{dz}\right)$

v_{th}: thermal speed of the particles

m: mass of particles

$$v_{th} \simeq \left(\frac{kT}{m}\right)^{1/2} = c_s$$

We can write

$$\text{momentum flux} = \text{stress} = \mu \frac{dU}{dz}$$

where the viscosity $\mu = \dfrac{1}{3}\, n m\, v_{th} \lambda$ (units g/cm s)

$$= \rho\, \frac{1}{3} v_{th} \lambda$$

$$= \rho \nu$$

where ν is the <u>kinematic viscosity</u> (units cm^2/s)

A fluid for which viscous stress $\propto$ (velocity gradient) is

known as a <u>Newtonian fluid</u>.
(see YouTube for amazing videos of <u>non</u>-Newtonian fluids!)

In general, the viscous stress can be written (see Landau & Lifschitz)

$$\sigma_{ik} = \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\,\delta_{ik}\,\underline{\nabla}\cdot\underline{u}\right) + \zeta\,\delta_{ik}\,\underline{\nabla}\cdot\underline{u}$$

where the momentum equation is

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}\right) = \underline{\nabla}\cdot\underline{\underline{T}}$$

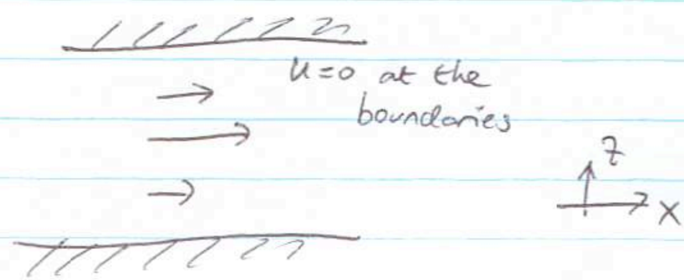
and $T_{ik} = -P\,\delta_{ik} + \sigma_{ik}$

The combination $\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$ is the shear part and excludes velocity gradients due to rotation, for which viscosity does not operate.

Typical values of μ and ν at 20°C

	μ	ν	
water	0.01	0.01	(cgs)
air	0.00018	0.150	
alcohol	0.018	0.022	
glycerine	8.5	6.8	
mercury	0.0156	0.0012	
molasses	($\simeq$ 50–100) $\longrightarrow$		

<u>Example: flow in a pipe</u>



fluid flows in a pipe in response to an applied pressure gradient in the x-direction

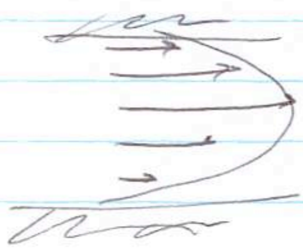
We seek a solution independent of x and steady-state.
The momentum equation is

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$$

write $\frac{\partial P}{\partial x}$ as $\frac{\Delta P}{L}$

with boundary conditions that $u=0$ at the wall, the flow is quadratic

$$\boxed{u = \frac{\Delta P}{\mu L}(H-z)\,z}$$



What is the viscosity of the Sun?

Estimate the viscosity due to collisions between ions
(assume protons - pure hydrogen composition)

$$\boxed{\nu \simeq \frac{1}{3} v_{th}\,\lambda} \qquad \lambda = \frac{1}{n\sigma} = \frac{1}{n}\frac{(kT)^2}{e^4}\frac{1}{\Lambda}$$

$$v_{th} = \left(\frac{2kT}{m_p}\right)^{1/2}$$

Λ: Coulomb logarithm from integrating over impact parameters

$$\Rightarrow \quad \nu \simeq \frac{(kT)^{5/2}(2m_p)^{1/2}}{\rho e^4 \Lambda}$$

$$= 0.7\ \mathrm{cm^2/s}\left(\frac{150\ \mathrm{g/cm^3}}{\rho}\right)\left(\frac{T}{10^7\,\mathrm{K}}\right)^{5/2}\left(\frac{8}{\Lambda}\right)$$ [T and ρ for the center of the Sun]

$$= 3\ \mathrm{cm^2/s}\left(\frac{10^{-1}\ \mathrm{g/cm^3}}{\rho}\right)\left(\frac{T}{10^6\,\mathrm{K}}\right)^{5/2}\left(\frac{8}{\Lambda}\right)$$ [Somewhere near the base of the convection zone]

Just like the mean density, the molecular viscosity of the solar material is $\simeq 1$ in cgs units.

