PHYS 643 Astrophysical Fluids

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Preamble

This document collects together notes for the graduate course PHYS 643 Astrophysical Fluids, as taught in the Winter terms of 2016 and 2018 at McGill University. The final two weeks, covering Rotating Fluids and Turbulence, are missing and (hopefully) will be added in a future update.

The course was organized around a different topic each week, with one class devoted to lecture and the other to student presentations on papers. The papers for each topic are listed at the end of each chapter. The material covered in class was put into action in the computational exercises, included here, which were completed as homework, and in a final numerical project.

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Week 1: Introduction

These notes are for the first week of PHYS 643 Astrophysical Fluids. The idea is to introduce the fluid equations, laying the groundwork for the specific topics in future weeks.

What is a fluid?

The fluid equations apply when the mean free path of particles λ is much smaller than the distances over which bulk properties, such as temperature and density, are varying.

For example, let's estimate λ in the Sun. At the center of the Sun, the temperature is $T \approx 10^7$ K and density $\rho \approx 150$ g cm^{-3}, so the matter is a completely-ionized plasma of (mostly) protons and electrons. The mean free path is given by $n\sigma\lambda = 1$ where n is the number density of scatterers and σ is the scattering cross-section. We can get n from the density, $n \approx \rho/m_p \sim 10^{26}$ cm^{-3}. The Coulomb cross-section is given roughly by writing down the scalings

$$\frac{e^2}{r} \sim k_B T \qquad \sigma \sim \pi r^2 \sim \frac{e^4}{(k_B T)^2}.$$

Plugging in numbers gives

$$\lambda \sim 10^6 \text{ cm} \, \frac{T^2}{n} \sim 10^{-6} \text{ cm}.$$

Much smaller than the radius of the sun $R_\odot \approx 7 \times 10^{10}$ cm, which is the scale on which the temperature varies. This large difference in scales means that the particles are in *local thermodynamic equilibrium* (LTE). For example, they have a Maxwell-Boltzmann distribution at the local temperature.

Under these conditions, we can treat the matter as a continuum and describe the matter with a set of conservation equations for mass, momentum and energy — the fluid equations. We don't have to worry about following the trajectories and interactions of individual particles (like in an N-body simulation of a star system for example), although there is a systematic derivation of the fluid equations from such a starting point (the Boltzmann equation), expanding in the small parameter λ/L. This is covered in the early chapters of Choudhuri.

Just to give a couple of situations in astrophysics where the fluid approximation is not so good, consider gas in a galaxy cluster with temperature $\sim 10^8$ K and $n \sim 10^{-3}$ cm^{-3}, then $\lambda \sim 10^{24}$ cm ≈ 0.3 Mpc, which is a large fraction of a typical cluster size. In the solar wind near Earth, $T \sim 10^5$ K and $n \sim 10$ cm^{-3} gives $\lambda \sim 10^{14}$ cm which is several AU.

The continuity equation, advective derivative, Eulerian and Lagrangian approaches, incompressible fluid

The continuity equation describes mass conservation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) \,.$$

This has a flux-conservative form: rate of change of a density on one side and the divergence of the flux of that quantity on the other side. The mass flux is $\rho\mathbf{u}$ (units: g cm^{-2} s^{-1}). Using the advective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

this can be rewritten

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}.$$

Make sure you are comfortable with the physical interpretation of this: e.g., if the flow converges, mass is flowing to a point and so the local density has to increase.

The advective derivative is also known as the Lagrangian derivative. It represents the rate of change of a quantity following along with the fluid element (Lagrangian approach) rather than asking what is the rate of change of the quantity at a fixed point in space (Eulerian approach).

An incompressible fluid (e.g. water) has a constant density, so that $D\rho/Dt = 0$ and $\nabla\cdot\mathbf{u} = 0$. (Think about how when a river widens, the water slows down so that the mass flow rate is the same). Incompressibility is a good approximation when the flow is subsonic $|u| \ll c_s$ because then any density variations will be rapidly smoothed out by sound waves much faster than the fluid motion.

Momentum equation; body and surface forces, viscosity, equation of state

The momentum equation in flux conservative form is

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = f_i + \frac{\partial}{\partial x_j}T_{ij}.$$

This describes conservation of the ith component of momentum density ρu_i (momentum per unit volume). The flux of the ith component of momentum in the j-direction is $\rho u_i u_j$.

On the right hand side, forces act to change the momentum. They are of two types. *Body forces* act on each particle in the fluid element. The body force per unit volume in the i-direction is f_i. Examples are gravity, $\mathbf{f} = \rho\mathbf{g} = -\rho\nabla\Phi$, and magnetic force, $\mathbf{f} = \mathbf{J}\times\mathbf{B}/c$.

The second term represents surface forces, and T_{ij} is a stress tensor. The diagonal elements of the stress tensor are forces that push inwards or outwards on the surface of a fluid element (direction along the normal to the surface). An example is pressure, described by

$$T_{ij} = -P\delta_{ij}$$

which gives

$$\frac{\partial}{\partial x_j}T_{ij} = -\frac{\partial}{\partial x_i}P = -\left(\nabla P\right)_i$$

the ith component of the pressure gradient. Fluid elements feel a force down the pressure gradient. Physically, the pressure force on one side of the fluid element outbalances the pressure force on the other, giving a net acceleration.

With pressure and gravity forces only, and using the continuity equation to simplify the left hand side, a common form of the momentum equation is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \rho \mathbf{g}.$$

(Think of this as $F = ma$ for a fluid element).

Viscosity in a fluid resists shear (as the random motions of particles transfer momentum between parts of the fluid moving with different velocities). It gives off-diagonal contributions to T_{ij}, i.e. the viscous force acts in a direction parallel to the surface of a fluid element rather than normal to it. In general, the viscous stress can be written

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \nabla \cdot \mathbf{u} \right) + \xi \delta_{ij} \nabla \cdot \mathbf{u},$$

where μ is the *shear viscosity* and ξ is the *bulk viscosity* (units of viscosity: g cm^{-1} s^{-1}). The velocity derivatives in the first term describe shearing motions of the fluid (as opposed to rotation of a fluid element which would have a minus sign — see vorticity below), and is the usual viscosity that we worry about. The bulk viscosity is not usually important, it describes irreversible processes that occur when a fluid element is compressed. The quantity $\nu = \mu/\rho$ is the *kinematic viscosity* (units: cm^2 s^{-1}). For an ideal gas this is roughly $\nu \sim \lambda^2/t_c \sim \lambda v_{th}$, where t_c is the collision time and v_{th} is the thermal velocity of the particles in the gas. If the fluid motions have $\nabla \cdot \mathbf{u} \approx 0$, the viscous term in the momentum equation simplifies to

$$\frac{\partial}{\partial x_j} T_{ij} \approx \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} \right)$$

or for constant μ,

$$\approx \mu \nabla^2 \mathbf{u}.$$

This last form shows that viscosity leads to diffusion of momentum.

The momentum and continuity equations describe the fluid motion completely if we know how to relate P and ρ, which depends on the equation of state of the fluid. For example, if the fluid flow is rapid enough that there is no time for heat flow between fluid elements, the motion is adiabatic and we can write $P \propto \rho^\gamma$, where γ is the adiabatic index. The opposite limit is extremely rapid heat transport so that the gas remains isothermal, $P \propto \rho$. Intermediate cases require that we also follow the temperature of the gas which requires a third equation, the energy equation. We'll come to that soon.

Magnetic fields: the MHD equations

For a magnetized plasma, we already mentioned the $\mathbf{J} \times \mathbf{B}$ force in the momentum equation[1] We also need to discuss how the magnetic field evolves. The electric field by

[1] If there is a non-zero charge density ρ_e, there will also be an electric force $\rho_e \mathbf{E}$, but usually in astrophysical situations the plasma is electrically neutral $\rho_e = 0$.

Ohm's law[2] is

$$\mathbf{E} = -\frac{\mathbf{u}\times\mathbf{B}}{c} + \frac{\mathbf{J}}{\sigma}.$$

The first term on the right hand side comes from the relativistic transformation from the fluid rest frame to the frame in which the fluid is moving with velocity $\mathbf{u}$. Faraday's law then gives the time-dependence of $\mathbf{B}$,

$$\frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E} = \nabla\times(\mathbf{u}\times\mathbf{B}) - c\nabla\times\left(\frac{\mathbf{J}}{\sigma}\right).$$

This is known as the *induction equation*.

The first term on the right hand side describes *flux freezing*. In the absence of ohmic dissipation (*ideal MHD*), magnetic field lines move with the fluid. A good way to see this is to derive an equation for the separation $d\ell$ between two fluid elements in a fluid. It turns out to be of the same form as the induction equation (without the ohmic term) but with $\mathbf{B}\rightarrow d\ell$. So if you take two fluid elements and follow them as they move through the flow, their separation vector and the local magnetic field vector evolve in the same way. That tells you that magnetic field lines are tied into the fluid.

To see the effect of the ohmic term, use Ampere's law[3], which gives the current density

$$\mathbf{J} = \frac{c}{4\pi}\nabla\times\mathbf{B}.$$

The induction equation is then

$$\frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E} = \nabla\times(\mathbf{u}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B}),$$

where the magnetic diffusivity $\eta = c^2/(4\pi\sigma)$. Since $\nabla\cdot\mathbf{B}=0$, $\nabla\times\nabla\times\mathbf{B} = -\nabla^2\mathbf{B}$ and so for constant η the ohmic term in the induction equation is

$$\frac{\partial\mathbf{B}}{\partial t} = \eta\nabla^2\mathbf{B},$$

a diffusion equation for $\mathbf{B}$. We see that ohmic dissipation gives rise to *ohmic diffusion* of the magnetic field. It breaks flux freezing, and leads to motion of the field lines within the fluid.

The fluid equations with the $\mathbf{J}\times\mathbf{B}$ force, the induction equation, and Ampere's law together form the equations of magnetohydrodynamics (MHD).

One more thing we can do is to look in more detail at the $\mathbf{J}\times\mathbf{B}$ force using Ampere's law to write $\mathbf{J}$ in terms of $\mathbf{B}$. Then

$$\frac{\mathbf{J}\times\mathbf{B}}{c} = \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi}.$$

[2] In fact, there are other terms that can appear in Ohm's law. See the Appendix for a more general derivation of Ohm's law using the two fluid equations.

[3] The term $\partial\mathbf{E}/\partial t$ in Ampere's law can be dropped as long as the timescale on which $\mathbf{B}$ is evolving is much longer than a light crossing time. Note that we then have $\nabla\cdot\mathbf{J}=0$, consistent with charge conservation.

The first term is the gradient of the magnetic pressure $B^2/8\pi$. The second term has two pieces. One piece has a direction along the field, and cancels the gradient along the field from the first term. The net effect is that the magnetic pressure acts only perpendicular to the field (as it must since the force is $\mathbf{J} \times \mathbf{B}$). So if you grab a flux tube and squeeze it, you will feel the magnetic pressure pushing back. The other piece of the $\mathbf{B} \cdot \nabla \mathbf{B}$ term is *magnetic tension*, which tries to make the fields lines straighten (like an elastic string). The magnitude of the tension force per unit volume is $B^2/4\pi R_c$, where R_c is the radius of curvature of the field line. We'll see later this force supports Alfven waves.

Energy equation

It helps to consider the bulk kinetic energy, internal energy, and magentic energy separately.

An equation for the kinetic energy density $(1/2)\rho u^2$ comes from carrying out the dot product

$$\mathbf{u} \cdot (\text{momentum equation}) \Rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u^2 u_j\right) = \mathbf{u} \cdot \mathbf{f} + u_i \frac{\partial T_{ij}}{\partial x_j}.$$

Again this is flux-conservative form and says that the kinetic energy density changes if there is mechanical work $\mathbf{u} \cdot \mathbf{f}$ on the fluid element, either from body or surface forces.

For internal energy, we can start with the 1st law of thermodynamics $dE = TdS - PdV$ which we write per unit mass as

$$de = Tds + \frac{Pd\rho}{\rho^2}. \tag{1.1}$$

For a given fluid element, the rate of change of entropy,

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}, \tag{1.2}$$

is the rate of change of heat content of the fluid element. It can come from internal heating or cooling (e.g. nuclear reactions that deposit energy in the gas or neutrinos that leave the volume and act as a volumetric cooling source), or from a heat flux at the surface of the fluid element. The heat flux $\mathbf{F}$ can often be written $\mathbf{F} = -K\nabla T$ where K is the thermal conductivity (heat flows down the temperature gradient). Including both contributions, we write

$$T\frac{Ds}{Dt} = \epsilon - \frac{1}{\rho}\nabla \cdot \mathbf{F},$$

the entropy equation.

We already mentioned the adiabatic approximation that $P \propto \rho^\gamma$ if there is no time for heat flow. Here, $\gamma = c_P/c_V$ is the ratio of specific heats. We can see this directly by demanding that fluid elements conserve entropy

$$\frac{Ds}{Dt} = \frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right) = 0.$$

In the second step, we used eq. (1.1) and $P = (\gamma - 1)\rho e$ to rewrite equation (1.2).

Adding the kinetic energy to the internal energy we get an equation for the total energy (neglecting magnetic energy)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x_j}\left(u_j\left[\frac{1}{2}\rho u^2 + \rho e + P\right]\right) = \left(\epsilon - \frac{\nabla \cdot \mathbf{F}}{\rho}\right) + \mathbf{u}\cdot\mathbf{f}.$$

(We write only the body force piece of the mechanical work for simplicity). Note that the *enthalpy* $h = e + P/\rho$ appears in the flux term. The enthalpy is often a more useful quantity than internal energy in flows at constant pressure, since it takes into account the PdV work done as the fluid moves around. It often comes up in chemistry, for example.

To include magnetic energy, we can dot **B** into the induction equation. This gives an equation for the magnetic energy density $B^2/8\pi$,

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\nabla\cdot\left(\frac{c\mathbf{E}\times\mathbf{B}}{4\pi}\right) - \mathbf{E}\cdot\mathbf{J}$$

which you may have seen before in electromagnetism. The first term on the right hand side is the divergence of the Poynting flux; the second is Ohmic dissipation.

Using Ohm's law, the $\mathbf{J}\cdot\mathbf{E}$ term can be written as two terms

$$-\mathbf{E}\cdot\mathbf{J} = -\frac{J^2}{\sigma} - \mathbf{u}\cdot\frac{\mathbf{J}\times\mathbf{B}}{c}.$$

The first is the energy dissipation rate from ohmic heating. This converts magnetic energy into internal energy: J^2/σ is the heating rate per unit volume. The second term has the same form as the mechanical work term $\mathbf{u}\cdot\mathbf{f}$ in the kinetic energy equation, but with opposite sign. This shows that the work done on the fluid by the $\mathbf{J}\times\mathbf{B}$ force takes energy from (or puts energy into) the magnetic field.

Examples

Here are two example problems to work on that will give you a chance to play around with the fluid and MHD equations:

1. *Magnetic field winding*. Consider a spherical star which is differentially rotating such that the fluid velocity is $\mathbf{u} = \hat{\phi}\ R\Omega(R)$, where we use cylindrical coordinates (R, ϕ, z) with z along the rotation axis. A poloidal magnetic field $(B_R(R, z), 0, B_z(R, z))$ threads the star initially.

(a) First assume that the velocity does not change over time. What does the induction equation imply for the subsequent evolution of the field? Explain your result physically.

(b) Now write down the momentum equation for the fluid and include the back reaction of the field on the fluid. What is the evolution in time?

2. *Electric field in an atmosphere*. Consider a plane-parallel atmosphere of fully ionized hydrogen gas. By writing down the momentum equations for the protons and electrons separately, show that (1) the structure of the atmosphere is given by $dP/dz = -\rho g$, where P is the sum of the electron and proton pressures, and (2) there is an electric field in the atmosphere. What is the value of the electric field, and what is its role?

Appendix: Two fluid equations

Another way to approach the MHD equations is to consider the electron and ions separately. Coupled by a collisional term, the momentum equations for each species are

$$n_e m_e \frac{D\mathbf{v_e}}{Dt} = -\frac{n_e m_e(\mathbf{v_e} - \mathbf{v_i})}{\tau_e} - n_e e\left(\mathbf{E} + \frac{\mathbf{v_e}\times\mathbf{B}}{c}\right) - \nabla P_e \tag{1.3}$$

$$n_i m_i \frac{D\mathbf{v_i}}{Dt} = -\frac{n_i m_i(\mathbf{v_i} - \mathbf{v_e})}{\tau_i} + n_i Ze\left(\mathbf{E} + \frac{\mathbf{v_i}\times\mathbf{B}}{c}\right) - \nabla P_i \tag{1.4}$$

where n_e, n_i, P_e and P_i are the electron and ion densities and pressures, m_e and m_i are the electron and ion masses, and τ_e and τ_i are the timescales on which the electron or ion velocity $\mathbf{v_e}$ or $\mathbf{v_i}$ relaxes due to collisions with the other species.

Charge neutrality implies that $n_e = Zn_i$. Momentum conservation also tells us that the collisional terms must cancel, i.e. $n_e m_e/\tau_e = n_i m_i/\tau_i$ or $\tau_e = (Zm_e/m_i)\tau_i$. This means that the electron velocity changes on a much faster timescale due to collisions with protons than vice versa. This makes sense if we consider two body collisions between particles with very different masses: the heavy particle undergoes a smaller velocity change by roughly the ratio of the particle masses.

