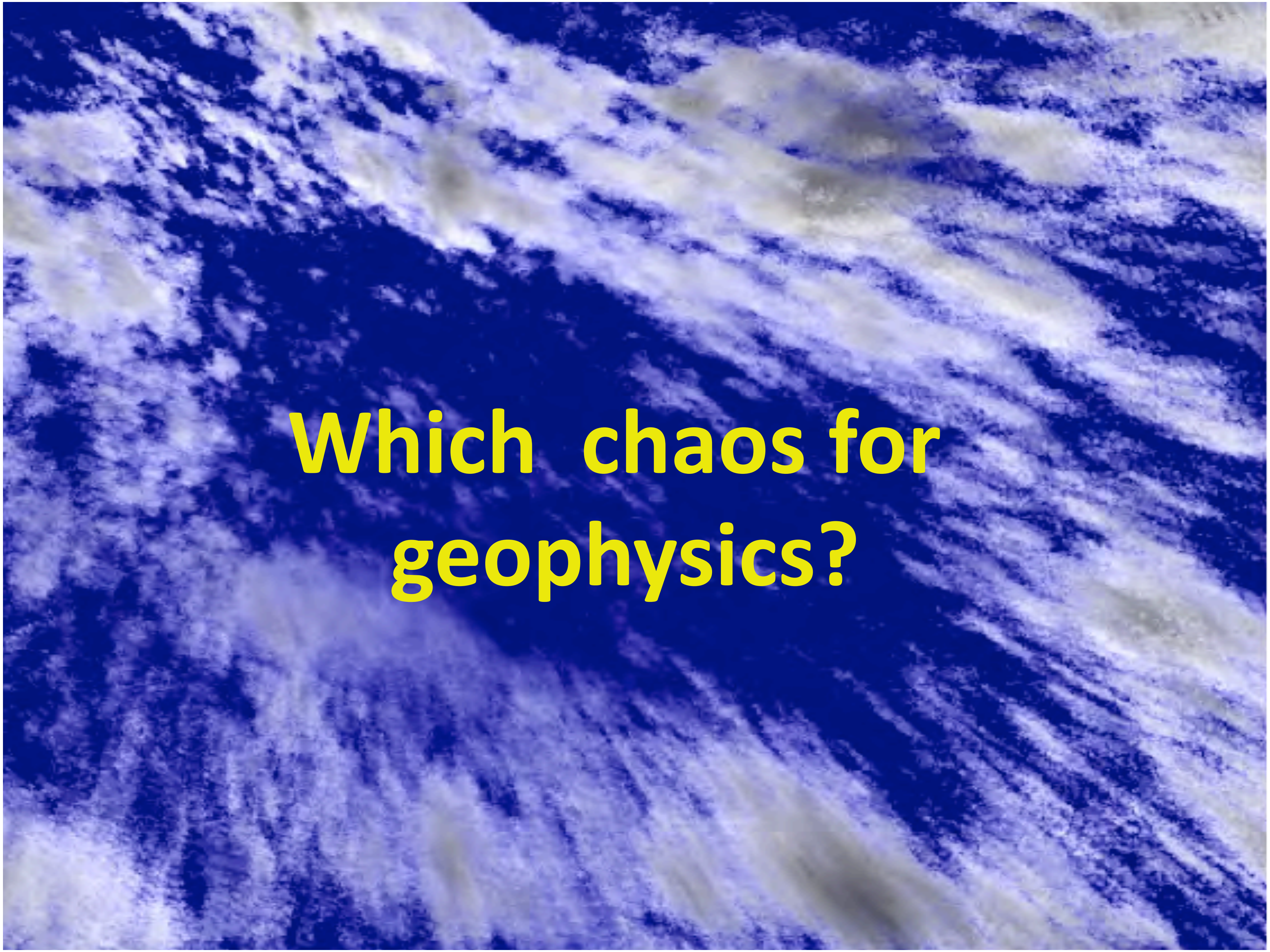
PHYS 616 Multifractals and Turbulence

Lecture 1:

Introduction:

Our multifractal world


Which chaos for
geophysics?

Deterministic Chaos?

Low Dimensional Nonlinear Dynamics I

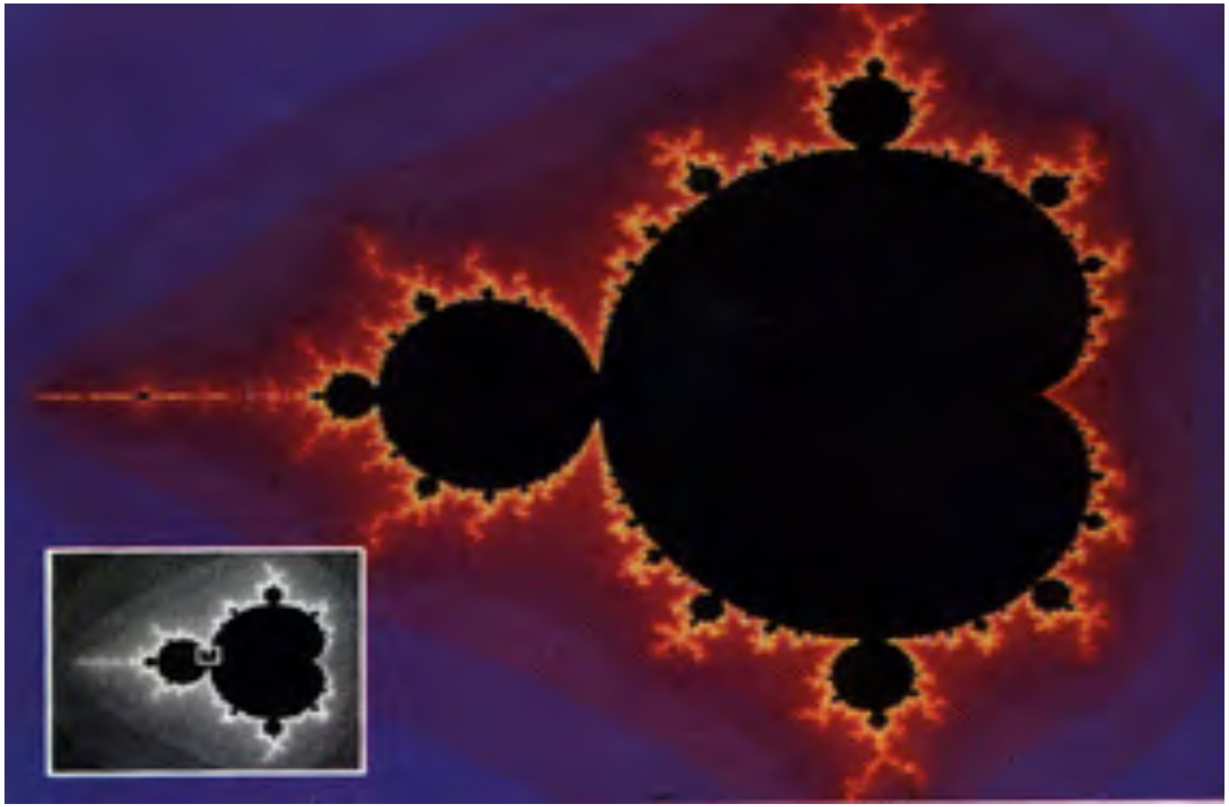
Nonlinear Mappings

Discrete time (=n) evolution of a few variables ($\underline{x}$):

Z, C are complex numbers

$$Z_{n+1} = Z_n^2 + C$$

The Mandelbrot set



Low Dimensional Nonlinear Dynamics II

Flows

Continuous time (=t) evolution of a few degrees of freedom ($\underline{X}$):