This value of viscosity implies that the Reynold's number for the flows in the convection zone is huge

$$Re = \frac{UL}{\nu}$$

we'll see where this number comes from next time

$$\simeq \frac{10^3\,\text{cm/s} \times 10^{10}\,\text{cm}}{3}$$

typical numbers for solar convection zone

$$\simeq \underline{\underline{3 \times 10^{12}}}$$

We therefore expect the flow to be <u>turbulent</u> - we'll consider the properties of turbulent flows next time.

An aside:

<u>Comparison of photon and ion-ion mean free paths in the Sun.</u>

As we discussed in class $\dfrac{\lambda_{photon}}{\lambda_{ion}} = \left(\dfrac{n_{ion}}{n_e}\right)\left(\dfrac{\sigma_{ion}}{\sigma_\gamma}\right)$

$\left(\dfrac{n_{ion}}{n_e}\right) \simeq 1$

$$\sigma_{ion} \sim \frac{e^4}{(kT)^2} \qquad \sigma_\gamma \simeq \sigma_T = \left(\frac{e^2}{mc^2}\right)^2$$

$$\Rightarrow \quad \frac{\sigma_{ion}}{\sigma_\gamma} = \left(\frac{mc^2}{kT}\right)^2 \approx \frac{3 \times 10^5}{T_7^2}$$

$\Rightarrow$ λ_γ is much greater than λ_{ion}. This is why conduction is not an important heat transfer mechanism in the Sun.

(or at least one of the reasons - also important are the relative speeds of e^- and γs and their number densities)

November 29, 2007.

PHYS 643 lecture 23

We saw last time in the movie that at large Re, flows become turbulent. Note that turbulence is a property of the <u>flow</u> as opposed to a property of the <u>fluid</u> such as viscosity.

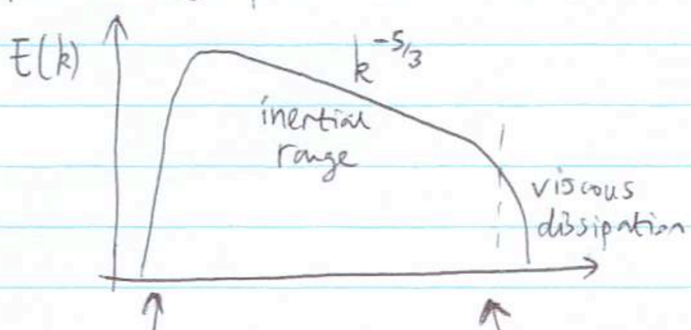
Characteristics of turbulence "symptoms"

- irregularity
- diffusivity
- large Re #'s
- 3D vorticity fluctuations
- dissipation

Energy cascade

Turbulence involves a cascade of energy from the largest to the smallest scales where viscosity dissipates the energy.

Typical energy spectrum (isotropic incompressible homogeneous turbulence)



outer scale — fluid stirred $\frac{UL}{\nu} \gg 1$

inner scale — $\frac{v_d l_d}{\nu} \sim 1$

if the energy transfer rate ε in the cascade is constant ("steady state") from dimensional arguments we can write

$\varepsilon \sim \dfrac{v^3}{l}$ at each scale l or $\boxed{v(l) \sim (\varepsilon l)^{1/3}}$ (*)

and in particular $\varepsilon \sim \dfrac{U^3}{L} \sim \dfrac{v_d^{\,3}}{l_d}$

But $v_d l_d \sim \nu \Rightarrow \boxed{l_d \sim \left(\dfrac{\nu^3}{\varepsilon}\right)^{1/4} \quad v_d \sim (\nu\varepsilon)^{1/4}}$

size of Eddy for which
viscous time = eddy turnover time

Also, $\left(\dfrac{L}{l_d}\right)^4 = L^3 \cdot \dfrac{\nu^3}{\varepsilon} \cdot \dfrac{\varepsilon}{\nu^3} = Re^3$

$\Rightarrow \boxed{l_d \sim \dfrac{L}{Re^{3/4}}} \quad \boxed{v_d \sim \dfrac{V}{Re^{1/4}}}$

eg. solar convection zone

outer scale $\sim 10^{10}$ cm $Re \sim 10^{12}$ (from last time).

inner scale $\sim \dfrac{10^{10}}{(10^{12})^{3/4}} \sim$ 10 cm

$v_d \sim \dfrac{10^3 \text{ cm/s}}{(10^{12})^{1/4}} \sim 1$ cm/s

$\Rightarrow \dfrac{l_d}{v_d} \sim 10$ seconds

~~and~~

whereas $\dfrac{L}{U} \sim \dfrac{10^{10}\text{ cm}}{10^3 \text{ cm/s}} \sim 10^7$ s

The famous scaling for $E(k) \propto k^{-5/3}$ follows:
the kinetic energy at scale k is $E(k)\, dk = E(k) k \, d\ln k$

$\sim v^2 \left(\dfrac{1}{k}\right)$

$E(k) \propto k^{-5/3}$ ✓ $\Leftarrow \quad \sim \varepsilon^{2/3} k^{-2/3}$

(using *)

If the stirring is kept the same but the viscosity varied, then the inertial range remains fixed but the scale of the viscous cutoff changes. We saw this in the movie where a turbulent jet looks identical at two different Re on large scales, but has much finer structure at larger Re.