The electrons and ions satisfy the continuity equations

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{v_e}) = 0$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\mathbf{v_i}) = 0.$$

Multiplying by the particle masses and adding, we find

$$\frac{\partial}{\partial t}(n_e m_e + n_i m_i) + \nabla\cdot(n_e m_e\mathbf{v_e} + n_i m_i\mathbf{v_i}) = 0$$

or

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,$$

where $\rho = n_e m_e + n_i m_i$ is the mass density and we define the fluid velocity $\mathbf{u}$ such that

$$\rho\mathbf{u} = n_e m_e\mathbf{v_e} + n_i m_i\mathbf{v_i}.$$

Note that since $m_e \ll m_i$, the fluid velocity is close to the ion velocity $\mathbf{u}\approx\mathbf{v_i}$. Subtracting the continuity equations and assuming charge neutrality gives $\nabla\cdot\mathbf{J} = 0$ as required for charge conservation.

Now add the two momentum equations (1.3) and (1.37). On the left hand side this gives

$$n_e m_e\frac{D\mathbf{v_e}}{Dt} + n_i m_i\frac{D\mathbf{v_i}}{Dt} = \rho\frac{D\mathbf{u}}{Dt}.$$

On the right hand side, the pressure gradient terms add $\nabla P_i + \nabla P_e = \nabla P$, where P is the total pressure, and the Lorentz force terms are

$$-n_e e\left(\mathbf{E} + \frac{\mathbf{v_e}\times\mathbf{B}}{c}\right) + n_i Ze\left(\mathbf{E} + \frac{\mathbf{v_i}\times\mathbf{B}}{c}\right) = n_e e\frac{(\mathbf{v_i} - \mathbf{v_e})\times\mathbf{B}}{c} = \frac{\mathbf{J}\times\mathbf{B}}{c}.$$

The final result is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \frac{\mathbf{J} \times \mathbf{B}}{c},$$

which is the familiar momentum equation for the fluid.

Ohm's law can be obtained from the electron equation of motion. We neglect the acceleration term on the left hand side, assuming that the electron velocity quickly adjusts to changes in Lorentz forces since the electrons are much less massive than the ions. Therefore

$$0 = -\frac{n_e m_e (\mathbf{v_e} - \mathbf{v_i})}{\tau_e} - n_e e \left(\mathbf{E} + \frac{\mathbf{v_e} \times \mathbf{B}}{c} \right) - \nabla P_e$$

or

$$\mathbf{E} = \frac{m_e \mathbf{J}}{n_e e^2 \tau_e} - \frac{(\mathbf{v_e} - \mathbf{v_i}) \times \mathbf{B}}{c} - \frac{\mathbf{v_i} \times \mathbf{B}}{c} - \frac{\nabla P_e}{n_e e}.$$

The electrical conductivity is $\sigma = n_e e^2 \tau / m_e$, and since $\mathbf{u} \approx \mathbf{v_i}$, we have

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} - \frac{\mathbf{u} \times \mathbf{B}}{c} + \frac{\mathbf{J} \times \mathbf{B}}{n_e e c} - \frac{\nabla P_e}{n_e e}. \tag{1.5}$$

Equation (1.5) is the Ohm's law we wrote down in the text, except for the last two terms which are the Hall term and battery term. The first of these is the Hall electric field that you may have come across before that arises when a current flows perpendicular to a magnetic field. The Lorentz force deflects the current-carrying charges until the Hall electric field grows to balance it. The battery term enters the induction equation as the cross product of the electron pressure and density gradients $\nabla P_e \times \nabla n_e$ (after taking the curl of $\mathbf{E}$) so that misalignment of the surfaces of constant electron density and constant electron pressure leads to magnetic field growth (the "battery effect").

Week 2: Cold Stars — White Dwarfs, Neutron Stars, and Planets

We start the course by discussing the topic of "cold stars", which encompasses white dwarfs, neutron stars, and planets. This is a good topic to start off with because we need only a couple of ideas: hydrostatic balance and the zero-temperature equation of state.

Hydrostatic balance

Stars and planets are in *hydrostatic balance* in which the pressure gradient from their interior to the surface balances their self-gravity[4]. Assuming spherical symmetry, the momentum equation in this situation is

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1.6}$$

where

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1.7}$$

and $m(r)$ is the mass contained within radius r. The boundary conditions are $m = 0$ at $r = 0$ and $P = 0$ at $r = R$. To solve the equations, we just need a relation between P and ρ. Under the assumption $P \propto \rho^\gamma$, the solutions are known as polytropes. A polytrope of index n has $\gamma = 1 + \frac{1}{n}$.

Stars and planets on the back of the envelope

A rough estimate of the structure is to write the two sides of the hydrostatic balance equation as

$$\frac{dP}{dr} \approx \frac{P_c}{R} \qquad \rho g \approx \frac{M}{R^3}\frac{GM}{R^2},$$

where P_c is the central pressure and R is the radius. This gives a formula for the central pressure in terms of the mass and radius of the object

$$P_c \approx \frac{GM^2}{R^4}.$$

For an ideal gas, we can get the central temperature also:

$$P_c \approx \frac{\rho_c k_B T}{m_p} \Rightarrow T_c \approx \frac{GMm_p}{k_B R}.$$

[4]To see that must be the case, look at the momentum equation

$$\frac{D\mathbf{u}}{Dt} = \frac{-\nabla P}{\rho} + \mathbf{g},$$

and imagine turning off the pressure gradients. The fluid would then accelerate in response to gravity. The time to collapse would be $\sim \sqrt{R/g} \sim \sqrt{R^3/GM}$, or about 30 minutes for the Sun, much less than its 5 billion year age. This implies the pressure gradient must balance gravity to a high degree of accuracy!

Plugging in numbers for the Sun gives $T_c \approx 2 \times 10^7$ K, pretty close to the central temperature of the Sun, 1.5×10^7 K.

For a polytropic relation $P \propto \rho^\gamma$, we can get the mass-radius scaling

$$P_c \approx \frac{GM^2}{R^4} \propto \rho^\gamma \propto \left(\frac{M}{R^3}\right)^\gamma \Rightarrow M^{\gamma-2} \propto R^{3\gamma-4}.$$

Interesting cases:

- *White dwarfs*. For non-relativistic degenerate electrons, $\gamma = 5/3 \Rightarrow R \propto M^{-1/3}$. As the white dwarf mass increases, the electrons become relativistic and $\gamma \rightarrow 4/3$. Then M becomes independent of R! The corresponding mass is the Chandrasekhar mass $M_{Ch} \approx 1.4\ M_\odot$, a maximum mass for white dwarfs.

- *Neutron star*. Degenerate neutrons hold up the star, but interactions between neutrons stiffen the EOS, giving $\gamma = 2$. Then R is independent of M, as seen in realistic calculations.

- *Incompressible material*. $\gamma \rightarrow \infty \Rightarrow M \propto R^3$, we expect this to hold for small "rocky" bodies such as moons or rocky planets. (We'll see later that gas giant planets like Jupiter lie between the $M \propto R^3$ and $M \propto R^{-3}$ limits.)

- *Isothermal sphere*. This has $\gamma = 1$ so that $P \propto \rho$. It is often used as a model of stellar systems such as globular clusters, although it has the property that $\rho \propto 1/r^2$ and therefore the mass contained within radius r grows $\propto r$, so it must be truncated to give the system a finite mass.

Equations of state

The following table summarizes the chemical potential μ, pressure P and internal energy density U for four cases of interest:

	μ	P	U
Ideal gas	$k_BT \ln\left(\frac{n}{n_Q}\right) \ll -1$ $n_Q = (2\pi m k_B T/h^2)^{3/2}$	nk_BT	$\frac{3}{2}P = \frac{3}{2}nk_BT$
Non-relativistic degenerate	$E_F = \frac{p_F^2}{2m} \propto n^{2/3} \gg k_BT$ $p_F = \hbar k_F = \hbar(3\pi^2 n)^{1/3}$	$\frac{2}{5}nE_F \propto n^{5/3}$	$\frac{3}{2}P$
Relativistic degenerate	$E_F = p_F c \propto n^{1/3}$	$\frac{1}{4}nE_F \propto n^{4/3}$	$3P$
Radiation	0	$\frac{1}{3}aT^4$	$3P = aT^4$

Mixtures

Usually in astrophysics we are dealing with a plasma consisting of a mixture of different chemical species. There is a whole terminology for dealing with this which you'll see used a lot, so we'll go through this here in some detail.

The starting point is that in a mixture we add the different contributions to the pressure from each species of particle. These depend on the number densities of different species, which can be obtained from the mass density if we know their number fraction Y_i or mean molecular weight μ_i, defined by $\rho Y_i = n_i m_p$, $\rho = \mu_i n_i m_p$, $Y_i = 1/\mu_i$. For the ions, we also define the mass fraction X_i as $\rho X_i = n_i A_i m_p$. Then $Y_i = X_i/A_i$.

As an example, consider a fully-ionized solar composition gas with hydrogen mass fraction $X_H = 0.7$ and helium mass fraction $X_{He} = 0.3$. The ion pressure is

$$P_{\text{ion}} = n_H k_B T + n_{He} k_B T = \frac{\rho k_B T}{m_p}\left(X_H + \frac{X_{He}}{4}\right) = \frac{\rho k_B T}{\mu_i m_p},$$

which defines $\mu_{\text{ion}} = (X_H + X_{He}/4)^{-1} \approx 1.3$. For a general mixture of ions,

$$Y_{\text{ion}} = \frac{1}{\mu_{\text{ion}}} = \sum Y_i = \sum \frac{X_i}{A_i}.$$

The electrons contribute $P_e = n_e k_B T$ to the pressure if they are non-degenerate. From charge neutrality, $n_e = \sum n_i Z_i$ and so

$$P_e = \frac{\rho k_B T}{m_p}\sum Y_i Z_i = \frac{\rho k_B T}{m_p}\sum \frac{X_i Z_i}{A_i} = \frac{\rho k_B T}{\mu_e m_p}.$$

For the H/He mixture, we infer $\mu_e = (X_H + X_{He}/2)^{-1} \approx 1.2$. The total pressure is

$$P = (n_e + n_H + n_{He}) k_B T = \frac{\rho k_B T}{m_p}\left(\frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_e}\right) = \frac{\rho k_B T}{\mu m_p}.$$

This defines the mean molecular weight $\mu^{-1} = \mu_e^{-1} + \mu_{\text{ion}}^{-1}$. For the solar mixture, $\mu^{-1} = 2X_H + 3X_{He}/4 \approx 0.6$.

Pure H has $\mu_e = \mu_i = 1$ and $\mu = 1/2$. Pure He has $\mu_e = 2$, $\mu_i = 4$, and $\mu = 4/3$. Heavier elements than helium also have $\mu_e \approx 2$ since $A \approx 2Z$ for all nuclei except hydrogen.

The ρ–T plane

The figure on the next page shows the different regions of the ρ–T plane, assuming a composition of pure helium. Electrons become degenerate when $E_F \approx k_B T$ (dashed line). For non-relativistic electrons, this is

$$\frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3} \approx k_B T \Rightarrow T_{d,nr} \approx 3\times 10^5\ \text{K}\ (\rho Y_e)^{2/3},$$

using $n_e = \rho Y_e/m_p$. (For ions to become degenerate, would need to lower the temperature by a factor $> m_p/m_e \sim 2000$.) Degenerate electrons become relativistic when

$$p_F = \hbar(3\pi^2 n_e)^{1/3} \approx m_e c \Rightarrow \rho Y_e \approx 10^6\ \text{g cm}^{-3}$$

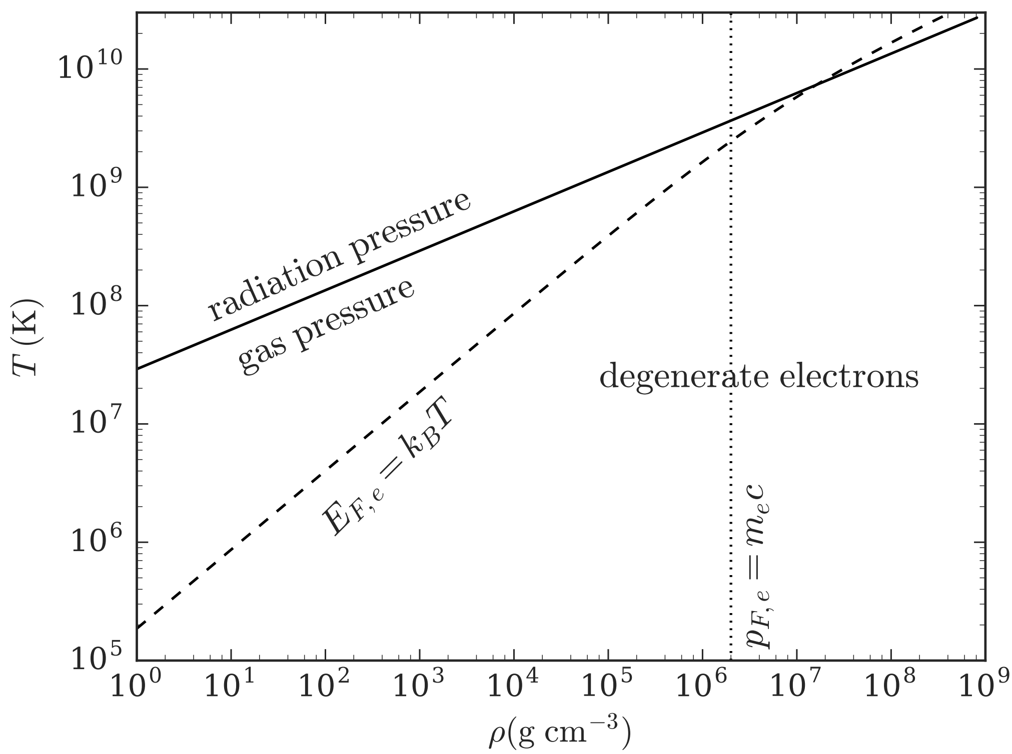
(vertical dotted line in the plot). The dashed line shown in the plot takes relativity into account by writing

$$E_F = m_e c^2\left(\sqrt{1+x^2} - 1\right) \approx k_B T$$

where $x = p_F/m_e c \approx (\rho Y_e/10^6 \text{ g cm}^{-3})^{1/3}$; notice it changes slope at $\rho \gtrsim 10^6$ g cm^{-3} once the electrons become relativistic.

The solid curve shows the boundary between radiation pressure and gas pressure, assuming the gas pressure is ideal:

$$\frac{1}{3}aT^4 = \frac{\rho k_B T}{\mu m_p} \Rightarrow T_{\rm rad} = \left(\frac{3\rho k_B}{\mu m_p a}\right)^{1/3} \approx 3\times 10^7 \text{ K} \left(\frac{\rho}{\mu}\right)^{1/3}.$$



White dwarf mass-radius relation

White dwarfs are stars held up by degenerate electron pressure. For low masses, the electrons are non-relativistic so that $P \propto \rho^{5/3}$, but as the mass approaches the Chandrasekhar mass the electrons become more and more relativistic and $\gamma \to 4/3$. (The positive ions also have a pressure, but it is much smaller than the electrons. That is because the ions are non-degenerate, so their pressure is a factor $\sim k_BT/E_F$ times smaller.)

As we mentioned earlier, the solutions of the stellar structure equations (1.6) and (1.7) for $P \propto \rho^\gamma \propto \rho^{1+1/n}$ are known as polytropes. You can look up the properties of polytropes for different values of polytropic index n, in particular the numerical solutions give the values of

$$\alpha_n = \frac{P_c}{GM^2/R^4} \qquad \beta_n = \frac{\rho_c}{\langle\rho\rangle},$$

where $\langle\rho\rangle = 3M/4\pi R^3$ is the mean density. For $\gamma = 5/3$, $n = 3/2$, $\alpha = 0.77$ and $\beta = 5.99$. For $\gamma = 4/3$, $n = 3$, $\alpha = 11.1$ and $\beta = 54.2$.

To get the white dwarf mass–radius relation, we write the equation of state at the center as $P_c = K_{nr}\rho_c^{5/3}$, where

$$K_{nr} = \frac{P}{\rho^{5/3}} = \frac{2}{5}\frac{nE_F}{\rho^{5/3}} = \frac{2}{5}\frac{n}{\rho^{5/3}}\frac{p_F^2}{2m} = \frac{2}{5}\frac{\hbar^2(3\pi^2)^{2/3}}{2m}\left(\frac{n}{\rho}\right)^{5/3} = 9.9\times 10^{12}\ \mathrm{cgs}\ Y_e^{5/3}.$$

Then using the $n = 3/2$ polytrope results for α and β gives the white dwarf mass-radius relation at low masses

$$R_{5/3} = M^{-1/3}\left(\frac{K_{nr}}{\alpha_{3/2}G}\right)\left(\frac{3\beta_{3/2}}{4\pi}\right)^{5/3} \approx 9\times 10^8\ \mathrm{cm}\ \left(\frac{M}{M_\odot}\right)^{-1/3}\left(\frac{Y_e}{0.5}\right)^{5/3}.$$

(We write $R_{5/3}$ to indicate that this is the white dwarf radius assuming $\gamma = 5/3$). As the star gets more massive, the radius shrinks. The central density increases rapidly with mass, $\rho_c \propto M/R^3 \propto M^2$.