$$\frac{d\underline{X}}{dt} = \underline{F}(\underline{X})$$

Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

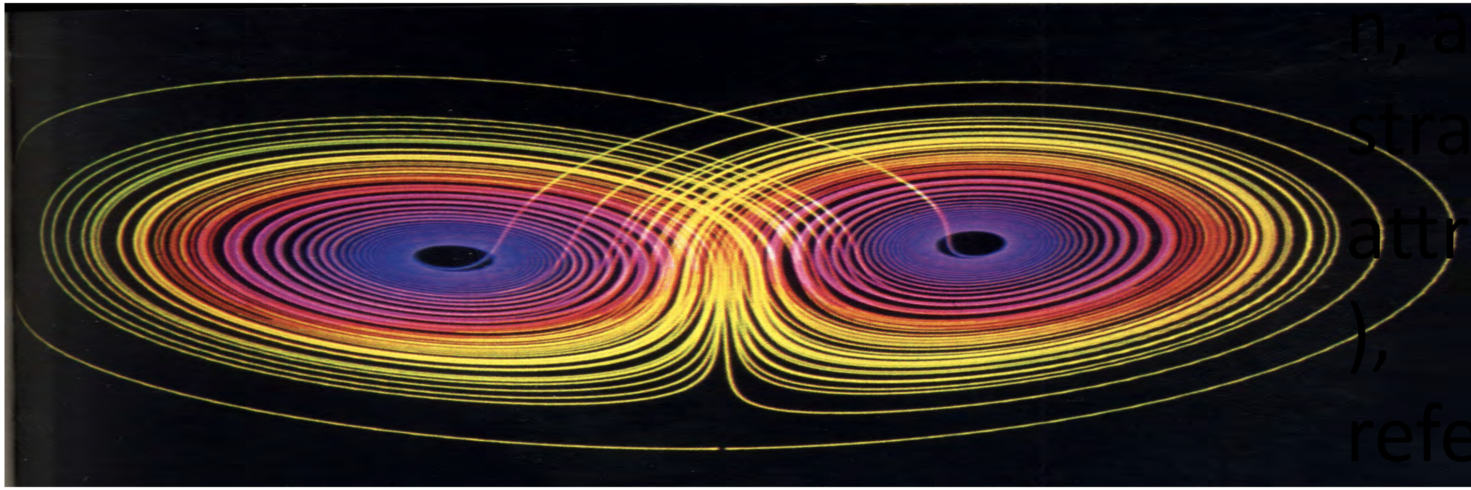
$$\frac{dy}{dt} = rx - y - xz$$

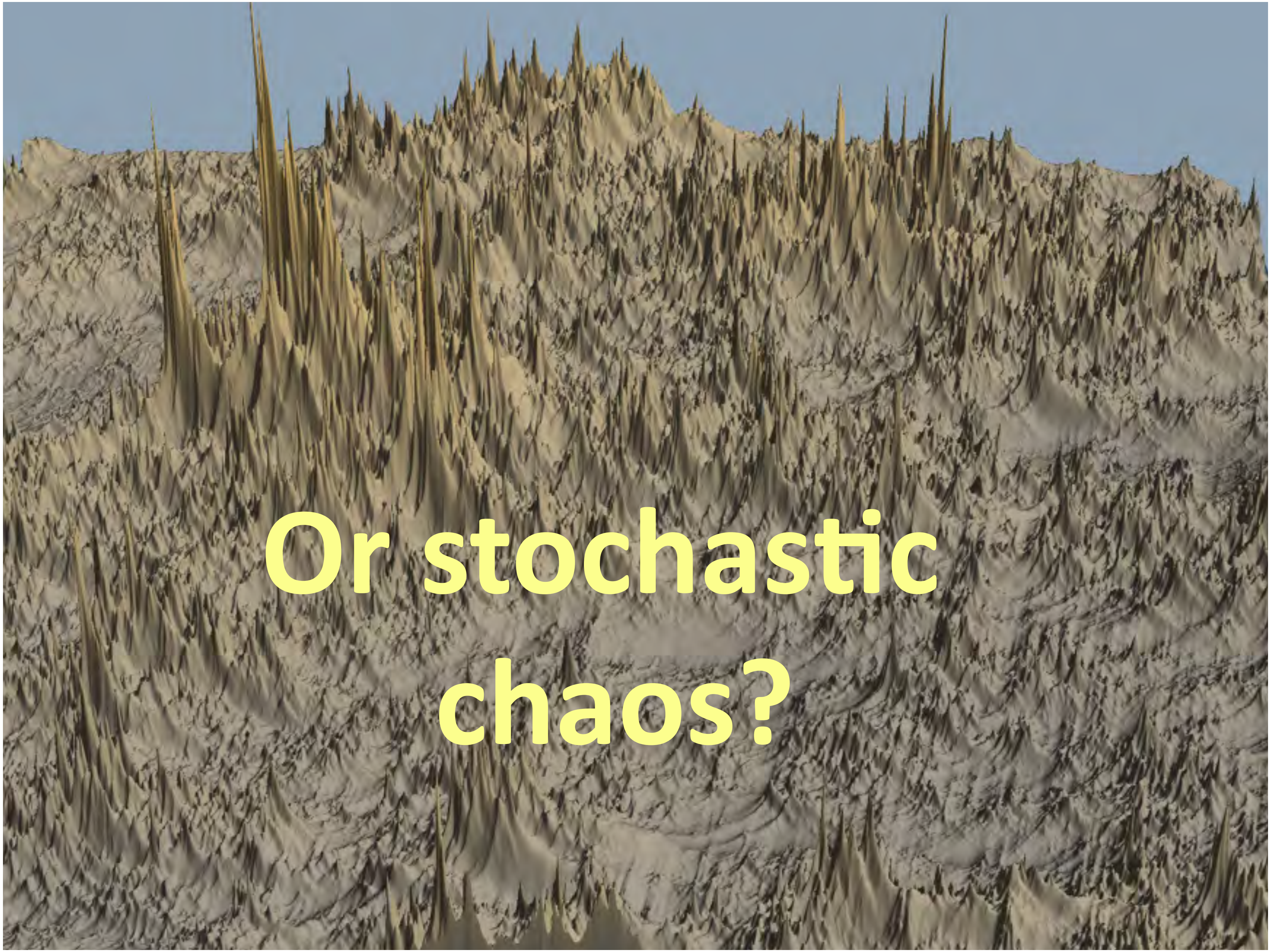
$$\frac{dz}{dt} = xy - bz$$

where r, b, σ are positive constants.

Few degrees of freedom… few applications

Give some details about chaos (definition, also strange attractors), references




Or stochastic
chaos?

High Dimensional Nonlinear Dynamics

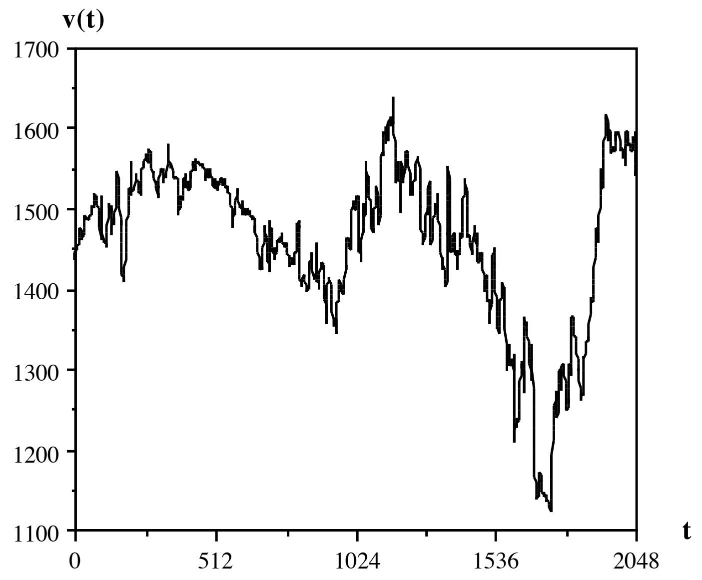
Nonlinear PDE 's

Fields/spatial structures evolving in time
Example: Navier-Stokes Equations:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\,\underline{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2\,\underline{v} + \underline{f}$$