We saw also that in freely decaying turbulence the small scales are erased first, consistent with the above picture. At time t, we expect the smallest lengthscale to be $\ell \simeq v(\ell) t \sim (\varepsilon \ell)^{1/3} t$

$$\Rightarrow \ell \propto t^{3/2}$$

Turbulent transport

The irregular motions lead to enhanced transport of momentum and other fluid properties, eg heat.

To analyse this, we adopt the Reynold's decomposition

$$u = U + u'$$

where $\overline{u} = U$ and $\overline{u'} = 0$

ie U is the mean flow; u' represents the velocity fluctuations

the averaging is $\overline{u'} = \frac{1}{\tau}\int_{t_0}^{\tau+t_0} dt\, u'$ with τ large

Assume incompressible flow then

$$\frac{\partial u_i}{\partial x_i} = 0 \Rightarrow \frac{\partial U_i}{\partial x_i} = 0$$

(fluctuations and mean flow are seperately incompressible)

The momentum equation is

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i = -\frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

Split into mean and fluctuating parts and take the average

$$\Rightarrow \quad \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\underbrace{\overline{u_j' \frac{\partial u_i'}{\partial x_j}}}_{\text{can write this as } \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)} - \frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

mean part of the pressure

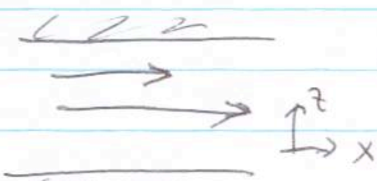
⇒ we can write the momentum equation for the mean flow as

$$\rho\left(\frac{\partial \underline{U}}{\partial t} + \underline{U}\cdot\nabla \underline{U}\right) = \nabla\cdot\underline{\underline{T}}$$

where
$$T_{ij} = -\delta_{ij} P - \underbrace{\rho\,\overline{u_i' u_j'}}_{\text{REYNOLD'S STRESS}}$$

Correlations in the velocity fluctuations can lead to transport of momentum.

eg. in the pipe



the term $\tau_{xz} = -\rho\,\overline{u_z' u_x'}$

acts to even out the flow, vertically transporting the x momentum.

If we had a relation (a closure relation) between $\overline{u_i' u_j'}$ and the mean flow we could solve for the mean flow.

It is often assumed for simplicity that

$$\overline{u_i' u_j'} = -\underbrace{D_T}_{\text{"Eddy viscosity"}}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

[i.e. same type of relation as microscopic viscosity]

Note the crucial difference with the viscous case, however: even if such a relation were valid (probably not), D_T is a property of the flow not the fluid.

Transport of a scalar

e.g. $$\rho c_p \left(\frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) = \frac{\partial}{\partial x_j} \left(K \frac{\partial T}{\partial x_j} \right)$$

Now decompose $\underline{u}$ and T

$$\underline{u} = \underline{U} + \underline{u}' \qquad T = T + T'$$

where $$\overline{T'} = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} T'(t)\, dt$$

$$\Rightarrow \quad \rho c_p \left(\frac{\partial T}{\partial t} + \underline{U} \cdot \nabla T \right) = \frac{\partial}{\partial x_j} \left(\underbrace{-\rho c_p \overline{T' u_j'}}_{\substack{\text{turbulent} \\ \text{heat flux} \\ \rho c_p \overline{T' u_j'}}} + K \frac{\partial T}{\partial x_j} \right)$$

Mixing length theory

A way to model the turbulent transport. In stars, it is applied to the transport of heat in stellar convection zones.

The idea is to think of a "blob" of fluid maintaining its identity ~~through~~ as it moves a distance ℓ the "mixing length"

The equation of motion is $$\frac{\partial u'}{\partial t} = -g \frac{\delta \rho}{\rho} \qquad \left[\text{ignore pressure perturbations} \right]$$

$$= -N^2 \xi_z$$

$$= +\frac{g}{H} \frac{\chi_T}{\chi_\rho} (\nabla - \nabla_{ad}) \xi_z$$

integrate this $\Rightarrow$ $$\boxed{u'^2 \simeq g \ell^2 \left(\frac{\nabla - \nabla_{ad}}{H} \right)}$$

The main assumption is that the temperature fluctuation is determined by the background variation : $\frac{T'}{T} = \ell \left(\frac{\nabla - \nabla_{ad}}{H}\right)$

Then $\overline{u'T'} \approx \sqrt{g\beta}\,\ell \cdot \beta \ell T$

where $\beta = \frac{\nabla - \nabla_{ad}}{H}$

or $\boxed{F_{conv} = \rho c_p T \sqrt{g}\, \beta^{3/2} \ell^2}$

an expression for the convective flux as a function of the free parameter ℓ.