Doing the same thing for the equation of state $P_c = K_r\rho_c^{4/3}$, the radius drops out and we get an expression for the Chandrasekhar mass

$$M_{Ch} = \left(\frac{K_r}{\alpha_3 G}\right)^{3/2}\left(\frac{3\beta_3}{4\pi}\right)^2 = 1.45\ M_\odot\ \left(\frac{Y_e}{0.5}\right)^2.$$

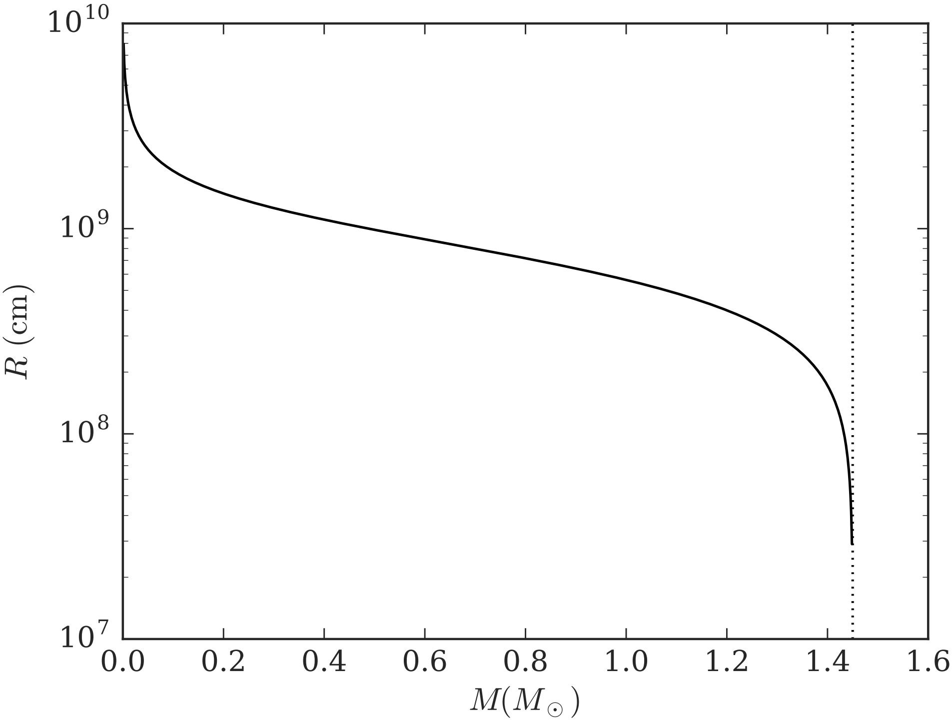
We can interpolate between the two limits by using the fitting formula obtained by Paczynski (1983) for the pressure of degenerate electrons

$$P_e^{-2} \approx P_{e,nr}^{-2} + P_{e,r}^{-2}, \tag{1.8}$$

which interpolates between non-relativistic and relativistic electrons (and Pacynski found was accurate to a few percent). If you use this formula for the central pressure, you will find

$$R \approx R_{5/3}\left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2}.$$

Here is a plot of this $M(R)$ relation:



As the mass approaches the Chandrasekhar mass, the central density increases dramatically (because of decreasing radius but also the increasing value of β_n as $\gamma \rightarrow 4/3$, see above). Once it gets to $\rho_c \sim 10^9$ g cm^{-3}, interesting things can happen. One possibility is carbon fusion leading to a Type Ia supernova. The other is that electrons can capture into the nuclei, removing pressure support and leading to collapse to a neutron star. (White dwarfs can reach these large masses either through merging or accretion, or through stellar evolution, e.g. the iron core of a massive star).

Neutron stars

We saw that the radius of a $\gamma = 5/3$ star is $R \propto M^{-1/3} K_{nr}$. The key point for neutron stars is that $K_{nr} \propto 1/m$ where m is the mass of the degenerate particle. For white dwarfs this is the electron mass; for neutron stars, the star is held up by degenerate neutron pressure and we should take $m = m_n$ the neutron mass. We expect the radius of a neutron star to be smaller than a white dwarf by a factor of $m_n/m_e \approx 2000$, or $R_{NS} \sim 10^9$cm$/2000 \approx 5$ km. This is about right. Detailed models give neutron star radii $\approx$ 10–13 km. They are a little larger because the neutrons repel each other when they are very close, so that the equation of state is stiffer than $\gamma = 5/3$, in fact closer to $\gamma \approx 2$. As we argued in the beginning, this gives radius almost independent of mass, which is seen in detailed calculations of mass–radius relations.

Coulomb pressure and planets

If you plug in Jupiter's mass $M_J \approx 10^{-3} M_\odot$ into the white dwarf mass-radius relation, you'll get a radius $\approx 10^{10}$ cm which is not too far off ($1\ R_J \approx 0.1\ R_\odot \approx 7\times 10^9$ cm). But clearly, as we reduce mass further something else must happen: eventually, we expect to see radius get smaller with decreasing mass. For example, if we scale up from the Earth assuming the same density ($M \propto R^3$) then that is also not so far off — Jupiter is about 10 times the radius of Earth and about 300 times the mass[5]. Somehow the mass–radius relation must turnover and change from $M \propto R^{-3}$ to $M \propto R^3$ at low masses.

What happens is that the Coulomb attraction of the positive ions and electrons in the plasma becomes important, leading to a negative contribution to the pressure, the *Couloumb pressure*. To calculate the size of this effect, first note that it is a good approximation to assume the electrons are uniformly distributed in space because $E_F \gg Ze^2/a$ where a is the interion spacing, so the electrons barely notice the ions. Then we can use the Wigner-Seitz approximation to calculate the energy associated with each ion. We consider an electrically-neutral sphere of radius R_Z around each ion that contains Z electrons, ie. $(4\pi R_Z^3/3)n_e = Z$. The electrostatic energy of the sphere has two contributions:

$$U_{ee} = \frac{3}{5}\frac{(Ze)^2}{R_Z} \qquad \text{electron} - \text{electron repulsion}$$

$$U_{ei} = -\frac{3}{2}\frac{(Ze)^2}{R_Z} \qquad \text{electron} - \text{ion attraction.}$$

The total energy per unit volume is then

$$U_C = -n_e\frac{9}{10}\frac{Ze^2}{R_Z} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3}e^2 n_e^{4/3}$$

(where I used $n_i = n_e/Z$). Notice that U_C becomes more negative as density increases, giving a negative pressure! The pressure is $-\partial(U_C V)/\partial V$ for volume V, giving $P_C = (1/3)U_C$ or

$$P_C \approx -6\times 10^{12}\ \text{erg cm}^{-3}\ (\rho Y_e)^{4/3} Z^{2/3}.$$

We can do two interesting things with this. The first is the *zero pressure solid*. We write down the total pressure from electrons and Coulomb:

$$P_{\text{tot}} = K_e\rho^{5/3} - K_C\rho^{4/3}. \tag{1.9}$$

There is a zero-pressure solution with density

$$\rho_0 = \left(\frac{K_C}{K_e}\right)^3 \approx 0.2\ \text{g cm}^{-3}\ ZA.$$

[5]I'm ignoring factors from composition differences in this paragraph. Earth is about 4 times denser than Jupiter, and the Y_e in a white dwarf is ≈ 0.5 whereas Jupiter is mostly hydrogen so will have Y_e closer to 1.

This overpredicts the density of terrestrial metals: for example, copper has $A \approx 64$ and $Z = 29$, giving $\rho_0 \sim 300$ g cm^{-3} (actual density is 9 g cm^{-3}), but the electronic configuration is much more complex than we have assumed in our simple model. The important point is that we have found a self-bound state which exists without any confining pressure. So in the low mass limit we might expect to see $M \sim \rho_0 R^3$ as expected.

The second thing is then to add in gravity:

$$\frac{GM^2}{R^4} \approx K_e \left(\frac{M}{R^3}\right)^{5/3} - K_C \left(\frac{M}{R^3}\right)^{4/3}.$$

Solving for R, we get

$$R = \frac{K_e}{GM^{1/3} + K_C M^{-1/3}}.$$

The two limits are $R = (K_e/G)M^{-1/3}$ "white dwarf" and $R = (K_e/K_C)M^{1/3}$ "rock". The maximum radius is where $M = (K_C/G)^{3/2} \approx 0.4\ M_J$.

The interplay between degeneracy pressure and Coulomb pressure, leading to the turnover of the $R(M)$ relation, is the reason why the radii of brown dwarfs are about the same as Jupiter, despite being 30–100 times more massive!

Papers

- Chandrasekhar 1931 "Maximum mass of white dwarfs" http://adsabs.harvard.edu/abs/1931ApJ....74...81C and "The density of white dwarf stars" http://www.tandfonline.com/doi/abs/10.1080/14786443109461710
- Holberg et al. 2012 WD "Observational constraints on the degenerate mass-radius relation" http://adsabs.harvard.edu/abs/2012AJ....143...68H, or more recent Parsons et al. 2017 "Testing the white dwarf mass-radius relationship with eclipsing binaries" https://ui.adsabs.harvard.edu/#abs/2017MNRAS.470.4473P/abstract
- Salpeter 1961 "Energy and pressure of a zero-temperature plasma" http://adsabs.harvard.edu/abs/1961ApJ...134..669S
- Seager et al. 2007 "Mass-radius relationships for solid exoplanets" http://adsabs.harvard.edu/abs/2007ApJ...669.1279S
- Lattimer and Prakash 2001 "Neutron star structure and the equation of state" http://adsabs.harvard.edu/abs/2001ApJ...550..426L

Exercises

1. *Gravitational energy of a star*. Even without any knowledge of the equation of state, there are certain integral relations that can be derived using only the fact that a star is in hydrostatic balance. Here is an example. The gravitational binding energy of a star is

$$\Omega = -\int \frac{Gm}{r} dm.$$

Using equations (1.6) and (1.7) and an integration by parts, show that

$$\Omega = -3 \int P dV,$$

where $dV = 4\pi r^2 dr$ is the volume element.

2. *Gravitational energy of a polytrope*. We can use the result from exercise 1 to derive an expression for the gravitational energy of a polytrope.

(a) First show by integrating by parts that

$$\int P dV = \int m \, d\left(\frac{P}{\rho}\right) = \left(\frac{\gamma - 1}{\gamma}\right) \int m \frac{dP}{\rho}.$$

(b) Next, use equation (1.6) to change integration variables to r and integrate by parts to find

$$\int P dV = \left(\frac{\gamma - 1}{\gamma}\right) \left[\frac{GM^2}{R} + 2\Omega\right].$$

(c) Now apply the result from exercise 1 to show that

$$-\Omega = \frac{3(\gamma - 1)}{5\gamma - 6} \frac{GM^2}{R} = \frac{3}{5 - n} \frac{GM^2}{R}.$$

As a check, what is the answer for an incompressible equation of state? Does it look familiar?

Appendix: TOV equations

When calculating the structure of a neutron star, general relativistic corrections are important, since

$$\frac{GM}{Rc^2} = 0.15 \left(\frac{M}{M_\odot}\right)\left(\frac{R}{10\ \mathrm{km}}\right)^{-1}.$$

The GR version of the stellar structure equations are known as the Tolman-Oppenheimer-Volkoff (TOV) equations. They are

$$\begin{aligned}
\frac{dm}{dr} &= 4\pi r^2\rho \\
\frac{dP}{dr} &= -\rho\frac{Gm}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{Gm}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \\
\frac{d\Phi}{dr} &= -\frac{1}{\rho c^2}\frac{dP}{dr}\left(1+\frac{P}{\rho c^2}\right)^{-1}.
\end{aligned}$$

As well as the continuity and momentum equations, there is an additional equation for the metric function Φ, which is defined such that the metric is

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega,$$

with

$$e^{2\lambda(r)} = \left(1-\frac{2Gm}{r}\right)^{-1}.$$

To match onto the exterior Schwarzschild metric, $\Phi(R) = (1/2)\ln(1-2GM/Rc^2)$ at the surface of the star. Note that r is defined such that it corresponds to the sphere with surface area $4\pi r^2$ (or circumference $2\pi r$). The proper distance between two shells is $dr(1-2Gm/rc^2)^{-1/2}$, giving a volume element

$$\left(1-\frac{2Gm}{rc^2}\right)^{-1/2} 4\pi r^2 dr.$$

The quantity $m(r)$ is the gravitational mass interior to coordinate r, equal to M at the surface.

Computational Exercise 1: The white dwarf mass–radius relation

Overview. The goal of this exercise is to calculate the mass-radius relation for $T = 0$ white dwarfs. This requires numerically integrating the equation of hydrostatic balance. You can do this using whatever method you wish, but to help you I describe a possible procedure below.

Equations and boundary conditions. The structure of the white dwarf is given by the equations of hydrostatic balance

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \qquad \frac{dm}{dr} = 4\pi r^2 \rho.$$

The boundary conditions are $m = 0$ at $r = 0$ and $P = \rho = 0$ at $r = R$.

Equation of state. The integration variables are m and P, so at each step you will need to compute the density from the pressure using the equation of state. Usually the equation of state is given the other way round, as a function $P(\rho)$, often as a numerical table for complex equations of state. Given $P(\rho)$, you can find the density ρ_0 corresponding to a particular pressure P_0 by solving the equation $P(\rho_0) = P_0$ numerically using a root-finding algorithm.

You can use this root-finding technique in your code, but there is also another option since we have an analytic expression for the equation of state. In this case, we can change integration variables from $P \to \rho$, ie. use the analytic equation of state to write dP/dr in terms of $d\rho/dr$. As the white dwarf mass increases, the electrons go from being non-relativistic ($P \propto \rho^{5/3}$) to relativistic ($P \propto \rho^{4/3}$). To take this into account, we can use the analytic fitting formula for the equation of state derived by Paczynksi (1983) that I mentioned in the notes:

$$P^{-2} = P_{nr}^{-2} + P_r^{-2},$$

where $P_{nr} = K_{nr}\rho^{5/3}$ is the non-relativistic degenerate electron pressure and $P_r = K_r\rho^{4/3}$ is the relativistic degenerate electron pressure. It is then straightforward to show that

$$\frac{d\ln P}{dr} = \frac{d\ln \rho}{dr}\left[\frac{5}{3}\left(\frac{P}{P_{nr}}\right)^2 + \frac{4}{3}\left(\frac{P}{P_r}\right)^2\right].$$

Use the class notes to determine the constants K_{nr} and K_r. Assume a carbon/oxygen white dwarf which has $Y_e = 0.5$.

Integration. Now the idea is to integrate outwards from the center of the star to the surface. At the center, there is a problem at $r = 0$ since the equation for dP/dr has an r in the denominator and we can't divide by zero! To avoid this, start the integration at a small distance $r = \epsilon$ from the center, where $\rho \approx \rho_c$ and $m \approx 4\pi\rho_c\epsilon^3/3$. Here I've written the central density as ρ_c.

Integrate outwards until the density falls to zero. The radius at which $\rho = 0$ is the radius of the star $r = R$, and the value of m at this point is the mass M of the star. How you do this step in practice depends on your integrator. Most likely you will have to tell your integration routine to integrate from $r = r_1 = 0$ to $r = r_2$. In that case, try different values of r_2 until you find the one that gives $\rho = 0$ at the edge.

Repeat this integration for several different choices for central density ρ_c logarithmically spaced from about 10^6 g cm^{-3} to 10^9 g cm^{-3}. You'll have to experiment to get the correct range in central density that covers a mass range up to the Chandrasekhar mass at $\approx 1.4\ M_\odot$.

Questions

1. *Mass-radius relation*. Plot the curve of radius against mass. For low masses, when the central density is small and $\gamma = 5/3$, you should find $R \propto M^{-1/3}$, but you'll see the slope changes at large masses. How does your answer compare with the analytic approximation from the lecture notes,

$$R = 8.7 \times 10^8\ \mathrm{cm}\ \left(\frac{M}{M_\odot}\right)^{-1/3} \left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2},$$

   and what do you determine to be the Chandrasekhar mass?

2. *Investigate the density profile.* Plot the density profile as a function of radius for different mass white dwarfs. How does the density profile change as you change the white dwarf mass?

3. *The extent to which the electrons are relativistic.* Plot the value of γ and E_F/m_ec^2 at the center of the white dwarf as a function of mass.

A possible extension is to add the Coulomb energy of the ions (see eq. [4] of the notes) to the pressure. Are there masses where the Coulomb energy becomes important?

Week 3: Hot Stars — Energy Transport, Nuclear Burning, and Stellar Evolution

We now move onto "hot stars" for which $k_B T \gg E_F$ and temperature matters in describing their structure. An important difference from cold stars is that hot stars can cool, so we need to understand energy sources and sinks and energy transport inside the star.

Radiative diffusion, opacity, and the luminosity of stars

The main energy transport mechanism in stars is diffusion of photons. The mean free path of a photon is $\lambda = 1/n\sigma$ where n is the number density of scatterers or absorbers and σ is the cross-section. In astrophysics, we usually write everything per gram, so that $\lambda = 1/\rho\kappa$ where κ is the cross-section per gram, or the *opacity*. For example, free electrons scatter photons with the Thomson cross-section

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.67 \times 10^{-25}\ \mathrm{cm}^2.$$

For pure hydrogen, the opacity is $\kappa = \sigma_T/m_p = 0.40\ \mathrm{cm}^2\ \mathrm{g}^{-1}$. The photon mean free path in the center of the Sun is then $\lambda \approx 10^{-2}$ cm (taking $\rho = 150\ \mathrm{g\,cm}^{-3}$). This is obviously much less than the solar radius, so photons are scattered or absorbed many times on traversing the Sun, but it is also much longer than the particle mean free path ($\sim 10^{-6}$ cm; as we discussed in Week 1), so that photons carry information about the temperature at their origin to the location where they are absorbed. Other important opacity sources in stars are *free-free* and *bound-free absorption*, associated with an electron absorbing a photon in the presence of a nucleus. Unlike electron scattering, the bound-free and free-free opacities depend on density and temperature, with the *Kramer's* scaling $\kappa \propto \rho T^{-7/2}$.

The heat flux carried by the diffusing photons is

$$F = -\frac{1}{3}\, c\, \lambda\, \frac{d}{dr}\left(aT^4\right) = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}.$$

The outwards luminosity at radius r is then $L = 4\pi r^2 F$. Note that in general the opacity depends on the local density, temperature and composition so we can write $\kappa(\rho, T, X_i)$ where X_i is a set of mass fractions describing the composition. The heat flux is of the form we discussed in Week 1, $F = -K\nabla T$ where $K \propto T^3/\kappa\rho$ is the thermal conductivity.