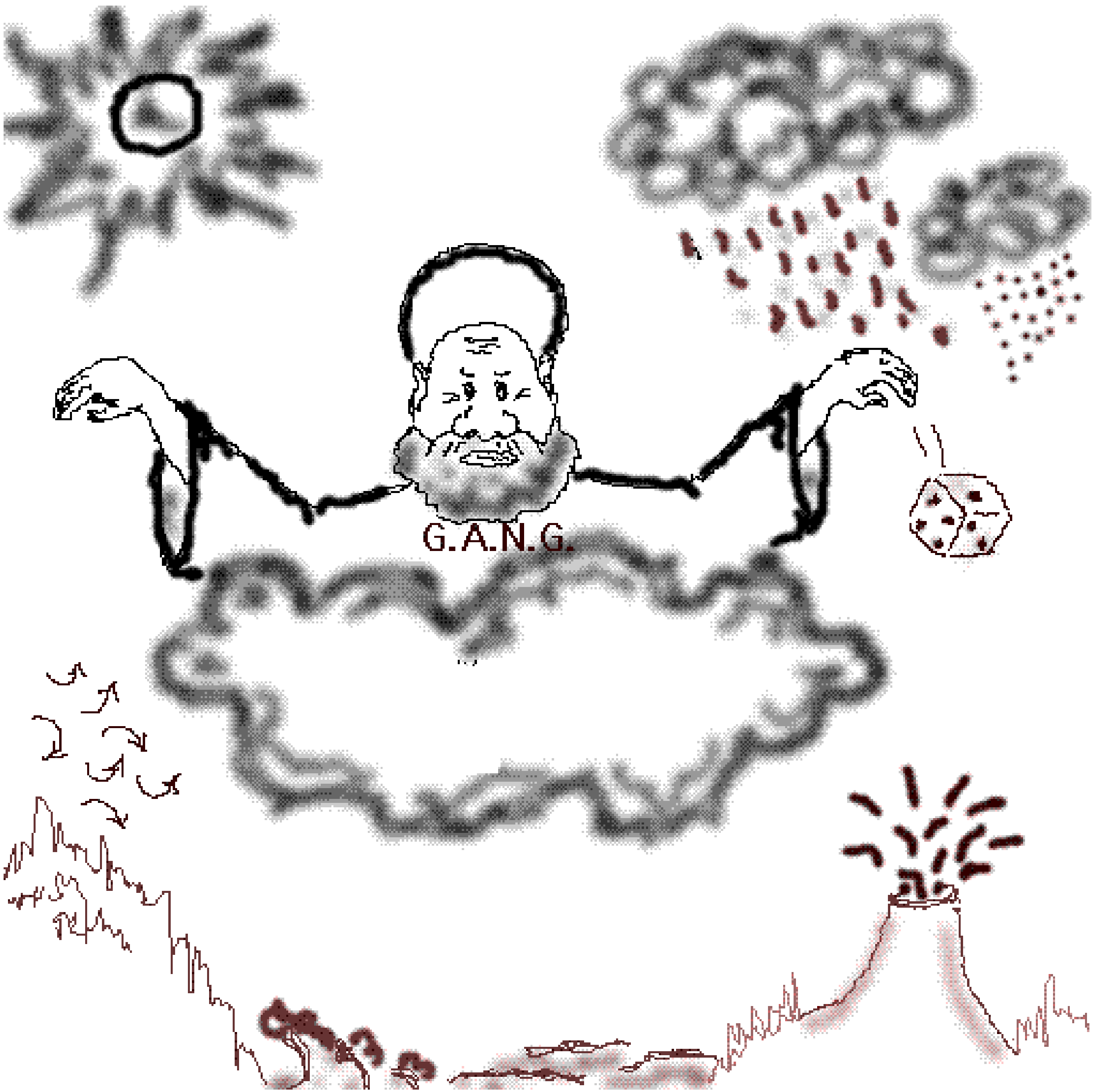
$$\nabla \cdot \underline{v} = 0$$

where $\underline{v}$= velocity, t = time, p = pressure, ρ = density, ν = viscosity, $\underline{f}$= body forces (e.g. stirring, gravity).



1 second of wind data

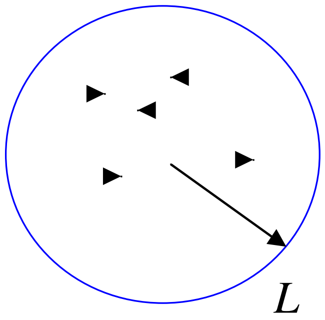
How does He play Dice?



Scale Invariance

The simplest scale invariant system:
Isotropic Scale Invariance and fractal sets

Fractal Dimension:



$$n(L) \propto L^{D}$$

Number of points

$$\rho(L) = \frac{n(L)}{L^{d}} =\propto L^{D-d} = L^{-C}$$

Density of points

d=dimension of space
D= fractal dimension of set
C=d-D= fractal codimension

Scale invariance:

$$n(\lambda L) = \lambda^{D} n(L)$$

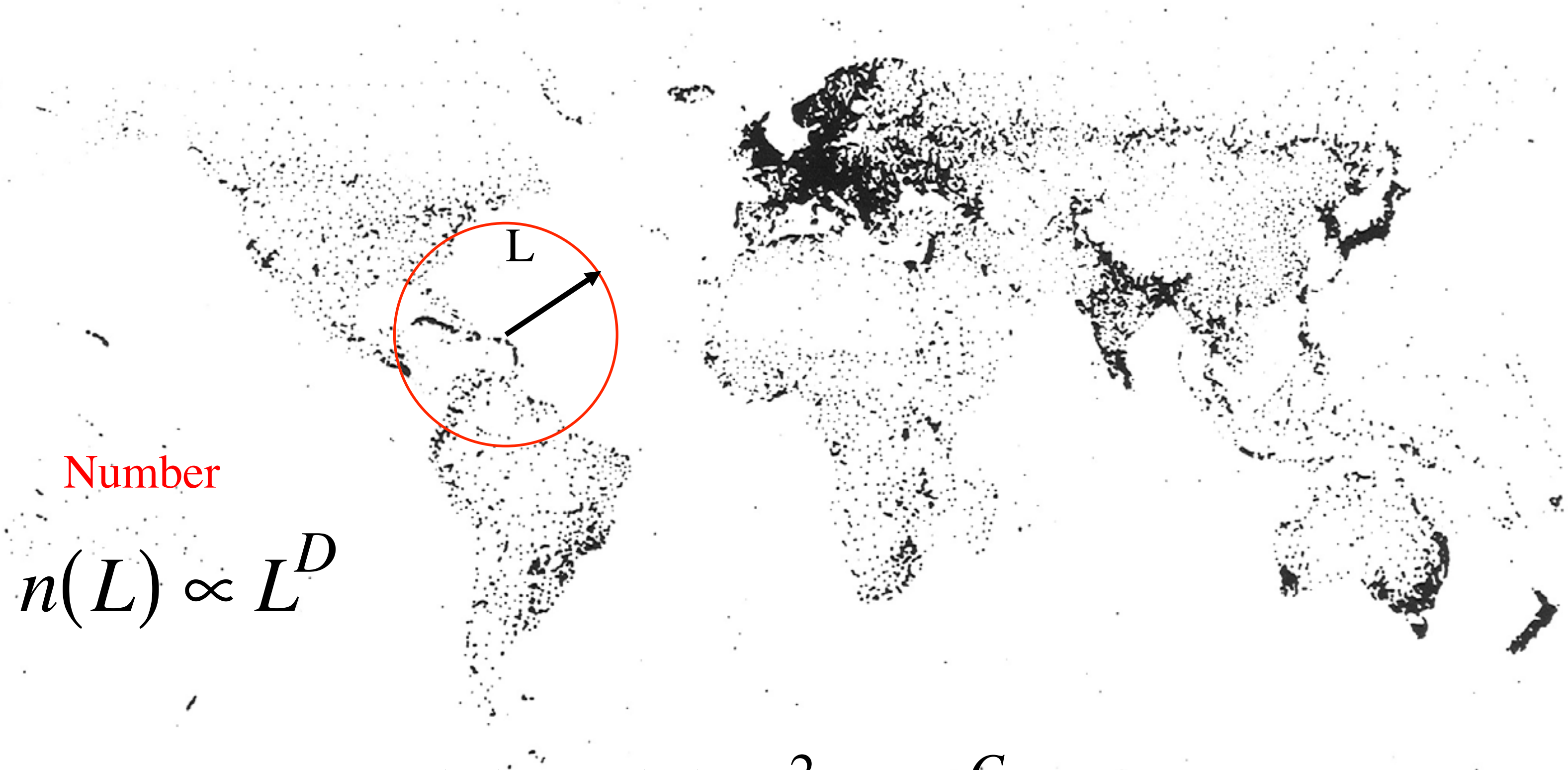
D=scale invariant

Same form after zoom by factor λ.

Meteorological measuring network

Fractal set: each point is a station

9962 stations (WMO)