Note that $\left(\frac{u'}{c_s}\right)^2 = \frac{\beta g \ell^2 \rho}{P} = \beta \ell \left(\frac{\ell}{H}\right)$

$= (\nabla - \nabla_{ad}) \left(\frac{\ell}{H}\right)^2$

Calibration of stellar models $\Rightarrow$ $\ell \sim H$

so that if $\boxed{u' \ll c_s \text{ then } \nabla \simeq \nabla_{ad}}$

this is called "efficient convection"

We can also write $F_{conv} = \underbrace{\rho c_p T}_{\approx P = \rho g H} \cdot \sqrt{g}\, \beta^{3/2} \ell^2$

$= \rho\, g^{3/2} \beta^{3/2} \ell^2 H$

$= (\rho u'^3) \left(\frac{H}{\ell}\right)$

eg. for the Sun. Assume $\ell = H$

then $v_{conv} \approx (F/\rho)^{1/3} \simeq \left(\frac{L}{4\pi R^2 \rho}\right)^{1/3}$

take $R = 7\times 10^{10}$ cm
$L = 4\times 10^{33}$ erg/s
$\rho = 1$ g/cm^3

$$\left.\begin{aligned} R &= 7\times 10^{10}\ \mathrm{cm} \\ L &= 4\times 10^{33}\ \mathrm{erg/s} \\ \rho &= 1\ \mathrm{g/cm^3} \end{aligned}\right\} \Rightarrow \quad v_{conv} = \frac{4000\ \mathrm{cm/s}}{\rho^{1/3}}\left(\frac{L_{33}}{4}\right)^{1/3}$$

Compare the sound speed $c_s = \left(\frac{kT}{m_p}\right)^{1/2} \simeq 30\,\mathrm{km/s}\ T_5^{1/2}$.

$$\Rightarrow \quad \nabla - \nabla_{ad} \approx \left(\frac{u'}{c_s}\right)^2 \sim 10^{-6}$$

The temperature gradient is extremely close to adiabatic - we can use this idea in stellar evolution codes - if the Schwarzschild criterion is violated, then limit the temperature gradient to be ∇_{ad}.

(for the temperature gradient calculated based on radiative heat transport).

December 4th, 2007.

PHYS 643 lecture 24

First, talk about turbulent transport (see notes from last time)
- Reynold's stress
- mixing length theory and stellar convection

Two further examples:

1) Stratified flow The important parameter in a stratified flow is the Richardson number Ri which we saw in the context of shear instabilities.

When $Ri \gg 1$, the turbulence is likely to be strongly anisotropic
eg. Spiegel & Zahn (1992) discussion of solar tachocline
Zahn (1992) "shellular rotation"

In a rotating flow, the tendency is for the flow to adopt constant velocity on cylinders – the Taylor-Proudman theorem. $\Omega(R)$

[To see this, consider an incompressible steady flow in the rotating frame $2\underline{\Omega} \times \underline{v} = -\frac{\nabla P}{\rho}$

take the curl: $\Rightarrow -(\underline{\Omega}\cdot\nabla)\underline{v} = 0$]

Zahn argues that turbulent mixing in the horizontal direction will lead to constant Ω on spherical shells instead $\Omega(r)$.

For the vertical viscosity, estimate $\nu \simeq \frac{1}{3} v l$

where $\left(\frac{N^2}{(dU/dz)^2}\right)\left(\frac{vl}{K}\right) \simeq Ri_c$ ← critical Ri number for instability

(the usual Ri number) →

← thermal conduction gives rise to instability even when $N^2 \gg (dU/dz)^2$

2) <u>Accretion disks</u> The idea is that turbulent viscosity provides the angular momentum transport that allows accretion to proceed.