Let's use the radiative diffusion equation to estimate the luminosity of a star. We mentioned last time that hydrostatic balance is enough to estimate the central temperature of a star if we know its mass and radius,

$$k_B T_c \approx \frac{GMm_p}{R} \Rightarrow T_c \approx 2 \times 10^7\ \mathrm{K}\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}.$$

The hot interior implies a luminosity

$$L \sim 4\pi R^2 \frac{4acT^4}{3\kappa R}\frac{3R^3}{4\pi M} \sim \frac{4acT^4R^4}{\kappa M},$$

where we write $r \approx R$, $\rho \approx (4\pi/3)(M/R^3)$, and $dT/dr \approx T/R$. Now putting in T_c for the temperature,

$$L \sim \frac{4ac}{\kappa}\left(\frac{Gm_p}{k_B}\right)^4 M^3 \sim 8\times 10^{36}\ \mathrm{erg\ s^{-1}}\left(\frac{M}{M_\odot}\right)^3\left(\frac{\kappa}{0.4\ \mathrm{cm^2\ g^{-1}}}\right)^{-1}.$$

This is about 1000 times too big for the Sun which has $L_\odot \approx 4\times 10^{33}$ erg s^{-1}; putting in an average temperature (e.g. at $r \approx 0.5R$ the temperature in the Sun is about $T_c/5$) would give a more reasonable value. The important thing is the scaling $L \propto M^3$ which is seen in models for stars with mass $M \gtrsim M_\odot$ for which the central temperature is large enough that electron scattering dominates the opacity. For $M \lesssim 1M_\odot$, free-free opacity dominates instead, introducing a temperature and density scaling into κ. These low mass main-sequence stars have a steeper dependence $L \propto M^{5.5}$.

An alternative energy transport mechanism in stars is *convection*, in which fluid motions transport heat. We'll look more into this when we talk about instabilities, but the basic idea is that if the temperature gradient is steep enough, the entropy gradient in the star can become negative (entropy decreases outwards). High entropy material underneath low entropy material is unstable to mixing and results in convection. Stars can be fully-convective (low mass stars $\lesssim 0.3M_\odot$), have a surface convection zone ($M \sim M_\odot$), or a convective core ($M \gtrsim M_\odot$).

Thermonuclear reactions

We've seen that a star must be hot to hold itself up against gravity $T_c \propto M/R$, and that implies a certain luminosity ($L \propto M^3$ for electron scattering). The luminosity is supplied by nuclear burning – at each stage of a star's life, the radius of the star adjusts to give the right central temperature at which nuclear burning can balance the luminosity.

For two nuclei to fuse, they must approach to a distance $\sim 10^{-13}$ cm (about the size of a nucleus) at which strong forces operate. In practise, this is not possible because of Coulomb repulsion between nuclei. For example, at the Sun's central temperature, the average energy of protons is ≈ 1 keV. We know that the binding energy of hydrogen e^2/a_0 is about 10 eV for $a_0 \sim 10^{-8}$ cm, so at 1 keV, the closest approach distance must be $\sim 10^{-10}$ cm. This is a factor of 1000 too large for fusion to occur. How then do nuclear reactions happen? The answer is that the protons tunnel through the Coulomb barrier.

We can estimate the probability for quantum tunnelling by saying that the wavefunction drops by a factor of e^{-kx} for a barrier of width x where $k \approx \sqrt{2mV_0}/\hbar$ is the wavevector of the evanescing wavefunction. For a closest approach r_c, the potential barrier height is $V_0 \sim e^2/r_c$ and the width of the barrier is $r_c \approx e^2/E$ where E is the center of mass energy of the two protons. Therefore

$$kx \sim \frac{e}{\hbar}\sqrt{2mr_c} \sim \sqrt{\frac{2mc^2}{E}\frac{e^2}{\hbar c}} = \sqrt{\frac{2\alpha^2 mc^2}{E}},$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant. A more detailed treatment which integrates through the barrier gives a similar result but with an extra factor of π in the prefactor. Also including the charges of the fusing nuclei Z_1 and Z_2, the final tunnelling probability is

$$\text{Prob} \propto \exp\left(-\sqrt{\frac{E_G}{E}}\right)$$

where

$$E_G = 2\pi^2\alpha^2 mc^2 (Z_1 Z_2)^2 \approx 1\ \text{MeV}\ Z_1^2 Z_2^2 \left(\frac{m}{m_p}\right)$$

is the *Gamow energy* and m is the reduced mass $m = m_1 m_2/(m_1 + m_2)$.

The higher the energy E, the more likely tunnelling is to occur, but the probability two particles have that energy is smaller, $\propto e^{-E/k_BT}$. The tunnelling rate is therefore a convolution between the tunnelling probability and Maxwell-Boltzmann distribution of particle energies. The tunnelling is most likely for energy E_0 where $\exp(-E_0/k_BT - \sqrt{E_G/E_0})$ has a maximum, or $E_0 = (k_BT)^{2/3}(E_G/2)^{1/3}$. For $k_BT \approx 1$ keV and $E_G \approx 1$ MeV, this is $E_0 \approx 6$ keV. The energies around E_0 where the reaction is most likely to occur is called the *Gamow window*. For many reactions, the energy-dependence of the cross-section must also be taken in to account, particularly when there is a resonance which boosts the cross-section at the resonant energy.

The fact that nuclear fusion is happening only for particles in the tail of the Maxwell-Boltzmann distribution means that thermonuclear reaction rates are extremely temperature sensitive. Another property of nuclear burning is that heavier nuclei have larger Z's and so a larger Coulomb barrier, and require higher temperatures to fuse. The larger Z nuclei have a larger factor in the exponent and so have reaction rates that are more temperature sensitive than lower Z nuclei. One impact of this for main sequence stars is that massive main sequence stars $M \gtrsim M_\odot$ which burn hydrogen via the CNO cycle have T_c roughly independent of mass and so $R \propto M$. (It's actually a bit shallower because T_c increases a little bit with M.)

Stellar evolution

The full set of equations that are needed to follow the evolution of a hot star are

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho$$

$$\frac{\partial P}{\partial r} = -\frac{\rho G m}{r^2}$$

$$T\frac{\partial S}{\partial t} = \epsilon_{\text{nuc}} - \epsilon_\nu - \frac{1}{4\pi r^2 \rho}\frac{\partial L}{\partial r}$$

$$\frac{\partial T}{\partial r} = \frac{\partial P}{\partial r}\frac{T}{P}\nabla$$

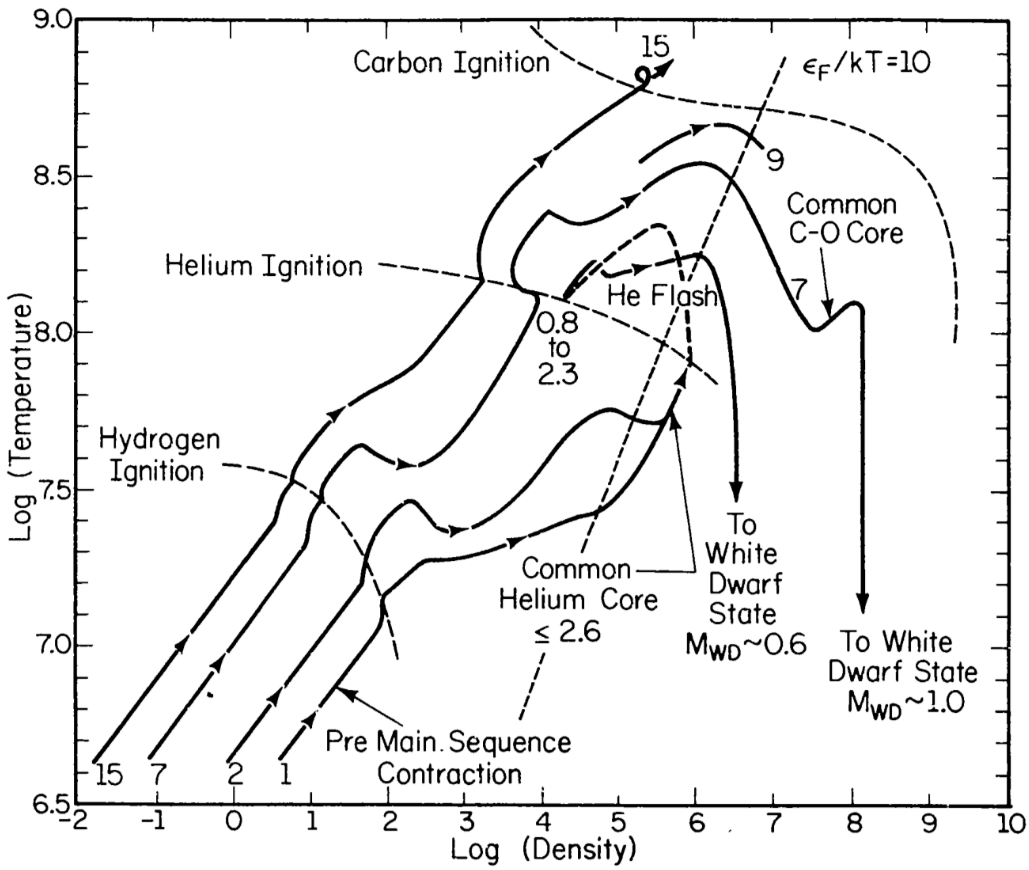
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\sum_j (r_{ji} - r_{ij}) + D\nabla^2 X_i$$

where the temperature gradient ∇ is determined by the energy transport process. If radiation is transporting energy,

$$\nabla = \nabla_{\rm rad} = \frac{3\kappa PL}{16\pi acGmT^4},$$

from the radiative diffusion equation. When convection operates, the temperature gradient is usually close to the adiabatic gradient $\nabla = \nabla_{\rm ad}$ (because this is the entropy-neutral gradient that marks the onset of convection). The nuclear energy generation rate per gram is written as $\epsilon_{\rm nuc}$ (units are erg $\mathrm{g^{-1}\ s^{-1}}$). In massive stars in late burning stages the temperature and density can be large enough that neutrinos become an effective cooling source. The local neutrino cooling rate is written as ϵ_ν. The last equation is actually a set of equations, one for each species, which follow the change in composition as nuclear reactions occur and as diffusion, convection or other processes mix composition in the star (for simplicity, I just put a diffusion term here).

Overall, the life of a star involves moving to higher central temperatures and densities, stopping at various nuclear burning stages, until the core becomes degenerate. This is illustrated in the figure below, taken from Iben (1985) http://adsabs.harvard.edu/abs/1985QJRAS..26....1I.



Several codes to follow stellar evolution are available. An interesting one to try is MESA (Modules for Experiments in Stellar Astrophysics). See their website at http://

mesa.sourceforge.net for references and more details. Here are two movies showing the evolution of main sequence stars:
1 solar mass – https://www.youtube.com/watch?v=oZY3TtA63sE
3 solar masses – https://www.youtube.com/watch?v=C4tucmhAaSk
Watching these movies you'll see that the nuclear burning is often unstable, leading to a rapid local rise in temperature within the star. This happens either when the nuclear burning is in a degenerate region (e.g. when helium ignites in the core of the solar mass star) or when the burning is in a thin shell (He burning or H burning shells in giants). In either of these situations, the star is not able to lower the presure by expansion in response to nuclear energy release. The temperature rises and the nuclear burning runs away.

Cores and envelopes

In stellar evolution, there is an interesting interplay between cores and envelopes. In a main sequence star like the Sun, the star is relatively compact, with a smooth change in density, temperature and composition from the center to the surface. However, once hydrogen runs out in the core and the main sequence lifetime ends, the star adopts a very different structure. The star contracts and heats up until the hydrogen at the edge of the helium core is hot enough to ignite. The ignition of a shell source has a dramatic effect on the hydrostatic structure of the star, which becomes a red giant, with a large, low density, extended hydrogen envelope sitting on top of a compact helium core in the center. This is a general feature: *if the nuclear burning is central, the star will be compact; if burning is in a shell source, the star adopts a giant structure.*

In a red giant, the core is isothermal at a temperature that is regulated by the shell H burning. An interesting aspect of an isothermal non-degenerate core is that there is a maximum mass envelope that it can support. The way to see this is to write an equation for the pressure at the surface of the core P_s. Integrating the hydrostatic balance equation from the center to the surface of the core gives

$$P_s = A\frac{T_c M_c}{R_c^3} - B\frac{GM_c^2}{R_c^4} \tag{1.10}$$

for constants A and B that depend on the internal density profile (the core has mass M_c and radius R_c). Think of this as saying that the surface pressure is the mean pressure in the core reduced by the weight of the core. For zero pressure at the surface, the radius is $R_0 = BGM_c/AT_c$ (which shows the $T \propto M/R$ scaling we've seen before).

Equation (1.10) has the interesting feature that there is a maximum pressure. At large core radius, both terms go to zero, so the surface pressure becomes small. At small core radius, the gravitational term increases faster than the mean pressure term, also reducing the pressure. The maximum surface pressure is

$$P_{s,\mathrm{max}} = \frac{27}{256} B\frac{GM_c^2}{R_0^4} \propto \frac{T_c^4}{M_c^2}$$

at a radius $R = (4/3)R_0 \propto M_c/T_c$.

The maximum surface pressure means that there is a maximum mass envelope that the core can support hydrostatically. This is known as the *Schönberg-Chandrasekhar limit*, and can be written as a ratio of core mass to total mass. This is because most of the mass of the star is contained in the envelope, so the pressure at the base of the envelope is $P_b \approx GM^2/R^4 \propto T_c^4/M^2$ since the (base of the) envelope is at the same temperature as the core and $T \propto M/R$. This means that $P_b/P_{s,\rm max} \propto (M_c/M)^2$. Typically the limit is found to be $M_c/M \lesssim 0.1$ for stability.

For red giants, this can lead to collapse of the helium core: as the hydrogen shell adds more and more helium to the core, it grows in mass. Once it reaches the Schönberg-Chandrasekhar mass, it collapses, initiating helium burning in the core. In practise, this happens only in a limited range of masses, because massive stars leave the main sequence with a helium core that already exceeds the Schönberg-Chandrasekhar limit.

Note that the surface pressure does not have this behaviour for a degenerate core: then the pressure $\propto 1/R^5$ rather than $1/R^3$ and the radius can always adjust to supply any surface pressure needed. In that case, the helium burning starts in an unstable way once the core temperature reaches a critical value, giving a core helium flash. This means that there is a separation in stellar evolution between stars that develop a degenerate helium core and undergo a helium core flash ($\lesssim 2\ M_\odot$) and those that have a non-degenerate helium core and do not undergo a core flash ($\gtrsim 2\ M_\odot$).

Papers

- Stevenson 1982 "Formation of the Giant Planets" http://adsabs.harvard.edu/abs/1982P%26SS...30..755S
- Ushomirsky et al. 1998 "Light element depletion in contracting brown dwarfs and pre-main-sequence stars" http://adsabs.harvard.edu/abs/1998ApJ...497..253U
- Deloye & Bildsten 2003 "The stellar structure of finite-entropy objects" http://adsabs.harvard.edu/abs/2003ApJ...598.1217D
- Woosley & Heger 2015 "The remarkable deaths of 9-11 solar mass stars" http://adsabs.harvard.edu/abs/2015ApJ...810...34W

Appendix: Gravothermal heat capacity

In class, we discussed the fact that the heat capacity of a star is negative: the temperature decreases in response to energy input. Here's how this works, following a similar argument to the one in Kippenhahn and Wiegert's book on stellar structure. We can ask: what is the response of the gas to entropy changes?

First, write the entropy change in terms of temperature and pressure:

$$dS = \left.\frac{\partial S}{\partial T}\right|_P dT + \left.\frac{\partial S}{\partial P}\right|_T dP.$$

Using the fact that the heat capacity at constant pressure is $c_P = T\, \partial S/\partial T|_P$, and the identity

$$\left.\frac{\partial S}{\partial T}\right|_P \left.\frac{\partial T}{\partial P}\right|_S \left.\frac{\partial P}{\partial S}\right|_T = -1,$$

we can write this as

$$TdS = c_P \left(dT - \frac{T}{P} \nabla_{\rm ad} dP \right), \tag{1.11}$$

where $\nabla_{\rm ad} = \partial \ln T/\partial \ln P|_S$.

So far, this is just thermodynamics, but now we put in the fact that the star is in hydrostatic balance, so that $P \propto 1/R^4$ and $\rho \propto 1/R^3$. This means that we must have

$$\frac{dP}{P} = \frac{4}{3}\frac{d\rho}{\rho}. \tag{1.12}$$

But the equation of state relates density to pressure and temperature changes through

$$d\ln P = \chi_T d\ln T + \chi_\rho d\ln\rho$$

where $\chi_X \equiv (\partial \ln P/\partial \ln X)$ with other variables held constant. Equation (1.12) becomes

$$\frac{\delta P}{P} = \frac{4\chi_T}{4-3\chi_\rho}\frac{\delta T}{T}. \tag{1.13}$$

Combining equations (1.31) and (1.12) gives

$$T\frac{dS}{dT} = c_P\left(1 - \frac{4\chi_T \nabla_{\rm ad}}{4-3\chi_\rho}\right) = c_\star,$$

where $c_\star$ is the effective heat capacity.