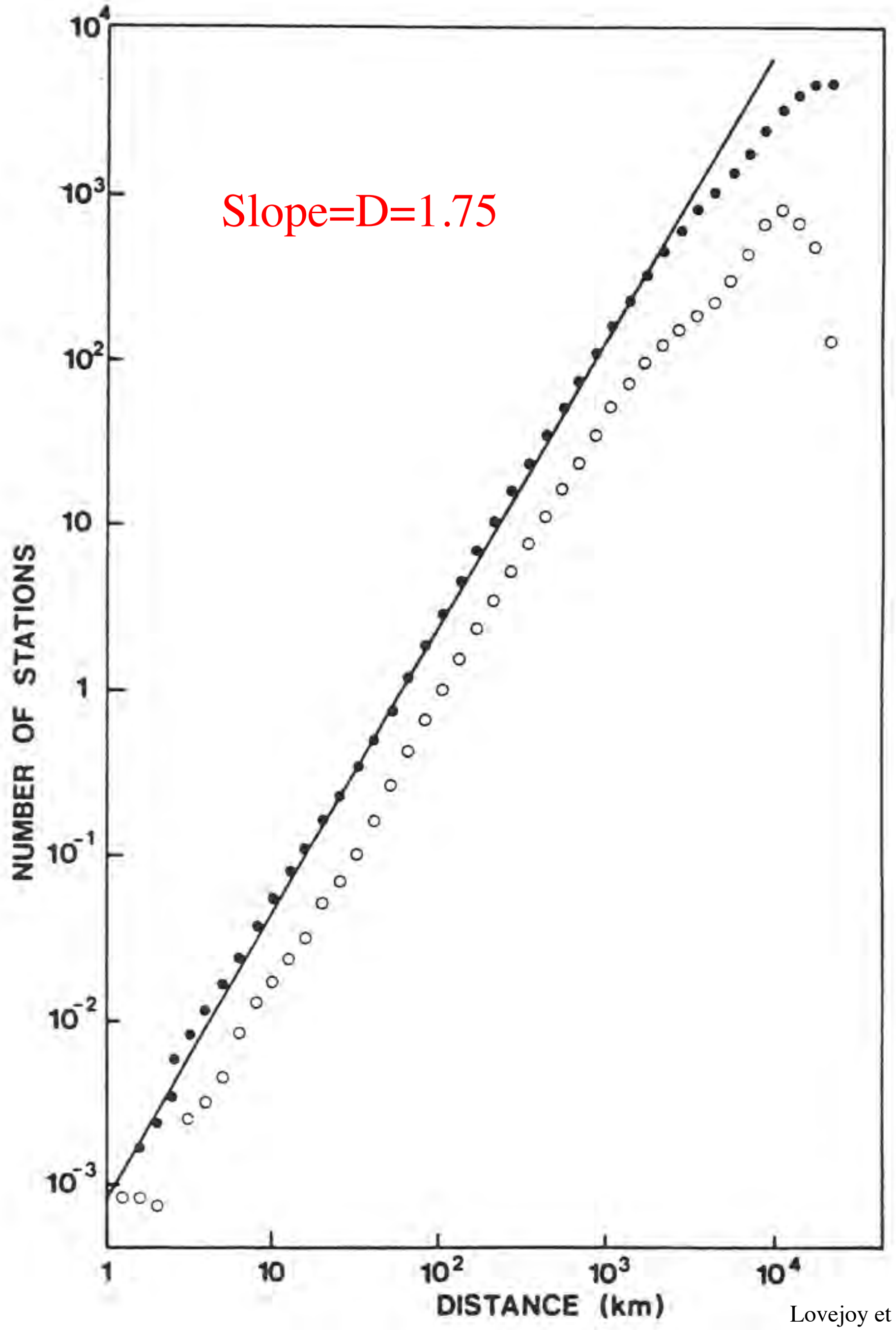
Number

$$n(L) \propto L^D$$

Density

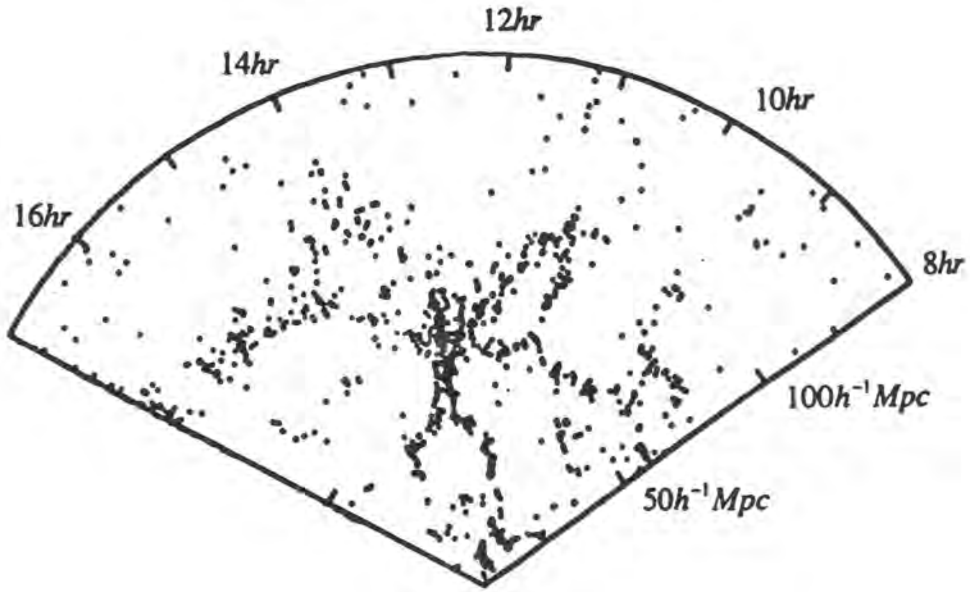
$$\rho(L) = n(L)L^{-2} \propto L^{-C}; \quad C = d - D; \quad d = 2$$