In the classical model of a thin accretion disk (Shakura & Sunyaev 70's) (see Pringle ARAA 1981) the viscosity is modelled as

$$\nu = \alpha c_s H$$

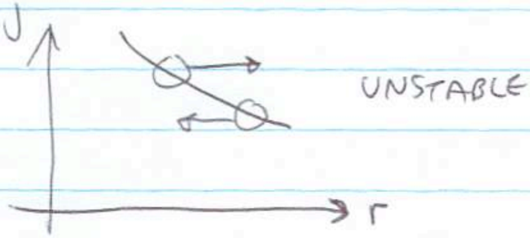
(the so-called "alpha prescription")

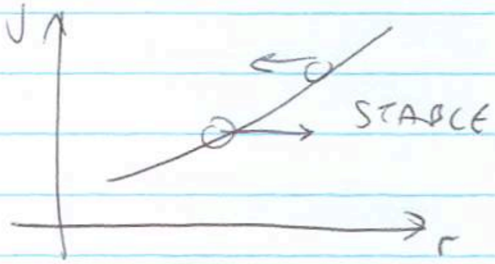
H: disk thickness or vertical scaleheight

with the same form for the viscous term as a Newtonian fluid (ie. $\sim \nu \, \partial^2 u / \partial R^2$)

In fact it turns out to be quite hard to get instability in a hydrodynamic disk

The classical instability criterion for a differentially-rotating system to axisymmetric perturbations is the <u>Rayleigh criterion</u> that angular momentum should <u>decrease</u> outwards.



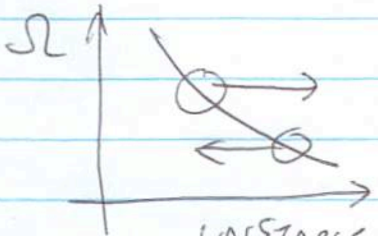


But a Keplerian accretion disk has $j = \sqrt{GMr}$ increasing outwards.

Numerical experiments which drive convective turbulence in the disk furthermore show that hydrodynamic turbulence actually transports angular momentum <u>inwards</u> not outwards!

[Application of this idea to solar convection zone]

The magnetic field is key here. One can show that introducing a magnetic field — no matter how weak — qualitatively changes the instability criterion to $\frac{d\Omega}{dr} < 0$ which is satisfied in the disk! ($\Omega_{\text{Keplerian}} \propto r^{-3/2}$) The point is that the magnetic field allows angular momentum transfer between fluid elements



UNSTABLE!

The fluid motions are now $\simeq$ at constant Ω rather than j.

The instability is known as the <u>Magnetorotational Instability</u> (MRI)

(Chandrasekhar, Balbus, Hawley)

Simulations which follow the non-linear development of the instability show that it transports angular momentum in the desired direction — <u>outwards</u>!

On that note we slightly abruptly bring the course to an end!

We'll finish on Thursday and Friday with the project presentations.

PHYS 643
1. Fluid basics

Idea of a fluid as having $\lambda \ll L$. The mean free path $\lambda = 1/n\sigma$.

Proving vector identities using index notation.

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Eulerian vs. Lagrangian descriptions. Advective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}$$

Continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\,(\rho\boldsymbol{u})$$

or

$$\frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\,\boldsymbol{u}$$

Momentum equation

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\rho\boldsymbol{g} - \boldsymbol{\nabla}P + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}$$

Acceleration due to gravity $g = -\boldsymbol{\nabla}\Phi$, gravitational potential Φ obeys Poisson's equation $\nabla^2\Phi = 4\pi G\rho$.

Hydrostatic balance. Plane-parallel atmosphere $dP/dz = -\rho g$. Isothermal atmosphere $\rho = \rho_0\exp(-z/H)$, scale height $H = k_BT/\mu m_p g$.

Energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho U\right) + \frac{\partial}{\partial x_j}\left(u_j\left[\frac{1}{2}\rho u^2 + \rho U + P\right]\right) = (\rho\epsilon - \boldsymbol{\nabla}\boldsymbol{\cdot}\,\boldsymbol{F}) + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{f}$$

The P term in the energy flux as representing PdV work.

Vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ and circulation $\Gamma = \int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{dl}$. Rigid rotation $\omega = 2\Omega$. Shear flow $\omega = du/dz$.

Kelvin's circulation theorem: $D\Gamma/Dt = 0$ for a barotropic fluid. The idea that vortex lines are carried bodily by the fluid. The local vorticity can change because of vortex stretching or vortex tipping.

Generation of vorticity by baroclinicity. The baroclinic vector $\boldsymbol{\nabla} P \times \boldsymbol{\nabla} \rho$.

Bernoulli's principle. $u^2/2 + \Phi + h =$constant along a streamline.

Magnetic force density

$$\frac{\boldsymbol{J}\times\boldsymbol{B}}{c} = -\boldsymbol{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{B}}{4\pi}.$$

The force is perpendicular to the field, and has two pieces - magnetic pressure and tension.

Ohm's law $\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}/c = \boldsymbol{J}/\sigma$. Ampere's law $\boldsymbol{J} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$.