Now look at different limits:

- For an ideal gas, $\chi_T = 1$, $\chi_\rho = 1$, and for a monatomic gas $\nabla_{\rm ad} = 2/5$, so that $c_\star = -(3/5)c_P < 0$. (The Sun is stable).
- For a degenerate gas, $\chi_T \sim k_B T/E_F \to 0$ so that the correction term becomes small and $c_\star \to c_P > 0$. (Helium core flash).
- If the burning is in a thin shell, equation (1.12) is no longer correct. To see this, consider a shell that has mass ΔM, thickness H and is located at radius r. If the shell changes its thickness by δH, the pressure change is of order $\delta H/r$, since pressure is $\sim GM\Delta M/4\pi r^4$. On the other hand the change in density is of order $\delta H/H$ (since mass conservation $\Rightarrow r^2\rho H =$constant). Therefore for a thin shell,

$$\frac{\delta P}{P} \sim \frac{H}{R}\frac{\delta\rho}{\rho},$$

  which means that $c_\star \approx c_P$ to first order in H/r. Burning in a thin shell is therefore unstable. This is the origin of the term *thin shell flash*.

Week 4: Compressible fluids — Sound waves and shocks

Sound waves

Compressions in a gas propagate as sound waves. The simplest case to consider is a gas at uniform density and at rest. Small perturbations in the density, velocity, and pressure

$$\rho \rightarrow \rho + \delta\rho, \qquad \mathbf{v} \rightarrow \mathbf{v} + \delta\mathbf{v}, \qquad P \rightarrow P + \delta P$$

then obey the equations

$$\frac{\partial \delta\rho}{\partial t} = -\rho\nabla\cdot\delta\mathbf{v} \tag{1.14}$$

and

$$\rho\frac{\partial \delta\mathbf{v}}{\partial t} = -\nabla\delta P, \tag{1.15}$$

where we have kept only terms first order in the perturbations. These equations show the physics of the wave: compression leads to a local increase in density and therefore pressure; the pressure gradient acts as a restoring force trying to remove the compression.

To see that there is a wave, we assume that the perturbations are rapid enough that there is no time for heat to flow into or out of a fluid element, so that the perturbations are adiabatic, with

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho}.$$

In that case, equations (1.14) and (1.15) can be combined into a wave equation

$$\frac{\partial^2\mathbf{v}}{\partial t^2} = \frac{\gamma P}{\rho}\nabla^2\mathbf{v} = c_s^2\nabla^2\mathbf{v},$$

where the *adiabatic sound speed*[6] c_s is given by $c_s^2 = \gamma P/\rho$. This is the sound speed that we usually think of – looking up values for atmospheric pressure $\approx 10^5$ Pa, density of air at STP ≈ 1.2 kg m^{-3}, and $\gamma = 7/5$ for a diatomic gas, I get 340 m/s.

Looking for plane wave solutions, ie. perturbations $\propto e^{-i\omega t+\mathbf{k}\cdot\mathbf{r}}$, we find a *dispersion relation*

$$\omega^2 = c_s^2 k^2.$$

The linear dispersion relation $\omega \propto k$ means that these waves are non-dispersive. They have frequency-independent and equal phase and group velocities: the phase velocity is $\omega/k = c_s$ and group velocity is $d\omega/dk = c_s$.

[6]Note that in general, the sound speed is

$$c_s^2 = \frac{\partial P}{\partial\rho}$$

with the partial derivative taken under whatever conditions are appropriate for the perturbations. We considered adiabatic perturbations so the derivative is taken at constant entropy. When heat transfer is very rapid for example, we would keep temperature constant when taking the derivative, giving the isothermal sound speed $c_T^2 = P/\rho$.

Things get more complicated when the fluid is magnetized. As we discussed in the first week, a magnetized plasma has a magnetic pressure that acts perpendicular to the field lines. Acoustic waves that are travelling across the magnetic field lines experience an extra restoring force and travel more quickly. For the perpendicular case $\mathbf{k} \perp \mathbf{B}$, the dispersion relation is

$$\omega^2 = k^2(c_s^2 + v_A^2)$$

where

$$\mathbf{v}_A = \frac{1}{\sqrt{4\pi\rho}}\mathbf{B}$$

is the *Alfven velocity*. This mode is known as the *fast magnetosonic mode*. An acoustic wave travelling along the field direction $\mathbf{k} \parallel \mathbf{B}$ does not feel the magnetic pressure and has the usual dispersion relation $\omega^2 = c_s^2 k^2$. These are known as *slow magnetosonic modes*.

Just to give a bit of the flavour of the calculation, the magnetic field enters through the $\mathbf{J} \times \mathbf{B}$ force. If the background field is uniform with $\mathbf{J} = 0$, the perturbations give a $\mathbf{J} \times \mathbf{B}$ force

$$\frac{\delta\mathbf{J} \times \mathbf{B}}{c} = \frac{(\nabla \times \delta\mathbf{B}) \times \mathbf{B}}{4\pi} = \frac{i}{4\pi}(\mathbf{k} \times \delta\mathbf{B}) \times \mathbf{B}.$$

We also need the induction equation

$$i\omega\delta\mathbf{B} = \nabla \times (\delta\mathbf{v} \times \mathbf{B}).$$

With these two extra ingredients, you can show that (try it!)

$$\delta\mathbf{v}(\omega^2 - (\mathbf{k} \cdot \mathbf{v}_A)^2) - (\mathbf{k} \cdot \delta\mathbf{v}) \left[\mathbf{k}(c_s^2 + v_A^2) - \mathbf{v}_A(\mathbf{k} \cdot \mathbf{v}_A)\right] + \mathbf{k}(\mathbf{k} \cdot \mathbf{v}_A)(\delta\mathbf{v} \cdot \mathbf{v}_A) = 0.$$

This is quite a complicated dispersion relation, which why it helps to think about particular limits. Setting $\mathbf{k} \cdot \mathbf{v}_A = 0$ makes several terms vanish, and you can straightforwardly show that $\omega^2 = k^2(c_s^2 + v_A^2)$, the fast magnetosonic mode mentioned above. If instead we assume $\mathbf{k} \parallel \mathbf{v}_A$, then things simplify to

$$\delta\mathbf{v}(\omega^2 - k^2 v_A^2) = (\mathbf{k} \cdot \delta\mathbf{v})\mathbf{k}(c_s^2 - v_A^2). \tag{1.16}$$

Dotting this equation with $\mathbf{k}$ gives $\omega^2 = k^2 c_s^2$ the slow magnetosonic wave mentioned earlier.

We should mention that there is also a non-compressive wave in the magnetized case, the *Alfven wave*. The tension of magnetic field lines supports a transverse wave similar to a wave on a string. To see this, set $\mathbf{k} \cdot \delta\mathbf{v} = 0$ (an incompressible perturbation) in equation (1.16). There is a solution if

$$\omega^2 = v_A^2 k^2$$

which is the dispersion relation for Alfven waves. You can use the induction equation to show that for these waves $\mathbf{k} \cdot \delta\mathbf{B} = 0$, ie. they are transverse to the magnetic field. They propagate at the Alfven speed $\mathbf{v}_A$.

Compressible vs. incompressible flow

An important point to make is that compressibility is a flow property as well as a material property. Flows that are subsonic, with fluid velocities much smaller than the sound speed, are incompressible with $\nabla \cdot \mathbf{v} \approx 0$, even though the material itself may be compressible. A way to think of this is that there is plenty of time for compressions to be smoothed out by propagation of sound waves if the flow is subsonic.

A simple illustration is given by a steady 1D isentropic flow. Isentropic means that we can write the pressure gradient term as $\partial P/\partial x = c_s^2 \partial\rho/\partial x$, where c_s is the isentropic sound speed. The momentum equation is then

$$v\frac{dv}{dx} = -\frac{c_s^2}{\rho}\frac{d\rho}{dx}$$

$$\Rightarrow \frac{v}{\rho}\frac{d\rho}{dv} = -\frac{v^2}{c_s^2}$$

$$\Rightarrow \frac{1}{\rho}\frac{d}{dv}\left(\rho v\right) = 1 - \frac{v^2}{c_s^2}. \tag{1.17}$$

Equation (1.17) shows that for subsonic flow, the mass flux ρv increases with velocity. This is what we would expect for an incompressible flow: at constant density, if you move faster the mass flux is larger. But note what happens at speeds faster than the sound speed. Then, the mass flux decreases as the flow speed increases. Despite moving faster, the density drops giving a smaller mass flux.

"Real-life" examples of these two limits are a river, which flows faster when the river narrows or slower when the river widens, and traffic on the freeway, which flows faster when the road widens and slows when the road narrows.

Steepening

When deriving the sound speed, we considered linear waves, ie. small perturbations to a background state. However, we know that the fluid equations have a non-linear term $(\mathbf{v}.\nabla)\mathbf{v}$, so that for large amplitudes it is not very useful to write the flow as a sum of plane waves. Whereas in a linear problem the plane waves evolve independently, and so it makes sense to use a Fourier decomposition, the non-linear terms couple the amplitudes of the different modes.

An important effect of the non-linear terms is that they lead to steepening of the velocity profile. We can see this by looking at the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0. \tag{1.18}$$

The general solution to this equation is

$$v = f(x - vt) = f(\xi),$$

where $f(\xi)$ is some arbitrary function of $\xi = x - vt$. To see this, change variables

$$\frac{\partial v}{\partial t} = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} = f'\left(-v - t\frac{\partial v}{\partial t}\right)$$

$$\Rightarrow \frac{\partial v}{\partial t} = \frac{-vf'}{1+f't}.$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{f'}{1+f't}. \tag{1.19}$$

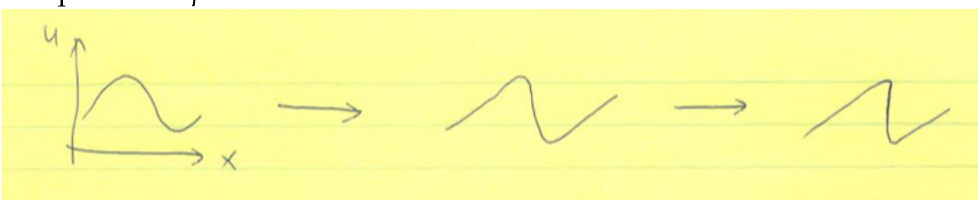
Combining these derivatives, we see that equation (1.18) is indeed satisfied.

More importantly, we see an interesting behaviour in the spatial derivative given by equation (1.19). An initial profile with $\partial v/\partial x|_{t=0} = f' < 0$ will reach $\partial v/\partial x \to \infty$ after a time

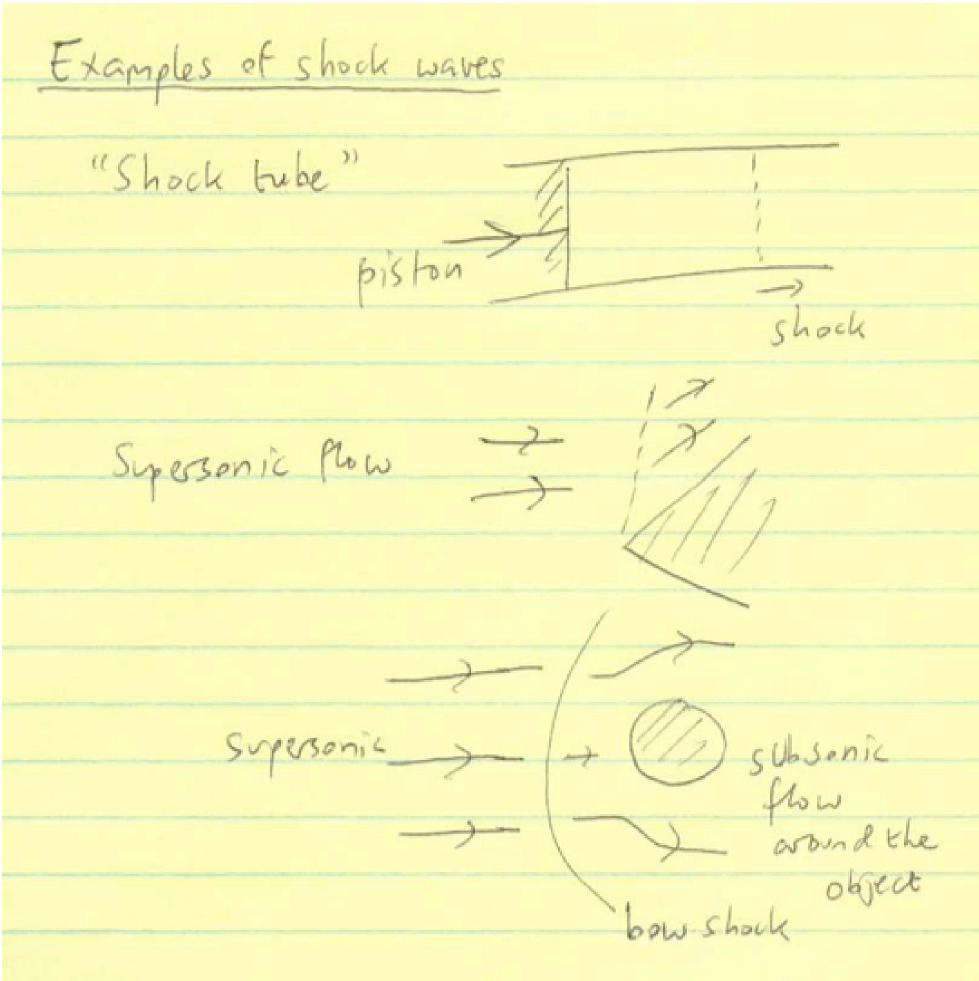
$$t = \left|-\frac{1}{f'}\right| = \left|-\frac{1}{\partial v/\partial x|_{t=0}}\right|$$

which we can think of as a local "turnover time" for the fluid.

The profile *steepens* in as illustrated in the sketch below.



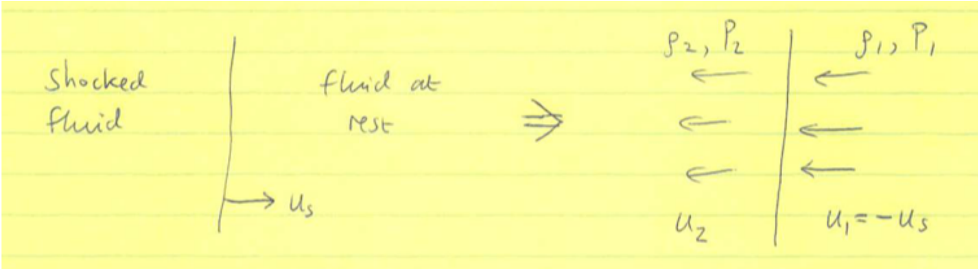
A *shock* forms in which the velocity v changes its value on a very short lengthscale. The thickness of the shock is set by the viscous term in the momentum equation which becomes important as dv/dx becomes large. Viscous stresses act to smooth out the velocity gradient and eventually will balance the steepening from the non-linear term. The lengthscale on which this happens is very short, of order the microscopic mean free path. In fact, we don't need to understand the details of what happens inside the shock, we can instead treat the shock as a discontinuity and relate the fluid velocity, density and temperature on each side using conservation of mass, momentum and energy. We do that in the next section.



A classical example of a situation in which a shock forms is the "shock tube" in which a piston moves into a cylinder. A shock propagates ahead of the cylinder, accelerating the fluid from rest to the speed of the cylinder, and at the same time compressing the gas. Another situation is supersonic flow around an object. A shock forms which acts to slow the fluid from supersonic to subsonic. The fact that the flow is subsonic near the object means that the sound crossing time can be shorter than the flow time, in this way the fluid flow around the obstacle.

Shock jump conditions

To derive the shock jump conditions, also known as the Rankine-Hugoniot relations, we first move into the frame of the shock as illustrated below. On the left, you see the shock moving to the right at speed v_s; on the right in the shock frame the unshocked fluid is moving to the left at speed v_s. Across the shock, the fluid changes velocity from $v_1 = -v_s$ to v_2, and density and pressure change from values ρ_1 and P_1 to ρ_2 and P_2.



We then integrate the fluid equations across the shock. For a steady 1D flow, continuity is

$$\frac{\partial}{\partial x}(\rho v) = 0.$$

Integrating,

$$\int_{-\epsilon}^{\epsilon} dx \frac{\partial}{\partial x}(\rho v) = [\rho v]_{-\epsilon}^{\epsilon} = 0$$

or

$$\rho_1 v_1 = \rho_2 v_2. \tag{1.20}$$

Momentum is

$$\rho v \frac{dv}{dx} = \frac{d}{dx}(\rho v^2) = -\frac{dP}{dx},$$

which when integrated gives

$$P_1 + \rho v_1^2 = P_2 + \rho v_2^2. \tag{1.21}$$

The total energy equation is

$$\frac{d}{dx}\left[v\left(\frac{1}{2}\rho v^2 + \rho e + P\right)\right] = 0$$

$$\Rightarrow \frac{1}{2}v_1^2 + e_1 + \frac{P_1}{\rho_1} = \frac{1}{2}v_2^2 + e_2 + \frac{P_2}{\rho_2}.$$

For an ideal gas, $P = (\gamma - 1)\rho e$, so we can rewrite this

$$\frac{1}{2}v_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{1}{2}v_2^2 + \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2}. \tag{1.22}$$

Equations (1.20)-(1.22) are the shock jump conditions, relating the "upstream" conditions (v_1, ρ_1, P_1) to the "downstream" ones (v_2, ρ_2, P_2) .

The jump conditions can be combined to derive a number of useful results. One of them is

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$

where $M_1 = u_1/c_1$ is the upstream Mach number, the shock velocity divided by the upstream sound speed. This shows that there is a maximum compression which occurs for a strong shock $(M_1 \gg 1)$, $\rho_2/\rho_1 = (\gamma + 1)/(\gamma - 1)$. This compression factor is 4 for a monatomic gas $(\gamma = 5/3)$.

While the compression is limited, note that the pressure and therefore temperature jump can be large. The pressure jump is

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

which is $\propto M_1^2$ for a strong shock.