The fractal dimension of the network= 1.75



Lovejoy et al 1986

Slice of the Universe

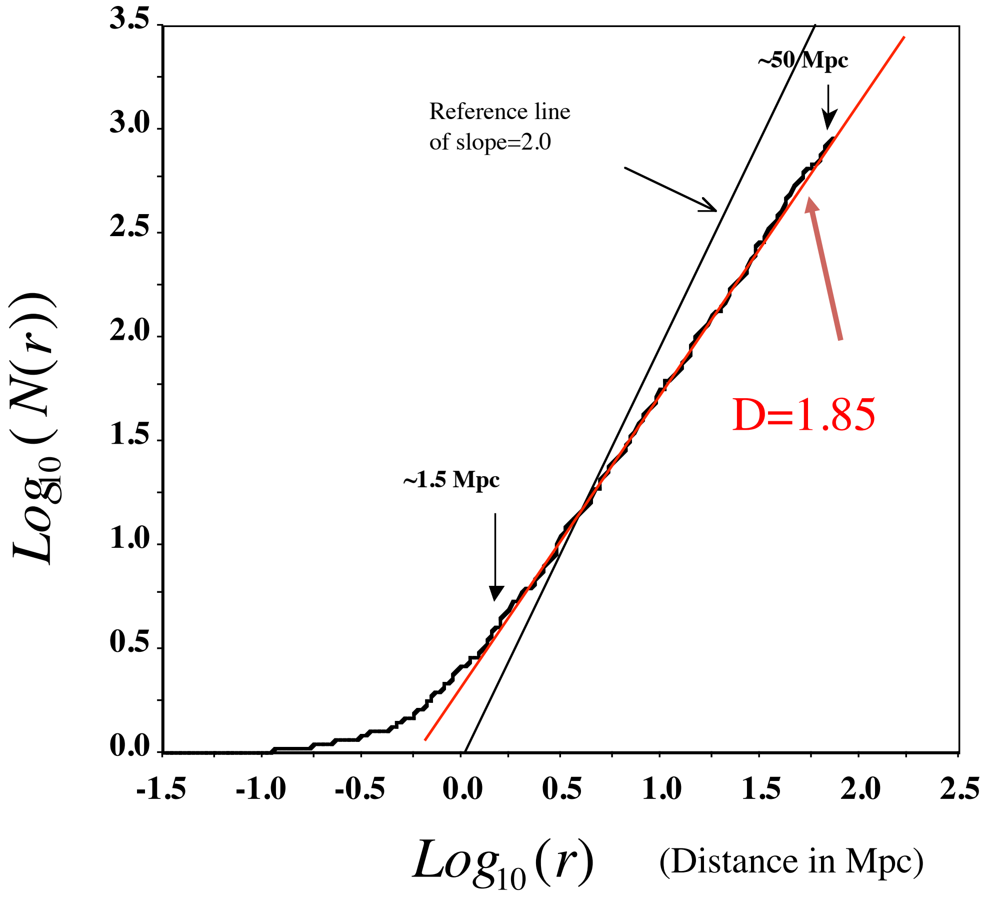


cfA2 catalogue

de Lapparent *et al* 1986

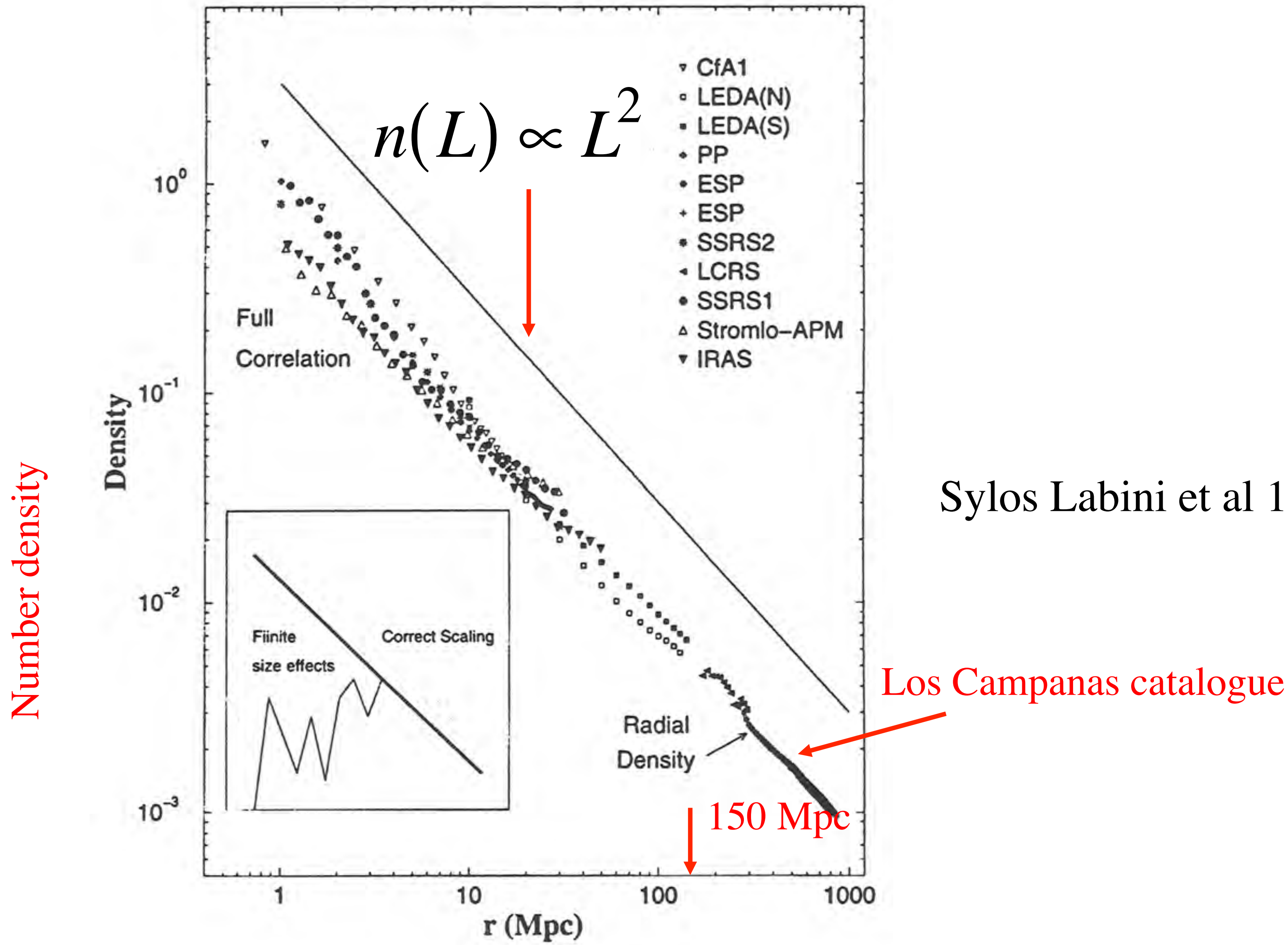
1068 galaxies with apparent magnitude m<15.5 and located in the region 8hr<α<17hr and 26.5”<δ<32.5. The sample’s depth is 150 Mpc (units of 100 km sec Mpc).

Fractal analysis of galaxies as points

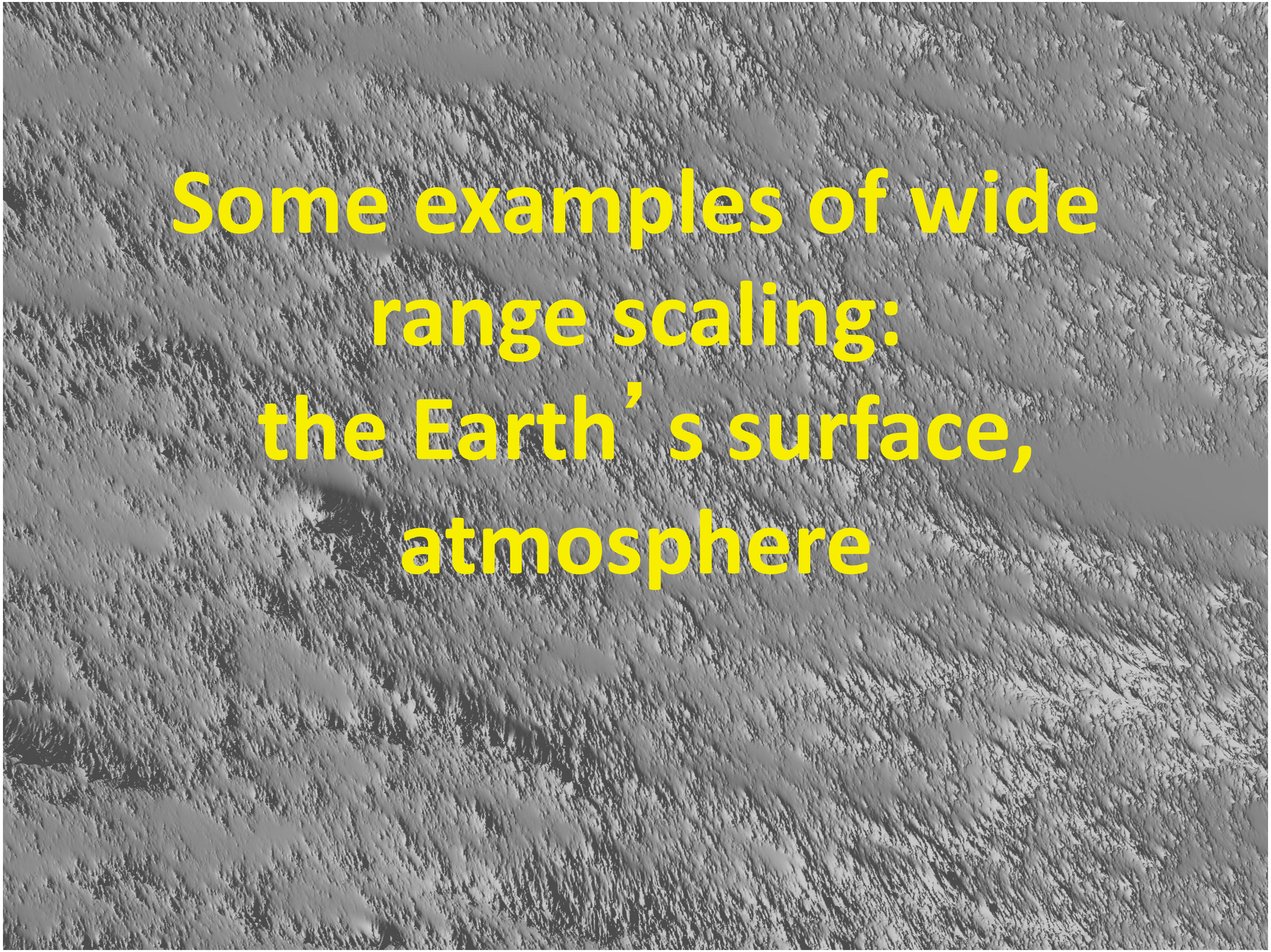


Scaling analysis of a "Slice of the Universe". The linear scaling range extends from a few up to about 50 Mpc, the size of the largest circle embedded in the sample. From Garrido, Lovejoy and Schertzer 1996.

Is the large scale structure of the universe scaling to 1000Mpc?



Sylos Labini et al 1998



Some examples of wide range scaling: the Earth’s surface, atmosphere

Multiscaling of the Navier-Stokes equations

Zoom factor λ $\quad \vec{x} \to \dfrac{\vec{x}}{\lambda}$

Rescaling of the velocity

H is an arbitrary scaling exponent $\quad \vec{v} \to \dfrac{\vec{v}}{\lambda^H}$

$$t \to \frac{t}{\lambda^{1-H}}$$

Rescaling of time, viscosity, forcing follow from dimensional considerations

$$\nu \to \frac{\nu}{\lambda^{1+H}}$$

$$\vec{f} \to \frac{\vec{f}}{\lambda^{2H-1}}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\vec{v} + \vec{f}$$

$$\nabla\cdot\vec{v} = 0$$

(constraint used to eliminate p) where $\vec{v}$ = velocity , t = time , p = pressure , ρ = density , ν = viscosity , $\vec{f}$ = body forces (stirring, gravity)

Kolmogorov's Law:

Considering $\varepsilon = -\dfrac{\partial v^2}{\partial t}$ energy flux to smaller scales to be invariant, we obtain $H = 1/3$, hence for mean shear

$$\Delta\vec{v} \approx \varepsilon^{1/3}\Delta x^{1/3}; \qquad E(k) = k^{-5/3}$$

This already leads to singularities:

$$\frac{\partial \vec{v}}{\partial x} = \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} \approx \Delta x^{-2/3} \to \infty$$