Induction equation.

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\,(\boldsymbol{u}\times\boldsymbol{B}) - c\boldsymbol{\nabla}\times\,\left(\frac{\boldsymbol{J}}{\sigma}\right).$$

When the first term dominates, "ideal MHD": the magnetic field lines are frozen into the fluid. The second term represents Ohmic diffusion, which allows field lines to move through the fluid. The magnetic diffusivity is $\eta = c^2/4\pi\sigma$. The relative importance of the two terms is measured by the magnetic Reynold's number $R_M = UL/\eta$.

Magnetic energy density $B^2/8\pi$ (same as the pressure). Magnetic energy equation:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\boldsymbol{\nabla}\cdot\,\left(\frac{c\boldsymbol{E}\times\boldsymbol{B}}{4\pi}\right) - \frac{J^2}{\sigma} - \boldsymbol{u}.\left(\frac{\boldsymbol{J}\times\boldsymbol{B}}{c}\right)$$

The different terms are: Poynting flux at the surface, Ohmic dissipation, energy transfer to kinetic energy via work done against the magnetic force.

Reading

Choudhuri §4, 14.1, 14.2

PHYS 643
2. Objects in hydrostatic balance

Simple scaling arguments: $P_c \approx GM^2/R^4$, $k_BT_c \approx GMm_p/R$.

Equation of state of an ideal gas of fermions. (a) A non-degenerate gas of non-relativistic particles has

$$P = nk_BT, \qquad U = \frac{3}{2}nk_BT = \frac{3}{2}P, \qquad \mu = k_BT\ln\left(\frac{n}{n_Q}\right),$$

where $n_Q = (2\pi mk_BT/h^2)^{3/2}$. The non-degenerate limit is when $\mu/k_BT \ll -1$ or $n \ll n_Q$.

(b) A fully-degenerate gas has $\mu = E_F \gg k_BT$. Fermi wavevector

$$k_F = (3\pi^2 n)^{1/3} = p_F/\hbar$$

Non-relativistic particles have

$$E_F = \frac{p_F^2}{2m} \propto n^{2/3}, \qquad P = \frac{2}{5}nE_F = K_{e,nr}\rho^{5/3} \qquad U = \frac{3}{2}P$$

Relativistic particles have

$$E_F = p_Fc \propto n^{1/3} \qquad P = \frac{1}{4}nE_F = K_{e,r}\rho^{4/3} \qquad U = 3P$$

Radiation

$$U = aT^4, \qquad P = \frac{1}{3}aT^4 = \frac{1}{3}U$$

where the radiation constant $a = 7.5657 \times 10^{-15}$ cgs.

Mean molecular weights.

$$\rho Y_i = n_i m_p \qquad \rho = \mu_i n_i m_p$$

defines Y_i and μ_i for species i. Mass fraction of ion species i defined by $X_i\rho = A_i n_i m_p$. Relation between Y_i and X_i is $Y_i = X_i/A_i$. Mean molecular weight $\mu^{-1} = \mu_e^{-1} + \mu_{\rm ion}^{-1}$. Ideal gas of ions and electrons has $P = \rho k_BT/\mu m_p$.

Different regimes for a mixture of ions, electrons, and radiation. When does each of these dominate the pressure? When are the electrons degenerate or non-degenerate, relativistic or non-relativistic?

White dwarfs. For low masses, $\gamma \approx 5/3$, and $R \propto M^{-1/3}$. Chandrasekhar mass $M_{Ch} = 1.45\ M_\odot (Y_e/0.5)^2$. Mass radius relation

$$R \approx 8 \times 10^8 \text{ cm } \left(\frac{M}{M_\odot}\right)^{-1/3} \left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2}$$

Neutron stars. A star held up by non-relativistic proton/neutron degeneracy pressure rather than electrons has a radius smaller by a factor $\approx m_p/m_e \approx 2000$. Typical model neutron star radii are $\approx$ 10–15 km. Interactions give an equation of state roughly $P \propto \rho^2$ which leads to a radius which is almost independent of mass.

Couloumb pressure in a degenerate gas. The electrons form an almost uniform background. Wigner-Seitz approximation:

$$U_C = -n_e \frac{9}{10} \frac{Ze^2}{R_Z}$$

The Coulomb pressure is $P = -K_C \rho^{4/3}$ with

$$K_C = 2.2 \times 10^{12} \text{ erg cm}^{-3} Z^{2/3} (Y_e/0.5)^{4/3}.$$

Density of zero-pressure matter $\rho = (K_C/K_{e,nr})^3$.