The P_2–ρ_2 relation is known as the shock adiabat or the Hugoniot curve. But note that the flow across the shock is definitely not adiabatic! There is a large jump in entropy as the ordered kinetic energy of the rapid upstream flow is converted into heat in the compressed slow-moving gas downstream. For example, for a strong shock with $\gamma = 5/3$ you should be able to show that the downstream temperature is

$$\frac{k_B T_2}{\mu_1 m_p} = \frac{3}{16} v_s^2.$$

More complex cases that you could look at are:

- an oblique shock, in which the flow direction is not perpendicular to the shock. These occur in flow around an object, where the shocks help to redirect the fluid.

- a magnetized shock. As you might expect from our discussion of fast and slow magnetosonic waves, the direction of the magnetic field relative to the shock front makes a difference. A magnetic field perpendicular to the flow and parallel to the shock is compressed and gives an extra pressure that must be included in the jump conditions. There is also a jump condition on B coming from integrating the induction equation across the shock. For example, you can show that the ratio B/ρ is the same on both sides when the magnetic field is parallel to the shock. Compression of the fluid also implies a larger field strength because of magnetic flux conservation.

- a radiative shock. Shocks in astrophysics are often very radiative: the temperature immediately after the shock is so great that it leads to rapid cooling of the shocked gas. The net result can be much larger compression factors than in the strong shock case. A limit to consider is the *isothermal shock* in which the cooling is strong enough to equalize the temperature of the pre-shock and post-shock gas. Then the compression ratio is $\rho_2/\rho_1 = u_1^2/c_T^2$ where c_T is the isothermal sound speed.

Papers

- Goldreich, Murray, & Kumar (1994) "Excitation of acoustic modes in the Sun" http://adsabs.harvard.edu/abs/1994ApJ...424..466G

- Bazer and Ericson (1957) "Hydromagnetic shocks" http://adsabs.harvard.edu/abs/1959ApJ...129..758B

- Taylor (1950) "The formation of a blast wave by a very intense explosion" http://adsabs.harvard.edu/abs/1950RSPSA.201..159T http://adsabs.harvard.edu/abs/1950RSPSA.201..175T

Week 5: Introduction to Numerical Methods

These notes give an introduction to numerical methods for solving the fluid equations. They draw upon material in the book *Numerical Recipes* by Press & Teukolsky (see in particular section 19.1 of that book), and the course on hydrodynamics by P. Dullemond at the University of Heidelberg [7]. I focus here on finite differencing because this is the technique mostly used in astrophysics, where the geometry of the flow is usually quite simple, e.g. spherical (star or outflow), cylindrical (accretion disk) or plane-parallel (local box). An alternative technique is finite elements, used a lot in engineering applications that have complex geometries (e.g. flow around an aeroplane).

Finite difference approximation for derivatives

We solve for fluid properties on a numerical grid, at locations $x_j = j\Delta x$ where j labels the grid point. For simplicity here, we assume constant grid spacing Δx, although the results can be generalized to non-constant spacing. Quantities on neighbouring grid points are related by a Taylor expansion

$$f_{j+1} = f_j + \Delta x f'_j + \frac{(\Delta x)^2}{2} f''_j + \mathcal{O}(\Delta x^3)$$

$$f_{j-1} = f_j - \Delta x f'_j + \frac{(\Delta x)^2}{2} f''_j + \mathcal{O}(\Delta x^3).$$

Considering either of these gives a first order expression for the first derivative,

$$f'_j = \frac{f_j - f_{j-1}}{\Delta x} + \mathcal{O}(\Delta x) \qquad f'_j = \frac{f_{j+1} - f_j}{\Delta x} + \mathcal{O}(\Delta x).$$

Adding and subtracting instead gives a second order expression for the derivative and second derivative,

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$f''_j = \frac{f_{j+1} - 2f_j + f_{j-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x^2).$$

The advection equation; numerical stability and numerical diffusion

First consider advection,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0.$$

Using our expressions for the derivatives, we might write

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v\frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x},$$

[7] You can find the notes at
http://www.ita.uni-heidelberg.de/~dullemond/lectures/num_fluid_2011/index.shtml

where n labels the timestep. This gives an expression for the quantity f at the next timestep $n+1$ in terms of the value at the current timestep n:

$$f_j^{n+1} = f_j^n - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right).$$

This is known as the forward-time centered-space (FTCS) scheme. This kind of scheme is referred to as *explicit* because the new values are written explicitly in terms of the old ones.

In fact, it turns out that this scheme is always numerically unstable. You can see this by looking for a solution

$$f_j^n = (\xi)^n e^{ikx_j},$$

where k is the wavevector and ξ is a complex amplitude. If $|\xi| > 1$ for any value of k, that mode will grow exponentially with increasing timestep n, and the numerical scheme is unstable. Trying a solution like this for the FTCS scheme gives

$$|\xi|^2 = 1 + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x),$$

which is indeed greater than unity for any value of k.

Fortunately, there is a simple way to write a stable method, the Lax method:

$$f_j^{n+1} = \frac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right).$$

This has

$$|\xi|^2 = 1 + \left[\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1\right]\sin^2(k\Delta x),$$

and so we see that the scheme is stable as long as

$$\frac{v\Delta t}{\Delta x} \leq 1.$$

This condition on the timestep is the *Courant-Friedrichs-Levy criterion* or "Courant condition". The criterion states that our timestep must not exceed the fluid travel time between two grid points, which makes sense physically because the information about fluid quantities is advected at that speed. Larger timesteps require information from grid points further away than Δx, not included in our update.

A way to understand why the scheme is stable is to separate out the FTCS part and see what additional terms have been added. The Lax method can be rewritten

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v\frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x} + \left(\frac{\Delta x^2}{2\Delta t}\right)\frac{f_{j+1}^n - 2f_j^n + f_{j-1}^n}{(\Delta x)^2}.$$

The additional term on the right is a diffusion term with diffusivity $(\Delta x)^2/2\Delta t$. This is known as *numerical diffusion*, it provides numerical dissipation that stabilizes the method. The damping is largest for short wavelengths where $k\Delta x \sim 1$ which are most unstable.

The Lax scheme provides a good illustration of different types of error:

- When $v\Delta t < \Delta x$, $|\xi| < 1$, giving an *amplitude error*: the amplitude of any given mode k decreases over time (it should stay constant under advection)

- *Phase error*. The factor ξ in the Lax scheme can be rewritten as

$$\xi = e^{-ik\Delta x} + i\left(1 - \frac{v\Delta t}{\Delta x}\right)\sin k\Delta x.$$

  For a timestep $\Delta t = \Delta x/v$, the phase of each mode is shifted by $k\Delta x$, equivalent to advecting by one grid point. But for timesteps $\Delta t < \Delta x/v$ the phase shift depends on k, so that different modes are advected at different speeds. Again, this should not happen under advection. The numerical method introduces dispersion as the component waves of the profile we are trying to advect move with different speeds.

- *Transport errors*: in the Lax scheme, the information from cells $j-1$ and $j+1$ propagates to cell j in the next timestep. But physically, if the velocity is to the right for example, only information in cell $j-1$ should be used to update cell j. A way around this is *upwind differencing* which avoids this problem, but at the expense of being first order:

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v_j \frac{f_j^n - f_{j-1}^n}{\Delta x} \qquad v_j^n > 0$$

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -v_j \frac{f_{j+1}^n - f_j^n}{\Delta x} \qquad v_j^n < 0.$$

Everything we've discussed here is first order in time, but there are higher order methods that you can read about in Numerical Recipes. A useful one is *staggered-leapfrog* which uses a second-order time-derivative

$$f_j^{n+1} = f_j^{n-1} - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right).$$

Numerically this requires storing the previous two timesteps in order to do the update. This method has the advantage that $|\xi| = 1$ for all modes no matter what timestep is used: the stability analysis gives

$$\xi = -i\frac{v\Delta t}{\Delta x}\sin k\Delta x \pm \sqrt{1 - \left(\frac{v\Delta t}{\Delta x}\sin k\Delta x\right)^2},$$

so while there is dispersion (the phase evolution is different for different modes), the amplitude of each mode stays constant, much better than the very dispersive first order Lax method. Note that staggered leapfrog also has a limit $\Delta t \leq \Delta x/v$ for stability.

The diffusion equation: implicit methods

In the case of diffusion, the simplest differencing that you might write down is stable for small enough timesteps. The update is

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = D\frac{f_{j+1}^n - 2f_j^n + f_{j-1}^n}{(\Delta x)^2},$$

with

$$\frac{D\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

for stability. The physical interpretation is that the timestep is constrained by the diffusion time between grid cells.

Solving diffusion problems with explicit schemes is particularly slow, because the distance diffused in time t grows slowly with time, as $L \propto t^{1/2}$. The number of timesteps needed to follow diffusion over a lengthscale L is $L^2/D\Delta t \geq 2(L/\Delta x)^2 \sim N^2$ where N is the number of grid points.

An alternative scheme that allows larger timesteps, at the expense of accuracy on small scales, is an *implicit* scheme

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = D\frac{f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}}{(\Delta x)^2},$$

in which we write the update in terms of the values at the next timestep rather than at the current timestep (hence the name implicit). Rearranging, we can write

$$-\alpha f_{j+1}^{n+1} + (1+2\alpha)f_j^{n+1} - \alpha f_{j-1}^{n+1} = f_j^n$$

where $\alpha = D\Delta t/(\Delta x)^2$. Written as a matrix equation this is

$$Af^{n+1} = f^n$$

for the vectors f^{n+1} and f^n, where the matrix A is tridiagonal, with entries $1 + 2\alpha$ on the diagonal and $-\alpha$ on the upper and lower diagonals. This system can be solved by finding the inverse of the matrix A, since then $f^{n+1} = A^{-1}f^n$.

This *fully-implicit* scheme has the feature that it goes to the steady-state solution for large time-steps $\Delta t \to \infty$. Although small scales are not followed accurately for large timesteps, they go the correct steady-state solution. An alternative *semi-implicit* scheme is Crank-Nicholson

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left[\frac{1}{2}\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right) + \frac{1}{2}\left(f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}\right)\right]$$

which is also stable for large timesteps. It has the advantage that it is second order in both space and time, whereas fully-implicit is second order in space, but first order in time.

Operator splitting

You will often have multiple operators in the equation you are solving. A simple example is the *advection-diffusion* equation

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + D\frac{\partial^2 f}{\partial x^2}.$$

One way to deal with this is to calculate the update for each operator separately. Starting with f^n, generate $f^{n+\frac{1}{2}}$ by updating with the diffusion operator with timestep Δt, then update $f^{n+\frac{1}{2}}$ with the advection operator with timestep Δt to obtain the final values f^n.

Flux-conservative schemes

We know that the fluid equations arise from the conservation laws for mass, momentum and energy. We can take advantage of that and work with the equations in flux-conservative form, so that the numerical method exactly conserves these quantities.

In *finite-volume methods*, we divide the volume into cells such that the grid points x_j are the locations of the cell centres, and the cell boundaries are at locations $x_{j\pm 1/2} = (1/2)(x_j + x_{j\pm 1})$. We then solve the equation

$$\frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x},$$

or in discretized form

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = -\frac{J^{n+\frac{1}{2}}_{j+\frac{1}{2}} - J^{n+\frac{1}{2}}_{j-\frac{1}{2}}}{\Delta x},$$

where we write the flux of quantity f at the cell boundaries $(j \pm 1/2)$ averaged over the timestep:

$$J^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\; J_{j+\frac{1}{2}}(t).$$

This formulation automatically conserves the quantity f, since the flux out of one cell equals the flux into the neighbouring cell.

The simplest choice for the flux J is to write

$$J_{j+\frac{1}{2}} = v_{j+\frac{1}{2}} f_j^n \qquad v_{j+\frac{1}{2}} > 0$$

$$J_{j+\frac{1}{2}} = v_{j+\frac{1}{2}} f_{j+1}^n \qquad v_{j+\frac{1}{2}} < 0$$

$$J_{j-\frac{1}{2}} = v_{j-\frac{1}{2}} f_{j-1}^n \qquad v_{j-\frac{1}{2}} > 0$$

$$J_{j-\frac{1}{2}} = v_{j-\frac{1}{2}} f_j^n \qquad v_{j+\frac{1}{2}} < 0$$

which is known as *donor cell advection* (equivalent to the upwind differencing discussed earlier). Depending on the sign of the velocity, the contents are either advected out of cell j or into cell j from the left or right neighbour. The assumption here is that the profile of f within the cell is well-approximated by a constant (given by the value at the center f_j). More complex assumptions about the profile of f give rise to higher order methods. For example, assuming f is linear across the cell (with slope chosen to be consistent with the difference in f between cell j and its neighbours) gives a scheme that is 2nd order in time. These piecewise linear schemes are discussed in detail in Chapter 4 of the Heidelberg notes I linked to earlier (see footnote on page 1).

Papers

This week, the paper discussion will involve researching a (magneto)hydrodynamics code such as PLUTO, PENCIL, FLASH, ZEUS, Castro.

You should discuss:

- What equations is the code solving?
- What numerical methods are used?
- What microphysics is included?
- What geometry can the code simulate?
- What are some of the applications that the code has been used for?

Computational Exercise 2: Steepening

Overview. The goal of this exercise is to write a 1D hydro code and use it to demonstrate the steepening of a sound wave.

Algorithm. The lecture notes on hydrodynamics from the University of Heidelberg give a simple algorithm that you can use to solve the 1D hydro equations (in Chapter 5). First, the fluid equations are written in flux-conservative form, with conserved quantities

$$f_1 = \rho$$

$$f_2 = \rho v$$

(the mass and momentum densities) and it is assumed that $P = c_s^2 \rho$ with constant sound speed c_s. The equations to solve are then

$$\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial x}(v f_1) = 0$$

$$\frac{\partial f_2}{\partial t} + \frac{\partial}{\partial x}(v f_2) = -\frac{\partial P}{\partial x}.$$

These are in flux-conservative form with the pressure gradient acting as a source term for the momentum density f_2. Note that given f_2 and f_1, the velocity at the grid centre can be obtained from the ratio f_2/f_1.

The algorithm has two steps:

1. Use donor-cell advection to update f_1 and f_2. To calculate the velocity at the cell boundaries, you can take an average of the velocity at the cell centres

$$v_{j+\frac{1}{2}} = \frac{1}{2}(v_j + v_{j+1}).$$

2. Add an additional update to the value of f_2 from step 1 to take into account the source term. You can do this by writing

$$\frac{\partial P}{\partial x} = c_s^2 \frac{\partial \rho}{\partial x}$$

and using a first order difference to calculate the density gradient in terms of the new values of f_1 you found in step 1.

Questions

1. Choose an initial condition that has a sinusoidal variation in density and/or velocity. Check that for small amplitudes, the wave propagates as expected.

2. Do you see steepening at larger wave amplitudes?

3. How large a timestep can you take and still be numerically stable?
4. Do you form a shock in your simulation? What sets its thickness?

Possible extensions

- An interesting extension is to add the energy equation to your code. We need a third quantity
$$f_3 = \rho e_{\rm tot}$$
where e is the specific total energy (sum of kinetic and internal energy). The energy equation is
$$\frac{\partial f_3}{\partial t} + \frac{\partial}{\partial x}\left(v f_3\right) = -\frac{\partial}{\partial x}\left(vP\right).$$
As for momentum, we can solve this by advecting f_3 in step 1 and then updating f_3 using the source term on the right hand side in step 2. The pressure is
$$P = (\gamma - 1)\rho e$$
where
$$e = e_{\rm tot} - \frac{v^2}{2}$$
is the thermal energy.

  With the energy equation included, you will be able to validate the shock jump conditions, e.g. check that the compression factor is $(\gamma+1)/(\gamma-1)$ for a strong shock.
- As a next step, you could modify your code to work in spherical symmetry (adding appropriate r^2 factors), and then model the Sedov-Taylor blast wave.
- Another area to investigate is to use a higher order approximation for the advection step (section 4.3 in the Heidelberg notes) and see how it improves the results.

Week 6: Inflows and Outflows

Outflows and inflows are important in astrophysics in systems with winds (star, galaxies, planets, accretion disks), jets (e.g. from accreting black holes) and accretion onto a central object (e.g. in compact binaries, star or planet formation, black hole growth). These notes discuss some different examples. We start with the simplest case of a spherically-symmetric wind from or accretion flow onto a point mass. We then consider some more complex examples that break spherical symmetry: stellar wind from a magnetized star and jets.

These flows have some common features that you should look out for. One is that there are conserved quantities that we can use to learn about the flow without necessarily solving for the detailed structure. Another feature is the existence of critical points at which the flow speed equals a wave speed, for example a sonic point at which the flow transitions from subsonic to supersonic.

Bondi accretion and Parker wind

Consider spherically-symmetric, steady, radial flow either onto or away from a point mass. The continuity equation is $\nabla \cdot \rho \mathbf{v} = 0$, which in spherical coordinates is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho v\right) = 0,$$

which shows that $r^2\rho v$ is constant throughout the flow. It is convenient to write this in terms of the mass loss rate or accretion rate

$$\dot{M} = 4\pi r^2 \rho v$$

(units of g s^{-1}). The momentum equation is

$$\rho v\frac{dv}{dr} = -\frac{dP}{dr} - \rho\frac{GM}{r^2}, \tag{1.23}$$

where we assume that the gravitational acceleration is dominated by the mass of the central star, ie. there is negligible mass in the flow itself.

As usual to simplify things we can make an assumption about the relation between P and ρ so that we don't have to worry about the energy equation. For an isothermal gas, $P = c_s^2\rho$ with c_s constant, giving

$$v\frac{dv}{dr} = -c_s^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2}. \tag{1.24}$$

From the continuity equation,

$$\frac{d\ln\rho}{dr} = -\frac{2}{r} - \frac{d\ln v}{dr}$$

and so, eliminating the density gradient from the momentum equation,

$$\left(v - \frac{c_s^2}{v}\right)\frac{dv}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2}.$$

Defining the sonic radius

$$r_s = \frac{GM}{2c_s^2}$$

we can rewrite this as

$$\left(1 - \frac{v^2}{c_s^2}\right) \frac{d\ln v}{d\ln r} = 2\left(\frac{r_s}{r} - 1\right).$$

This shows that if the flow makes a transition from subsonic to supersonic or vice-versa, that must happen at $r = r_s$ in order for the velocity gradient to remain finite. This is known as the *sonic point*.