The Emergence of physical laws

Quantum mechanics → Classical Mechanics

Large scales (usually)

stochastic → deterministic

Statistical mechanics → Continuum mechanics, thermodynamics

Large numbers of particles

stochastic → deterministic

Low level (fundamental) — high level

The emergence of atmospheric dynamics

(Classical)

Continuum mechanics

deterministic

Low level (fundamental)

Large Re

Laws of turbulence

Classical:

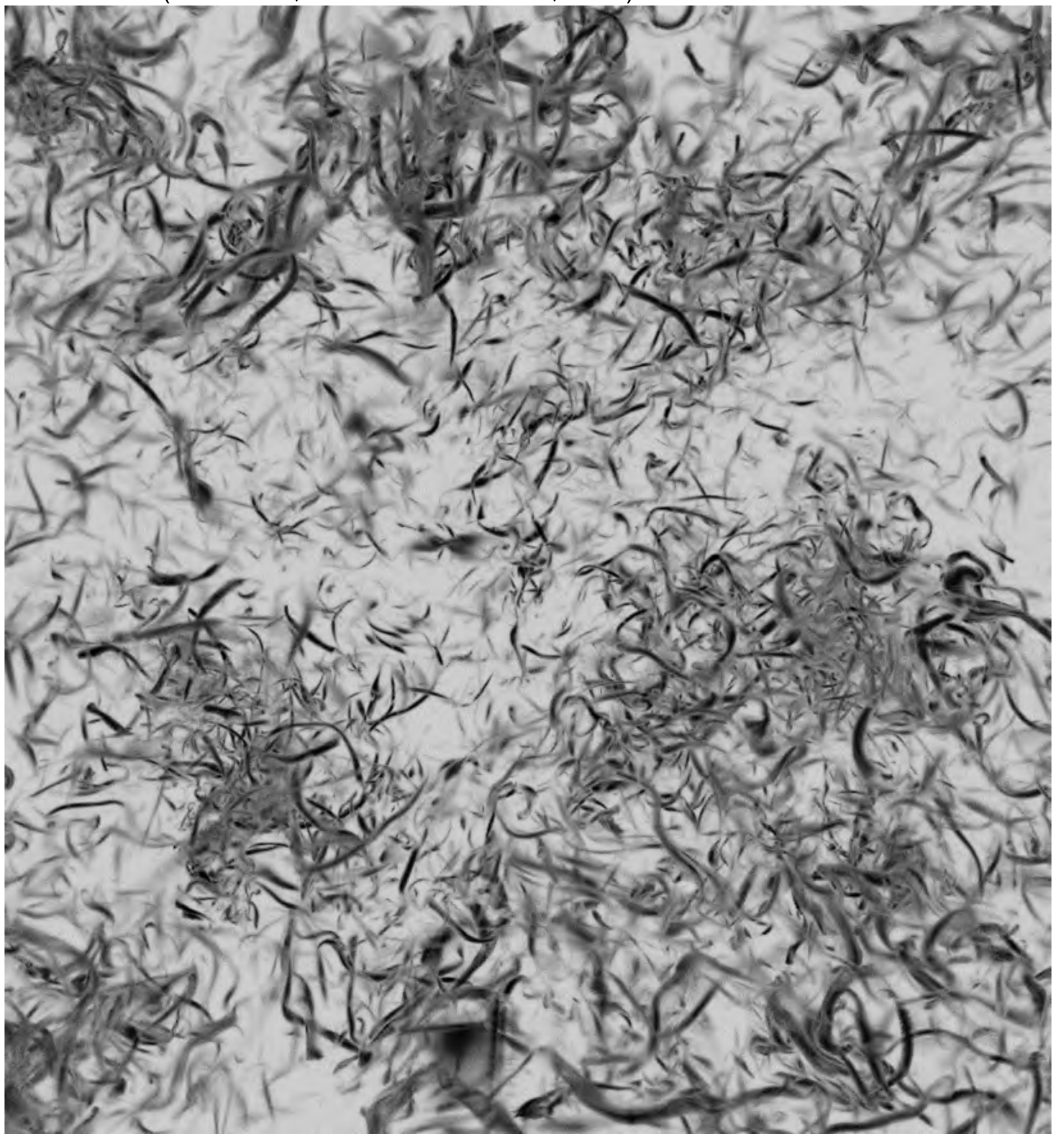
Richardson, Kolmogorov, Corrsin, Obukhov, Bolgiano

High level

stochastic

Vortices in strongly turbulent fluid

(M. Wiczek, numerical simulation, 2010)



$$\Delta v\left(\underline{\Delta r}\right) = \varphi\left|\underline{\Delta r}\right|^{H}$$

e.g. Kolmogorov $\varphi=\varepsilon^{1/3}$, $H=1/3$

a) $\left|\underline{\Delta r}\right| <\approx 100m$ b) isotropic

c) $\varphi \approx$constant, quasi Gaussian

Emergent laws and Complexity

The relative simplicity of the high level laws is due to a *reduction of the complexity* of the system

If all existing emergent laws are used to describe a system, the remaining complexity is *irreducible*

Emergence of Atmospheric laws

$$\text{Fluctuations} \approx (\text{turbulent flux}) \times (\text{scale})^{H}$$

- Fluctuations ← Differences, tendencies, wavelet coefficients
- (turbulent flux) ← Cascading Turbulent flux
- (scale) ← Anisotropic Space-time Scale function
- H ← Fluctuation /conservation exponent

Fourier domain:

$$\left(\frac{Variance_{observables}}{wavenumber}\right) = \left(\frac{Variance_{flux}}{wavenumber}\right)(wavenumber)^{-2H}$$

$$= (wavenumber)^{-\beta}$$

Space: $E(k) \approx k^{-\beta}$

Time: $E(\omega) \approx \omega^{-\beta}$

The weather regime: The emergent laws hold up to planetary scales (Horizontal scaling)

$$E(k) = k^{-\beta}$$

Energy Spectra

Scaling geometric sets of points = fractals
Scaling fields=multifractals

$$E(k) \propto k^{-\beta}$$

k=2π/L= wavenumber, β=spectral exponent

Scale invariance

$$E(\lambda^{-1}k) = \lambda^{\beta} E(k)$$

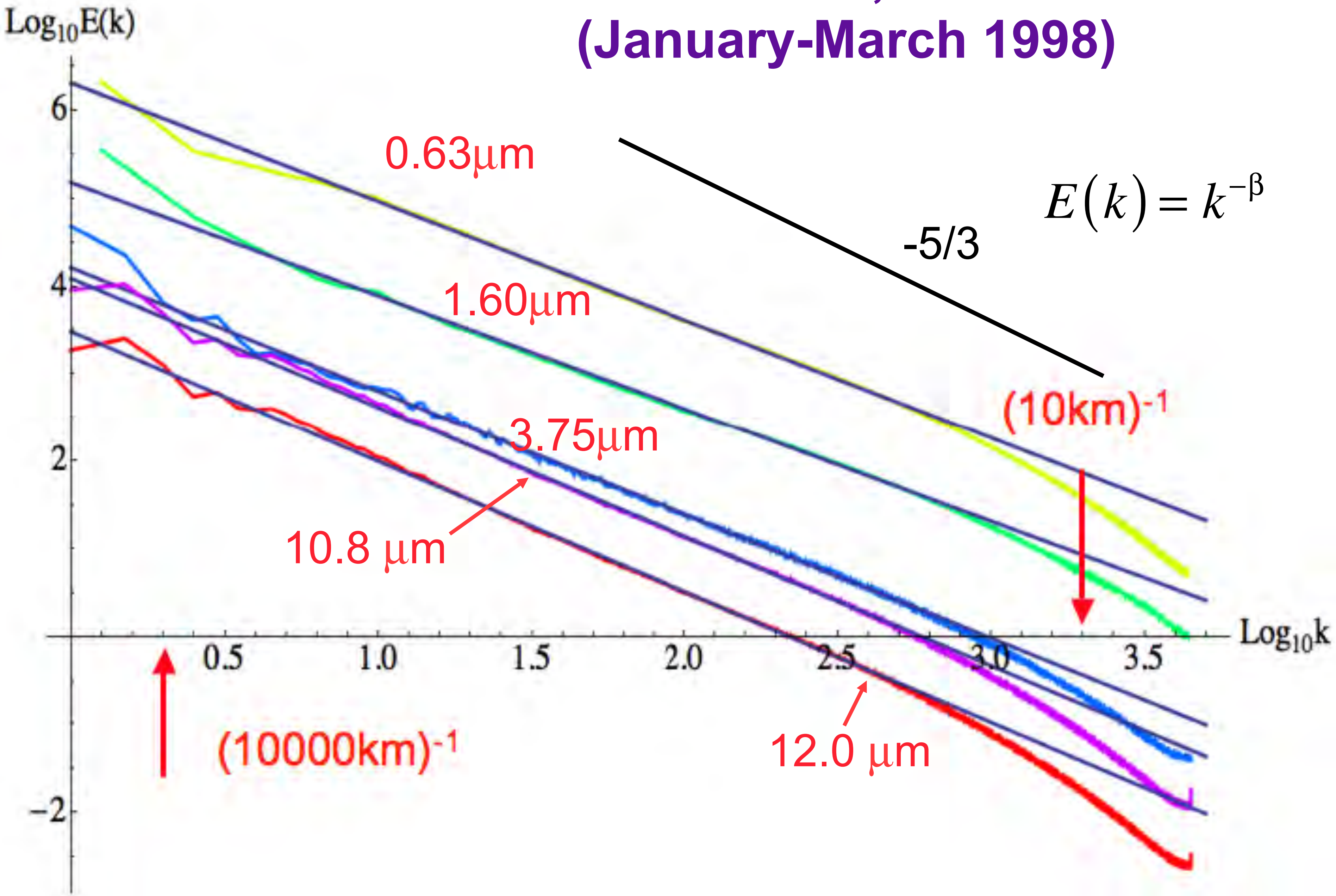
β Invariant under zoom by factor λ in real space.