Mass radius relation

$$R = \frac{K_e}{GM^{1/3} + K_C M^{-1/3}}$$

“Hot” objects. $k_B T$ sets the pressure rather than E_F. Central temperature $T_c \approx GM\mu m_p/k_B R$. Heat transport

$$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}$$

For constant opacity (e.g. electron Thompson scattering) $L \propto M^3$.

The idea that a core can only support a finite size envelope. Application to helium cores (Schönberg-Chandrasekhar limit) and planet formation (runaway accretion to form Jupiter).

Reading

This part of the course is not covered in Choudhuri. The best places to look are books on stellar structure, in particular:

Clayton, “Principles of Stellar Evolution and Nucleosynthesis”, Chapter 2.

Hansen & Kawaler, “Stellar Interiors” (the latest edition of this book is Hansen, Kawaler, & Trimble), mostly Chapter 3.

White dwarf mass-radius relation compared to observations: Provencal et al. 1998, ApJ, 494, 759

Neutron star mass-radius relations: Lattimer & Prakash 2001, ApJ, 550, 426

Mass-radius relations for low mass stars and planets: Deloye & Bildsten 2003, ApJ, 598, 1217; Fortney et al. 2007, ApJ, 659, 1661

PHYS 643
3. Compressible Fluids

Sound waves. Sound speed $c_s^2 = (\partial P/\partial\rho)$. Adiabatic sound speed $c_s^2 = \gamma P/\rho$, isothermal sound speed $c_s^2 = P/\rho$. Dispersion relation $\omega = \pm c_s k$.

Waves in a magnetized fluid. The Alfven velocity $\boldsymbol{v_A} = \boldsymbol{B}/\sqrt{4\pi\rho}$. Fast magnetosonic wave (compressible wave across field lines) $\omega^2 = k^2(c_s^2 + v_A^2)$. Slow magnetosonic wave (compressible wave along field lines) $\omega^2 = k^2 c_s^2$. Alfven wave (transverse wave restored by magnetic tension) $\omega^2 = v_A^2 k^2$.

General solution to linearized wave equation. $\delta\rho = f(x - ct) + g(x + ct)$.

Characteristics for compressible flows. Riemann invariants for isentropic flow $J_\pm = u \pm \int dP/\rho c_s = u \pm 2c/(\gamma - 1)$. Along a C_+ characteristic, J_+ is constant, and J_- determines the shape of the curve (vice-versa for C_-). The example of a piston moving into a shock tube. Formation of shocks.

Shock jump conditions derived from conservation laws. The density contrast as a function of Mach number $M = u_1/c_1$. A strong shock ($M^2 \gg 1$) has

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1}$$

which is 4 for $\gamma = 5/3$. The entropy increases across the shock as energy in bulk motion goes into internal energy. Radiative shocks. Isothermal jump conditions. Thin, slow moving shell associated with radiative shocks.

Self-similar flows. Sedov-Taylor solution for a spherical blast wave. Constant pressure, low density interior. Most of the mass is behind the shock. Application to supernova remnants. Three phases of SNR evolution: ballistic phase, energy-conserving self-similar phase, snowplough momentum-conserving phase.

Transition from subsonic to supersonic flow. There is a maximum mass flux density in a 1D flow, which occurs at the sonic point. Subsonic flow has increasing flux with increasing velocity; supersonic flow has decreasing flux with increasing velocity. The de Laval nozzle. Using Bernoulli's principle to calculate the velocity as a function of pressure. Blandford & Rees (1974) application to outflows from active galaxies.

Spherical accretion and winds. Bondi-Hoyle accretion rate

$$\dot{M} \approx \pi (GM)^2 \rho_\infty / c_\infty^3$$

or for moving star

$$\dot{M} \approx \pi(GM)^2\rho_\infty/v^3$$

Parker's solution for the solar wind.

Relativistic hydrodynamics. Energy momentum tensor

$$T^{\mu\nu} = \frac{wu^\mu u^\nu}{c^2} + P\eta^{\mu\nu}$$

where $w = e+P$ is the enthalpy, e is the energy density, and P the pressure, all measured in the rest frame of the fluid element. The four-velocity is $u^\mu = \gamma(c, \boldsymbol{u})$, $\eta^{\mu\nu}$ is the metric.

Sound waves. The sound speed is $c_s^2 = c^2(\partial e/\partial P)_S$. For an ultrarelativistic gas, this is $c_s = c/\sqrt{3}$.

Bernoulli's constant for relativistic flows is $\gamma w/n$.

Relativistic shocks. In the frame of the shock, matter flows into the shock with $\beta \approx 1-1/2\Gamma^2$ and leaves with $\beta \approx 1/3$. In the frame of the undisturbed fluid, the postshock material has $\gamma_2 = \Gamma/\sqrt{2}$. The (rest frame) density increases by a factor $\approx \Gamma$ across the shock. The energy density increases by a factor Γ^2 across the shock.