The velocity and density profiles $v(r)$ and $\rho(r)$ can be obtained by integrating the continuity and momentum equations. (In fact, in the case of an isothermal flow, this integration can be done analytically). Two solutions are possible which go through the sonic point: (i) subsonic flow close to the star $v < c_s$ with v increasing outwards, becoming supersonic at $r > r_s$, or (ii) subsonic flow beyond the sonic point $r > r_s$ and v increasing inwards, becoming supersonic at $r < r_s$. Option (i) corresponds to a wind, discussed by Parker (1958) in the context of the Sun, whereas option (ii) with $v < 0$ corresponds to accretion of mass onto the central star, first discussed by Bondi (1952).

The mass accretion rate or mass loss rate can be written in terms of the sound speed (or equivalently temperature) by evaluating equation (1.23) at the sonic point where $v = c_s$ and $\rho = \rho_s$:

$$\dot{M} = 4\pi r_s^2 c_s \rho_s = \pi \frac{(GM)^2}{c_s^3} \rho_s.$$

To find ρ_s for a given situation, we map from the boundary as appropriate. Integrating equation (1.24) gives

$$\frac{1}{2}v^2 + c_s^2 \ln\rho - \frac{GM}{r} = B = \text{constant}.$$

For the accretion case, $v \to 0$ and $GM/r \to 0$ at large distance, so $B = c_s^2 \ln\rho_\infty$. At the sonic point therefore

$$\frac{1}{2}c_s^2 + c_s^2 \ln\rho_s - \frac{GM}{r_s} = c_s^2 \ln\rho_\infty,$$

or $\rho_s = \rho_\infty e^{3/2}$. This gives the *Bondi accretion rate*

$$\dot{M} = \pi e^{3/2} \frac{(GM)^2}{c_s^3} \rho_\infty,$$

the accretion rate onto a point mass[8] placed in a gas with sound speed c_s and density ρ_∞ (assuming the flow is isothermal). For the case of a wind, we apply the boundary

[8] Hoyle & Lyttleton (1939) derived a similar formula but they considered accretion by a star moving through the interstellar medium (they were interested in whether accretion could power stellar luminosities). Their result is $\dot{M} \sim \rho_\infty (GM)^2/v_\star^3$, the same scalings but with $c_s \to v_\star$. The geometry of the flow is quite different in that case, with the incoming matter being gravitationally-focussed behind the star (in the star's frame) and then falling in.

conditions at the stellar surface $r = R$ where we take $v = 0$ and $\rho = \rho_\star$, giving

$$B = c_s^2 \ln \rho_\star - \frac{GM}{R}.$$

Therefore

$$\ln\left(\frac{\rho_s}{\rho_\star}\right) = \frac{3}{2} - \frac{GM}{Rc_s^2} = \frac{3}{2} - \frac{2r_s}{R} < 0, \tag{1.25}$$

and the *mass loss rate in the wind* is

$$\dot{M} = \pi e^{3/2} \frac{(GM)^2}{c_s^3} \rho_\star e^{-GM/Rc_s^2}.$$

Magnetized stellar wind and angular momentum loss

Magnetic fields can play an important role in stellar winds from rotating stars. In particular, through magnetic forces in the azimuthal direction, the magnetic field determines the angular momentum loss rate in the wind, and therefore how quickly the star spins down.

Weber & Davis (1967) made a simple model in which they considered only the equatorial plane and assumed that the wind had "combed out" the magnetic field, so that the magnetic field lies in the equatorial plane

$$\mathbf{B} = B_r(r)\hat{\mathbf{e}}_r + B_\phi(r)\hat{\mathbf{e}}_\phi.$$

Everything is assumed to be axisymmetric and so only depends on r, and steady. Because the magnetic field has to be divergence free,

$$\nabla \cdot \mathbf{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) = 0,$$

$r^2 B_r$ must be constant, i.e. $B_r \propto 1/r^2$. The velocity is

$$\mathbf{v} = v_r(r)\hat{\mathbf{e}}_r + v_\phi(r)\hat{\mathbf{e}}_\phi.$$

As in the Parker wind, mass continuity tells us that $r^2 \rho v_r = \dot{M}/4\pi$ is constant. With $\mathbf{v}$ and $\mathbf{B}$ being in the equatorial plane, and $\partial/\partial\phi = 0$, the only possible non-zero component of the induction equation is

$$\frac{\partial B_\phi}{\partial t} = -c(\nabla \times \mathbf{E})_\phi = -\frac{1}{r}\frac{d}{dr}\left[r\left(v_r B_\phi - v_\phi B_r\right)\right].$$

In a steady state, we must therefore have

$$r\left(v_r B_\phi - v_\phi B_r\right) = \text{constant} = -R^2 \Omega B_r(R) = -r^2 \Omega B_r$$

where Ω is the spin of the star. This tells us that

$$\frac{B_\phi}{B_r} = \frac{v_\phi - \Omega r}{v_r} \tag{1.26}$$

so that the flow is along the magnetic field lines everywhere in the rotating frame. There is a steady pattern in the rotating frame.

To focus on the angular momentum, let's look at the ϕ component of the momentum equation. This is

$$\rho v_r \frac{1}{r}\frac{d}{dr}(r v_\phi) = \frac{1}{c}(\mathbf{J}\times\mathbf{B})_\phi = \frac{B_r}{4\pi r}\frac{d}{dr}(rB_\phi),$$

where we use

$$\mathbf{J} = \frac{c}{4\pi}\nabla\times\mathbf{B} = -\frac{c}{4\pi}\frac{1}{r}\frac{d}{dr}(rB_\phi)\,\hat{\mathbf{e}}_\theta.$$

But $\rho v_r \propto 1/r^2$ and $B_r \propto 1/r^2$, and so we can integrate this:

$$r v_\phi - \frac{B_r}{4\pi\rho v_r} r B_\phi = \text{constant} = L. \tag{1.27}$$

We call the constant L because we see from the first term that this is the angular momentum per unit mass. If the magnetic field was not present, rV_ϕ would be constant, but the magnetic torques cause this to change across the flow.

The Alfven velocity $v_A^2 = B_r^2/4\pi\rho$ can be used to define a radial Alfven Mach number

$$M_A = \frac{v_r}{v_A} = \frac{\sqrt{4\pi\rho}v_r}{B_r}.$$

Equation (1.27) becomes

$$r v_\phi - \frac{1}{M_A^2} r v_r \frac{B_\phi}{B_r} = L. \tag{1.28}$$

We see that since $B_r \propto \rho v_r$, $M_A^2 \propto v_r/B_r \propto 1/\rho$, so the Alfven Mach number increases through the flow.

Together, equations (1.26) and (1.28) can be used to solve for v_ϕ:

$$r v_\phi = \frac{L M_A^2 - r^2\Omega}{M_A^2 - 1}.$$

Close to the star, $M_A \ll 1$, $v_\phi \approx r\Omega$ which corresponds to rigid rotation at the stellar spin frequency. Far from the star, $M_A \gg 1$ and $r v_\phi = L$, so the flow has constant angular momentum per unit mass. What's happening is that close to the star the magnetic field is strong enough to keep the fluid moving rigidly with the star; far from the star the magnetic torques are no longer important and so the fluid moves outwards with constant angular momentum. The transition occurs at a particular radius, the *Alfven radius* $r_A = (L/\Omega)^{1/2}$ at which $M_A = 1$.

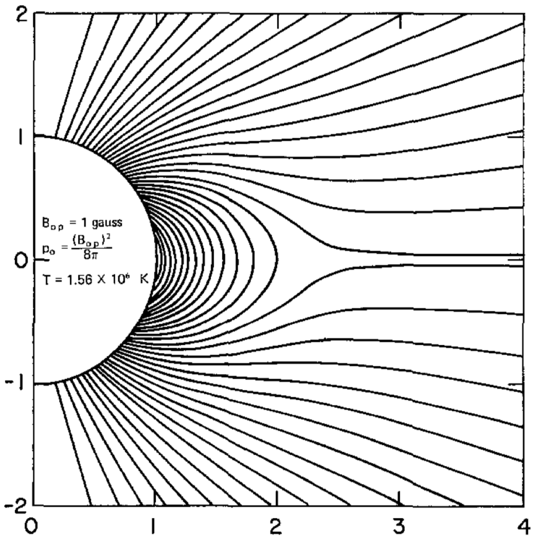
The angular momentum loss in the wind is therefore

$$\dot{M} r_A^2 \Omega$$

which can be much greater than $\dot{M} R_\star^2 \Omega$ which would be the angular momentum loss rate for the Parker wind. The magnetic field keeps the plasma rotating rigidly out to $r = r_A$ which gives a larger "lever arm" for the torque.

In their paper, Weber & Davis go on to look at the radial structure of the wind. As well as the Alfven point (at $\sim 30R_\odot$ in their model), there is also a sonic point as in the spherical solution (located much closer to the Sun at a few solar radii). (In fact, there are multiple critical points where the flow velocity matches one of the wave speeds as they discuss in detail in the paper).

The plot below is taken from Pneuman & Kopp (1973) which was an early paper doing a multi-D model of the wind, with the stellar field assumed to be a dipole. We see the same ideas apply: a closed zone close to the star where the magnetic torques dominate, and open field lines further out where the field becomes flow-aligned.

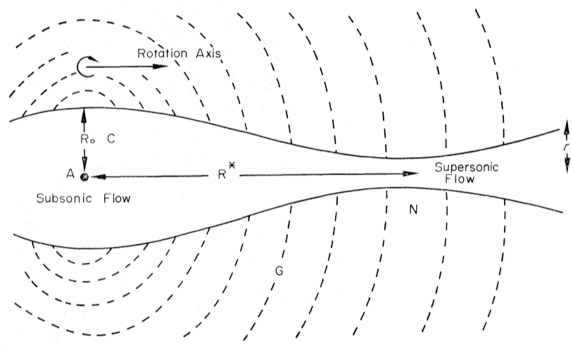


Jets as nozzles

Next, let's discuss an example which is more one-dimensional: the collimation of a jet. Active galaxies in particular show jets that remain remarkably collimated over huge scales (much larger than the size of the host galaxy). An early idea discussed by Blandford & Rees (1974) is that the pressure of the gas around the source could act to collimate the jet and achieve a supersonic outflow.

We showed earlier that for a 1D isentropic flow, the mass flux increases with velocity for subsonic flows (incompressible flow) but decreases with velocity for supersonic flows (compressible flow; see eq. [4] of the Week 4 notes). One place where this comes up is in designing a nozzle through which gas can flow and become supersonic. If a sonic transition is to occur with a steady flow, the product of area and mass flux must be constant. This means that the nozzle must be designed to have a decreasing area at first while the flow is subsonic, but then increase again later so that the flow can continue to accelerate. This kind of nozzle is known as a *de Laval nozzle*.

Blandford & Rees proposed that a similar effect is happening in radio galaxies, except the area of the nozzle is not specified in advance but rather that the confining pressure from the external gas sets the area of the flow. The figure below from their paper shows the overall idea:



For a 1D isentropic flow, the momentum equation can be written

$$\frac{d}{dx}\left(\frac{1}{2}v^2 + \frac{c_s^2}{\gamma - 1}\right) = 0$$

where $c_s^2 = \gamma P/\rho$ is the adiabatic sound speed and $P \propto \rho^\gamma$. The Bernoulli constant

$$B = \frac{1}{2}v^2 + \frac{c_s^2}{\gamma - 1}$$

is a constant of the flow. If there is some pressure and density P_0 and ρ_0 at which $v = 0$ (in the nozzle context, this is the pressure and density in the container; for the jet it's the pressure and density at the base of the flow) then keeping B constant implies

$$\frac{v^2}{c_s^2} = \frac{2}{\gamma - 1}\left[1 - \left(\frac{P}{P_0}\right)^{(\gamma-1)/\gamma}\right]$$

which gives the velocity as a function of pressure. For a large enough pressure drop in the external gas, the flow will make a transition to supersonic flow. Blandford & Rees made essentially this argument, although they used relativistic equations since the flow speed is a significant fraction of c for the radio jets.

Papers

- Blandford & Rees 1974 "A 'twin-exhaust' model for double radio sources" https://ui.adsabs.harvard.edu/#abs/1974MNRAS.169..395B/abstract
- Bondi 1952 "On Spherically Symmetrical Accretion" https://ui.adsabs.harvard.edu/#abs/1952MNRAS.112..195B/abstract
- Hoyle & Lyttleton 1939 "The effect of interstellar matter on climatic variation" https://ui.adsabs.harvard.edu/#abs/1939PCPS...35..405H/abstract
- Parker 1958 "Dynamics of the Interplanetary Gas and Magnetic Fields" https://ui.adsabs.harvard.edu/#abs/1958ApJ...128..664P/abstract
- Pneuman & Kopp 1973 "The solar wind and the temperature-density structure of the solar corona" https://link.springer.com/article/10.1007/BF00152928

- Weber & Davis 1967 "The Angular Momentum of the Solar Wind" https://ui.adsabs.harvard.edu/#abs/1967ApJ...148..217W/abstract

Week 7: Oscillations and Instabilities

We've already seen two examples of waves in a fluid system: sound waves in a uniform gas, and fast and slow magnetosonic waves and Alfven waves in a magnetized plasma. Here I go through two examples of linear stability analysis as examples of more complex situations: first what happens when we include the energy equation explicitly for sound waves, and second how to deal with a background that has a gradient.

Sound waves with thermal conduction

Earlier, we derived the dispersion relation for sound waves by assuming a relation between the pressure and density perturbations

$$\delta P = \frac{\partial P}{\partial \rho}\delta\rho = c_s^2\delta\rho.$$

The partial derivative can be taken at constant entropy, in which case $c_s^2 = \gamma P/\rho$ is the adiabatic sound speed, or at constant temperature, giving the isothermal sound speed $c_s^2 = P/\rho = k_B T/\mu m_p$. These two cases can be understood as limits of either very inefficient heat transfer (adiabatic) or efficient heat transfer (isothermal) on the timescale of the sound wave period ($2\pi/\omega = 2\pi/c_s k = \lambda/c_s$).

Instead of making this assumption, let's instead include the energy equation in the calculation. We will assume ideal gas, in which case the pressure, density and temperature perturbations are related by

$$\frac{\delta P}{P} = \frac{\delta\rho}{\rho} + \frac{\delta T}{T} \tag{1.29}$$

(since $P \propto \rho T$). The entropy equation is

$$T\frac{Ds}{Dt} = -\frac{1}{\rho}\nabla\cdot\mathbf{F} = \frac{1}{\rho}\nabla\cdot\left(K\nabla T\right),$$

where K is the thermal conductivity (we will assume this is a constant).

Perturbing the entropy equation gives

$$-i\omega T\delta s = -\frac{k^2 K\delta T}{\rho} \tag{1.30}$$

(the background is stationary, so only the time derivative term of D/Dt contributes at linear order). This is the extra equation that we need to eliminate δT and derive the relation between δP and $\delta\rho$. To do this, write

$$Tds = T\left.\frac{\partial s}{\partial T}\right|_P dT + T\left.\frac{\partial s}{\partial P}\right|_T dP. \tag{1.31}$$

We then use the fact that $T\,\partial S/\partial T|_P = c_P$ the heat capacity at constant pressure, the identity

$$\left.\frac{\partial s}{\partial P}\right|_T \left.\frac{\partial P}{\partial T}\right|_s \left.\frac{\partial T}{\partial s}\right|_P = -1,$$

and the adiabatic index

$$\frac{\gamma-1}{\gamma} = \left.\frac{\partial \ln T}{\partial \ln P}\right|_s$$

to rewrite equation (1.31) as

$$T ds = c_p \left(dT - \frac{\gamma-1}{\gamma}\frac{T}{P} dP \right). \tag{1.32}$$

This allows us to write down δs in terms of δT and δP. Equation (1.30) then gives

$$c_P T \left(\frac{\delta T}{T} - \frac{\gamma-1}{\gamma}\frac{\delta P}{P} \right) = \frac{k^2 K T}{i\omega\rho}\left(\frac{\delta T}{T}\right).$$

Using the ideal gas relation from equation (1.29) to eliminate δT in favour of δP and $\delta\rho$, we find

$$\boxed{\frac{1}{\gamma}\frac{\delta P}{P} - \frac{\delta\rho}{\rho} = \frac{k^2 K}{i\omega\rho c_P}\left(\frac{\delta P}{P} - \frac{\delta\rho}{\rho}\right)} \tag{1.33}$$

I've put a box around this result because it is the relation between δP and $\delta\rho$ that we've been looking for. The quantity $D = K/\rho c_P$ is the thermal diffusivity (units of cm^2/s), since we can write

$$T\frac{Ds}{Dt} = c_P \frac{DT}{Dt} = \frac{K}{\rho}\nabla^2 T \Rightarrow \frac{DT}{Dt} = \frac{K}{\rho c_P}\nabla^2 T = D\nabla^2 T$$

(working at constant pressure for simplicity and again assuming constant K). The thermal timescale associated with the perturbation is therefore $1/(k^2 D)$. When the mode frequency ω is either large or small compared with $k^2 D$, we recover the adiabatic or isothermal limits discussed earlier. However, in general, we see that the relation between δP and $\delta\rho$ has a complex prefactor. The dispersion relation will be

$$\frac{\omega^2}{k^2} = c_s^2 = \frac{\gamma P}{\rho}\left(\frac{i\omega - k^2 D}{i\omega - \gamma k^2 D}\right).$$

In general we see that k^2 will be complex, so that for a given ω there will be a propagating wave (real part of k) but with a decaying amplitude (imaginary part of k).