Examples in the spatial domain

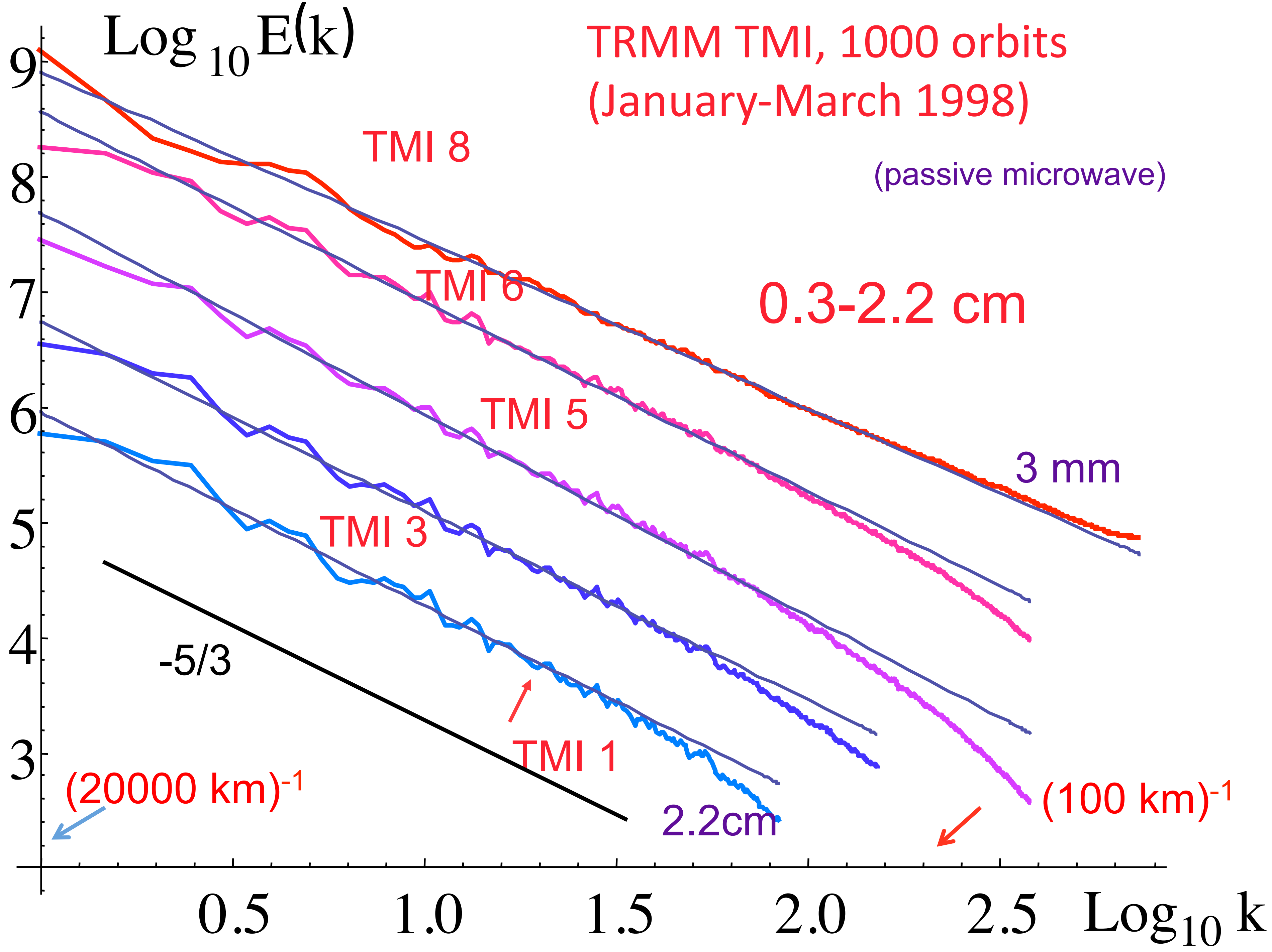
The Atmosphere

1) horizontal

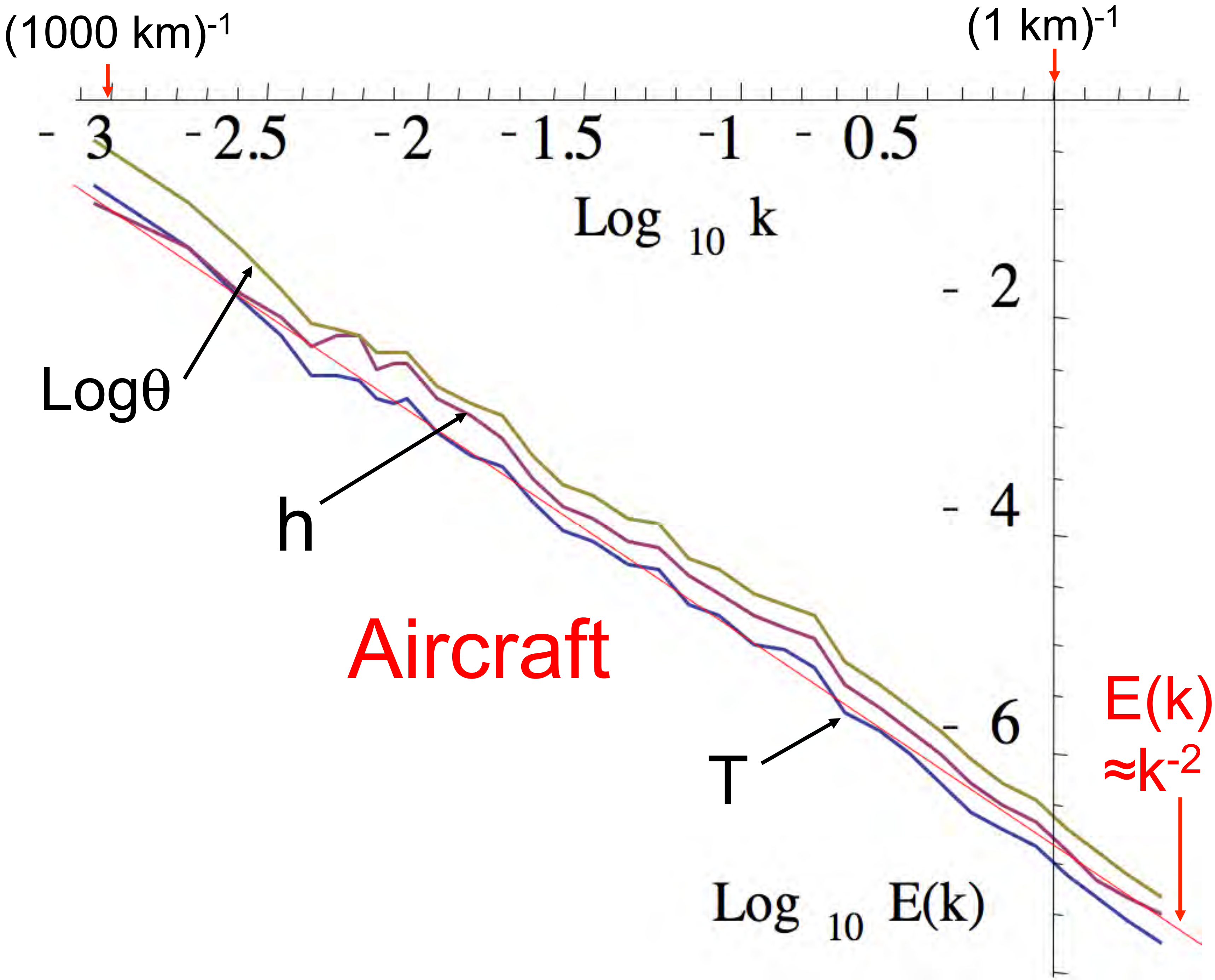
TRMM VIRS, 1000 orbits (January-March 1998)