Reading

Choudhuri Chapter 6.

The best places to look for a discussion of characteristics and shock development are:

Landau & Lifshitz, Fluid Mechanics (Course of Theoretical Physics Volume 6)

Zeldovich & Raizer, Physics of Shock Waves and High Temperature Hydrodynamic Phenomena

Blandford & Rees (1974) AGN outflows as relativistic de Laval nozzles

For a start on radiative shocks, see Shu's volume 2.

Taylor's two papers are (1950) Proc Roy Soc London A201, 159

Models of supernova remnants: Mansfield & Salpeter (1974) McKee (1974, reverse shock), Chevalier (1974)

Spherical accretion and winds: it's really worth reading the original papers, especially to get the motivation and context. Bondi (1952), Hoyle & Lyttleton (1939), Parker (1958)

There is a brief discussion of relativistic hydrodynamics in Choudhuri, but see Landau & Lifshitz Chp XV for a good treatment.

Blandford & McKee (1976) Phys Fluids give jump conditions for relativistic shocks, and derive self-similar solutions for a relativistic blast wave.

PHYS 643
4. Oscillations and Instabilities

Eulerian and Lagrangian perturbations. $\Delta f = \delta f + \boldsymbol{\xi}\cdot\boldsymbol{\nabla} f$. Velocity perturbation $\Delta \boldsymbol{u} = D\boldsymbol{\xi}/Dt$. The perturbed continuity equation is

$$\frac{\Delta\rho}{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi} \qquad \delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi})$$

valid for a static background or when there is a background flow.

Surface gravity waves in an incompressible fluid. In the deep limit $k_\perp H \gg 1$, the wave frequency is given by $\omega^2 \approx gk_\perp$, and the motions are approximately circular, $\xi_z \approx \xi_\perp$. In the shallow limit $k_\perp H \ll 1$, the wave frequency is $\omega^2 \approx gk_\perp^2 H$, and the motions are mostly horizontal, $\xi_z/\xi_\perp \approx (k_\perp H) \ll 1$.

Oscillations in a stratified fluid. The Brunt-Väisälä frequency or buoyancy frequency N, given by $N^2 = -gA$ with

$$A = \frac{d\ln\rho}{dr} - \frac{1}{\gamma}\frac{d\ln P}{dr}$$

The propagation diagram for the modes. p-modes are high frequency ($\omega > c_s k_\perp$, $\omega > N$) modes with $\omega \approx c_s k$, g-modes are low frequency ($\omega < N$, $\omega < c_s k_\perp$) modes with $\omega \approx N(k_\perp/k)$.

Convective instability occurs when $A > 0$ or $N^2 < 0$. This is the Schwarzschild criterion for convection. In terms of temperature, instability occurs if

$$\frac{d\ln T}{d\ln P} > \nabla_{ad}$$

or if entropy decreases outwards (direction opposite to gravity).

Interchange and Parker instability, The idea of magnetic buoyancy. An isolated flux tube is buoyant with respect to its surroundings. The MHD energy principle as a way to assess stability.

Shear instabilities. Rayleigh's inflexion point theorem d^2U/dz^2 must change sign somewhere in the flow. Fjortoft's theorem that the vorticity must have a maximum. Howard's semicircle theorem that somewhere in the flow the phase velocity of the unstable mode equals the fluid velocity. In a stratified fluid, $Ri < 1/4$ for instability, where the Richardson number $Ri \equiv N^2/(dU/dz)^2$ compares the work done against gravity to the energy available in the shear.

Reading

Choudhuri Chapter 7 and parts of Chapter 14 (for discussion of magnetic buoyancy and Parker instability).

Two classic books on stellar pulsations are J. P. Cox (1980) "Theory of Stellar Pulsation" and Unno et al. (1989) "Nonradial Oscillations of Stars"

Two classic books on instabilities are "Hydrodynamic and Hydromagnetic Stability" by Chandrasekhar, and "Hydrodynamic Stability" by Drazin and Reid.

The onset and non-linear development of convection is covered in books on stellar structure and evolution, e.g. "Stellar Interiors" by Hansen, Kawaler, & Trimble.

I included some plots from "Lecture Notes on Stellar Oscillations" by J. Christensen-Dalsgaard which you can find on the web.

Papers on the MHD energy principle and interchange/Parker instabilities are Bernstein et al. (1958), Greene & Johnson (1968), Newcomb (1961), Parker (1966).

Shear instabilities: Miles, Howard, Chimonas