I will leave this as an exercise, but for example one limit to consider is when the wave is *almost* adiabatic, so that $k = k_R + ik_I$ with $k_I \ll k_R$. In this limit, $\omega^2 \approx k_R^2 c_{\mathrm{ad}}^2$, where c_{ad} is the adiabatic sound speed, and

$$\frac{k_I}{k_R} \approx \frac{\gamma-1}{2}\frac{k_R^2 D}{\omega} \approx \frac{\gamma-1}{2}\frac{\omega D}{c_{\mathrm{ad}}^2}.$$

For air, $c_s \approx 330\ \mathrm{m\ s^{-1}}$ and $D \approx 2\times 10^{-5}\ \mathrm{m^2/s}$, giving a decay length of about 10^6 wavelengths.

Gravity waves

As a second example, let's look at the waves in a plane-parallel atmosphere. The additional ingredient here is gravity, since the atmosphere is in hydrostatic balance

$$\frac{dP}{dz} = -\rho g$$

so that there are background pressure, density and temperature gradients.

When dealing with problems with background gradients, it can be useful to write things in terms of Lagrangian perturbations. The perturbations we have been writing down so far are Eulerian, since at any given time t they give the difference between the perturbed and unperturbed flows at the same point in space, e.g.

$$\delta\rho\left(\mathbf{r},t\right) = \rho\left(\mathbf{r},t\right) - \rho_0\left(\mathbf{r},t\right),$$

where ρ is the perturbed density and ρ_0 is the unperturbed density. Instead we could define the Lagrangian perturbation

$$\Delta\rho\left(\mathbf{x_0},t\right) = \rho\left(\mathbf{r}(\mathbf{x_0},t),t\right) - \rho_0\left(\mathbf{r_0}(\mathbf{x_0},t),t\right),$$

where $\mathbf{x_0}$ is a Lagrangian label that identifies the fluid element, for example a good choice would be the initial location of the fluid element. The difference in the positions of the fluid element in the unperturbed flow $\mathbf{r_0}(\mathbf{x_0},t)$ and perturbed flows $\mathbf{r}(\mathbf{x_0},t)$ is the Lagrangian displacement

$$\xi = \mathbf{r}(\mathbf{x_0},t) - \mathbf{r_0}(\mathbf{x_0},t).$$

The Eulerian and Lagrangian perturbations at a particular spatial location $\mathbf{r}$ are related by

$$\Delta\rho\left(\mathbf{r}(\mathbf{x_0},t),t\right) - \delta\rho\left(\mathbf{r},t\right) = -\rho_0\left(\mathbf{r_0}(\mathbf{x_0},t),t\right) + \rho_0\left(\mathbf{r},t\right) \approx \xi\cdot\nabla\rho_0(\mathbf{r}),$$

or

$$\boxed{\Delta\rho = \delta\rho + \xi\cdot\nabla\rho}$$

As an application, consider the perturbed continuity equation

$$-i\omega\delta\rho = -\nabla\cdot\left(\rho\delta\mathbf{v}\right).$$

We assume $\mathbf{v} = 0$ in the background, in which case we can also write

$$\delta\mathbf{v} = \frac{\partial\xi}{\partial t} = -i\omega\xi$$

and so

$$\delta\rho = -\nabla\cdot(\rho\xi) = -\rho\nabla\cdot\xi - \xi\cdot\nabla\rho$$

$$\Rightarrow \boxed{\frac{\Delta\rho}{\rho} = -\nabla\cdot\xi} \qquad (1.34)$$

If the Lagrangrian displacements have a non-zero divergence, it implies a Lagrangian density change. Note that if the background is moving then δv and ξ have a more complex relation.

Going back to the plane-parallel atmosphere, the perturbed continuity equation is therefore equation (1.34). We consider adiabatic perturbations which we can immediately write down taking advantage of the Lagrangian formalism as

$$\frac{\Delta P}{P} = \gamma \frac{\Delta \rho}{\rho}.$$

Therefore

$$\frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho} - \xi_z \left[\frac{d \ln P}{dz} - \gamma \frac{d \ln \rho}{dz} \right]$$

or

$$\frac{\delta P}{\rho c_s^2} = \frac{\delta \rho}{\rho} - \frac{N^2 \xi_z}{g}, \tag{1.35}$$

where we define the Brunt-Väisälä frequency N and *convective discriminant* $\mathcal{A}$ according to

$$N^2 = -g\mathcal{A} = -g \left[\frac{d \ln \rho}{dz} - \frac{1}{\gamma} \frac{d \ln P}{dz} \right].$$

The momentum equations are

$$-\rho \omega^2 \xi_z = -\frac{d\delta P}{dz} + g\delta\rho \tag{1.36}$$

$$-\rho \omega^2 \xi_x = -ik_x \delta P. \tag{1.37}$$

Note that whereas we have assumed an x-dependence for the perturbations of $e^{ik_x x}$, we do not specify a functional form for the z-dependence; it will be determined by how the background changes with height. To solve the equations in a realistic atmosphere or star requires integration of the equations over height z. The equations form an eigenvalue problem: in general, only a certain set of frequencies ω_n give solutions that satisfy the boundary conditions at $z = 0$ or $z = \infty$ ($r = 0$ and $r = R$ in the case of a star).

A useful limit to consider however is when the vertical wavelength of the waves is much smaller than the pressure or density scale heights. Then the coefficients in the equations remain constant on the scale of a wavelength, and we can write a local WKB solution $e^{ik_z z}$. Substituting this into the continuity equation, momentum equations and adiabatic condition (eqs. [1.34], [1.35], [1.36], and [1.37]) gives the dispersion relation

$$c_s^2 k_z^2 = \left(\omega^2 - N^2\right) \left(1 - \frac{k_x^2 c_s^2}{\omega^2}\right).$$

A vertically-propagating wave requires $k_z^2 > 0$ so that k_z is real. This can happen in two ways. The first is $\omega^2 \gg N^2$, when $\omega^2 = c_s^2(k_z^2 + k_x^2) = c_s^2 k^2$. These are the sound waves or acoustic waves we have encountered before. We see them again in the plane-parallel atmosphere.

The second solution for propagating waves is when $\omega^2 < N^2$ and $\omega^2 < c_s^2 k^2$. Then both terms on the right hand side of the dispersion relation are negative and so $k_z^2 > 0$. The dispersion relation when $\omega^2 \ll N^2$ is

$$\omega^2 = N^2 \left(\frac{k_x}{k}\right)^2.$$

These waves are *internal gravity waves*. They are incompressible waves, ie. they satisfy $\nabla\cdot\xi \approx 0$ (if you repeat the calculation setting $\nabla\cdot\xi = 0$ exactly, you'll find that the sound waves go away but the gravity waves survive). The restoring force for the wave is from horizontal pressure gradients that arise from horizontal variations in the hydrostatic column that arise as the fluid moves. One interesting fact about gravity waves is that the phase and group velocities of a wavepacket are orthogonal (try proving this using the dispersion relation). In the context of stars, standing gravity waves or acoustic waves can exist that occupy the entire stellar volume in some cases or may propagate in only a limited region of the stellar interior (where $k^2 > 0$). In this context, gravity waves are referred to as *g-modes* and the acoustic waves as *p-modes* (g for gravity and p for pressure).

The convective discriminant $\mathcal{A}$ is so-named because it indicates whether the atmosphere is unstable to convection. In a situation in which $\mathcal{A} > 0$, $N^2 < 0$ and $\omega^2 < 0$ indicating instability. The way to understand this is to consider moving a fluid element upwards slowly enough that it stays in pressure equilibrium with its surroundings, but quickly enough that the motion is adiabatic. The density contrast between the fluid element and its surroundings after moving a vertical distance Δz is

$$\Delta z \left.\frac{\partial \rho}{\partial P}\right|_S \frac{dP}{dz} - \Delta z \frac{d\rho}{dz} = -\rho \Delta z \left[\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz}\right] = -\rho\Delta z\mathcal{A}.$$

If $\mathcal{A} > 0$, we see that the fluid element will be less dense than its surroundings and so will buoyantly rise further: the atmosphere is unstable to vertical perturbations. The criterion $\mathcal{A} > 0$ is the *Schwarzschild criterion* for convection.

Papers

There are many different types of waves and instabilities relevant for astrophysical objects. Here is a selection of recent papers that give some nice examples:

- Fuller (2014) "Saturn ring seismology: Evidence for stable stratification in the deep interior of Saturn"
  http://adsabs.harvard.edu/abs/2014Icar..242..283F
- Showman & Polvani (2011) "Equatorial Superrotation on Tidally Locked Exoplanets"
  http://adsabs.harvard.edu/abs/2011ApJ...738...71S
- Philippov et al. (2016) "Spreading Layers in Accreting Objects: Role of Acoustic Waves for Angular Momentum Transport, Mixing, and Thermodynamics"
  https://ui.adsabs.harvard.edu/#abs/2016ApJ...817...62P/abstract
- Levin (2007) "On the theory of magnetar QPOs"
  https://ui.adsabs.harvard.edu/#abs/2007MNRAS.377..159L/abstract

Appendix: Perturbation equations in spherical geometry

For stellar or planetary oscillations, we need the perturbation equations in spherical geometry. We assume the background is spherically-symmetric, so pressure or density depend only on r. We take density perturbations of the form

$$\delta\rho = \delta\rho(r)e^{im\phi}P_\ell^m(\cos\theta)e^{-i\omega t}$$

and similarly for the pressure perturbation δP and radial displacement ξ_r. The non-radial displacements have a different angular dependence:

$$\xi_\theta = \xi_\theta(r)e^{im\phi}\frac{dP_\ell^m(\cos\theta)}{d\theta}e^{-i\omega t}$$

$$\xi_\phi = \xi_\phi(r)e^{im\phi}\frac{imP_\ell^m(\cos\theta)}{\sin\theta}e^{-i\omega t}$$

With these choices for the angular dependences, the perturbation equations then depend only on r, as follows:

Adiabatic perturbations

$$\frac{\Delta\rho}{\rho} = \frac{1}{\gamma}\frac{\Delta P}{P}$$

$$\Rightarrow \frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\delta P}{P} + \frac{N^2\xi_r}{g}$$

(using the definition of N^2 from the text; $g(r) = Gm(r)/r^2$).

Continuity

$$\frac{\Delta\rho}{\rho} = -\nabla\cdot\xi$$

$$\frac{1}{\gamma}\frac{\delta P}{P} + \frac{\xi_r}{\gamma}\frac{d\ln P}{dr} = -\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} - \frac{\xi_\theta}{r\sin\theta P_\ell^m}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P_\ell^m}{\partial\theta}) + \frac{m^2\xi_\phi}{r\sin^2\theta}$$

$$\Rightarrow \frac{1}{\gamma}\frac{\delta P}{P} + \frac{\xi_r}{\gamma}\frac{d\ln P}{dr} = -\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} + \ell(\ell+1)\frac{\xi_\theta}{r}$$

Momentum

$$-\rho\omega^2\xi_r = -\frac{d\delta P}{dr} - g\delta\rho$$

$$-\rho\omega^2\xi_\perp = -\frac{\delta P}{r}$$

$$\xi_\phi = \xi_\theta = \xi_\perp$$

We have made the approximation that the perturbations do not change the gravitational potential, so $\delta g = 0$; this is known as the Cowling approximation.

These simplify to give two ODEs to integrate:

$$\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} = \frac{g}{c_s^2}\xi_r - \frac{\delta P}{\rho}\left[\frac{1}{c_s^2} - \frac{\ell(\ell+1)}{\omega^2 r^2}\right] \tag{1.38}$$

$$\frac{d\delta P}{dr} = -\frac{g}{c_s^2}\delta P + \rho(\omega^2 - N^2)\xi_r. \tag{1.39}$$

We have used the fact that $\gamma H = \gamma P/\rho g = c_s^2/g$, where c_s^2 is the adiabatic sound speed and $H = -dz/d\ln P$ is the pressure scale height.

The boundary condition at the stellar surface is that the Lagrangian pressure perturbation should vanish there

$$\frac{\Delta P}{P} = 0 \Rightarrow \frac{\delta P}{P} = \frac{\xi_r}{H} \qquad \text{at } r = R. \tag{1.40}$$

At the center $r = 0$, we see that there are terms that diverge, so we need to step away from the origin and begin our integration at a small non-zero value of r. The boundary conditions for $\ell > 0$ (non-radial oscillations) are

$$\frac{d\delta P}{dr} = \frac{\ell}{r}\delta P \qquad \frac{d\xi_r}{dr} = \frac{\ell - 1}{r}\xi_r,$$

or for $\ell = 0$ (radial oscillations)

$$\frac{d\delta P}{dr} = 0 \qquad \frac{d\xi_r}{dr} = \frac{\xi_r}{r}.$$

An equivalent way to write the first $\ell > 0$ boundary condition is

$$\xi_r = \ell \xi_\perp, \tag{1.41}$$

where $\xi_\perp$ can be expressed in terms of δP using the horizontal momentum equation.

Equations (1.38) and (1.39) and the boundary conditions of equations (1.40) and (1.41) define an eigenvalue problem for the mode frequency ω.

There are three "quantum numbers" that label the modes: ℓ, m, and the number of radial nodes n. However note that the azimuthal wavenumber m doesn't enter into the equations, so the frequency of the mode depends on the number of radial nodes n and the angular quantum number ℓ, but not m. This changes if spherically symmetry is broken, eg. a rotating or magnetized star.

Computational Exercise 3: Oscillation modes of the Sun

Overview. The goal of this exercise is to compute the oscillation frequencies and eigenmodes of the Sun. You will be able to identify the p-modes and g-modes and study their frequencies, and their eigenfunctions and how they relate to the internal structure.

Perturbation equations. The equations describing adiabatic perturbations of a spherically-symmetric star are derived in the Appendix of the notes for Week 7 — see equations (10) and (11) of those notes, as well as equations (12) and (13) which give the boundary conditions at the centre and surface of the star.

The perturbation equations consist of two ODEs for the radial displacement ξ_r and the pressure perturbation δP in the mode. They form an eigenvalue problem in that the boundary conditions at the centre and surface of the star can only be satisfied for a discrete set of mode frequencies (the eigenvalues). The idea here is to integrate these equations using background quantities (N^2, g, c_s, ρ) from a model of the Sun, and to find the eigenvalues and corresponding eigenfunctions.

Solar model. You can find a model of a 1 solar mass star here:
http://www.physics.mcgill.ca/~cumming/teaching/643/code/solar_model.dat
This file is a `profile` file from the `1M_pre_ms_to_wd` test suite in the MESA stellar evolution code. I ran this with the `max_age` parameter set to 5 Gyr to make this model, so it should be a reasonable approximation to the Sun.

The first few lines of the file give some parameters of the model, and then you will see the radial profile of various quantities in the star. Here are descriptions of the different columns taken from MESA (the `profile_columns.list` file in MESA):

```
zone       ! numbers start with 1 at the surface
mass       ! m/Msun. mass coordinate of outer boundary of cell.
logR       ! log10(radius/Rsun) at outer boundary of zone
logT       ! log10(temperature) at center of zone
logRho     ! log10(density) at center of zone
logP       ! log10(pressure) at center of zone
x_mass_fraction_H
y_mass_fraction_He
z_mass_fraction_metals
log_g      ! log10 gravitational acceleration (cm sec^2)
pressure_scale_height ! in Rsun units
gamma1     ! dlnP_dlnRho at constant S
csound     ! sound speed
brunt_N2   ! brunt-vaisala frequency squared
```

It is a good idea to make some plots of these different quantities (for example against

r). Do they match what you would expect for the Sun?

Shooting method. One way to solve the equations for the eigenvalues and eigenmodes is to start at the centre of the star, make a guess for the frequency ω, integrate out to the surface and check whether the boundary condition is satisfied there. If not, modify the guess for ω and integrate again, repeating until the outer boundary is satisfied. This method is known as a *shooting method*.

When integrating the ODEs, you will need to evaluate the background quantities such as N^2 at different locations r. To do this, you can interpolate between the values of r given in the model file. A useful routine for this is `scipy.interpolate.interp1d`.

By writing a function that calculates how well the outer boundary condition is satisfied as a function of ω, ie. it calculates $\delta P/P - \xi_r/H$ at $r = R$ for any given value of ω, you can find the modes using a root finder such as `scipy.optimize.brentq`.

Exploring the mode spectrum. Once you have your code working, explore the spectrum of modes. One way to begin is to first fix $\ell = 1$ and try to find the modes with a small number of radial nodes n (places where ξ_r crosses zero — you will see this if you plot the eigenfunction, but you can also get your code to calculate that for you). Typical frequencies for these should be of order $\sim 100\ \mu$Hz. Then work your way up and down in frequency from there.

Questions:

1. What do the eigenfunctions look like for the different modes? Can you separate the g and p modes?
2. How do the frequencies of the g and p modes scale with the number of radial nodes? Does this scaling match what you would expect from the dispersion relation?
3. Do you see trapping of the modes in particular regions of the star? To help with this, you can plot N^2 and $\ell(\ell+1)c_s^2/r^2$ on the same plot as your eigenfunction. A guide to where the modes should propagate is where k^2 from the analytic dispersion relation is positive.
4. We mentioned that g modes are incompressible modes. Check this using your solutions.

There is a nice set of lecture notes on stellar oscillations by Jørgen Christensen-Dalsgaard, which you can find here: `http://astro.phys.au.dk/~jcd/oscilnotes/`. Chapter 5 shows some results for the Sun that you can compare against (in particular Figs. 5.2, 5.6, 5.8, and 5.10).