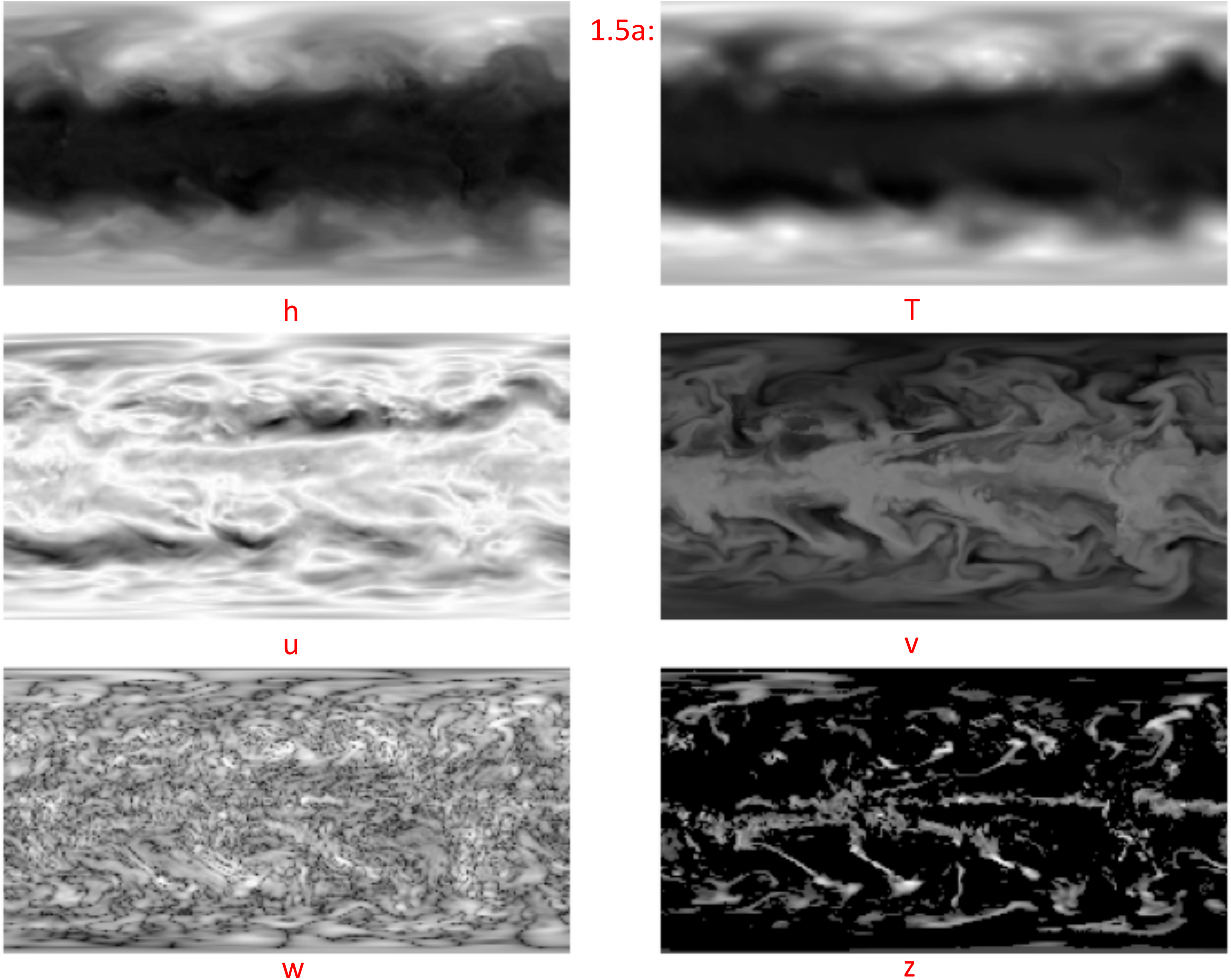
Visible, near infra red, thermal infra red

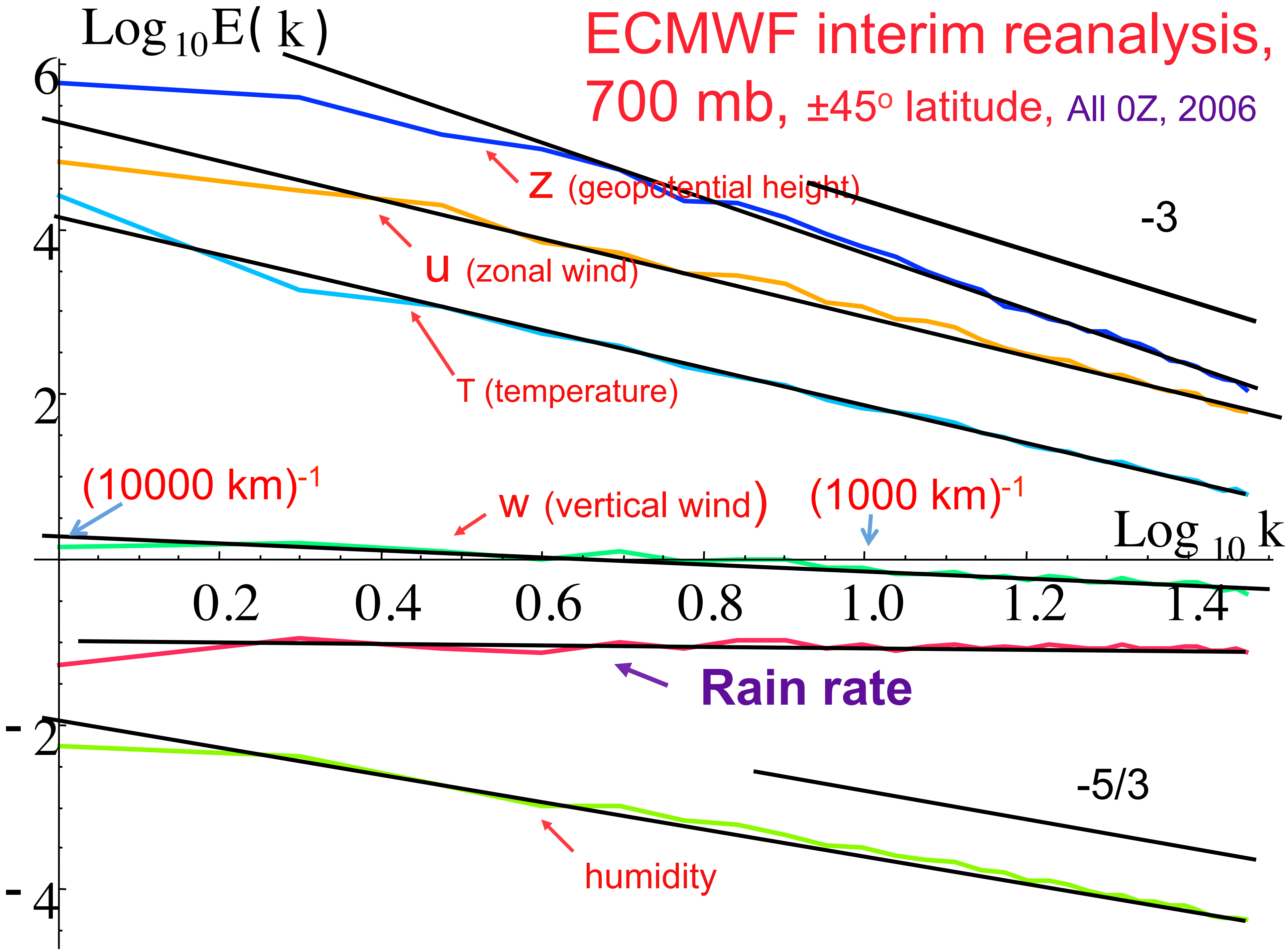


Log 10 E(k)
TRMM TMI, 1000 orbits
(January-March 1998)
(passive microwave)
TMI 8
TMI 6
0.3-2.2 cm
TMI 5
3 mm
TMI 3
-5/3
TMI 1
(20000 km)-1
2.2cm
(100 km)-1
9
8
7
6
5
4
3
0.5
1.0
1.5
2.0
2.5
Log10 k

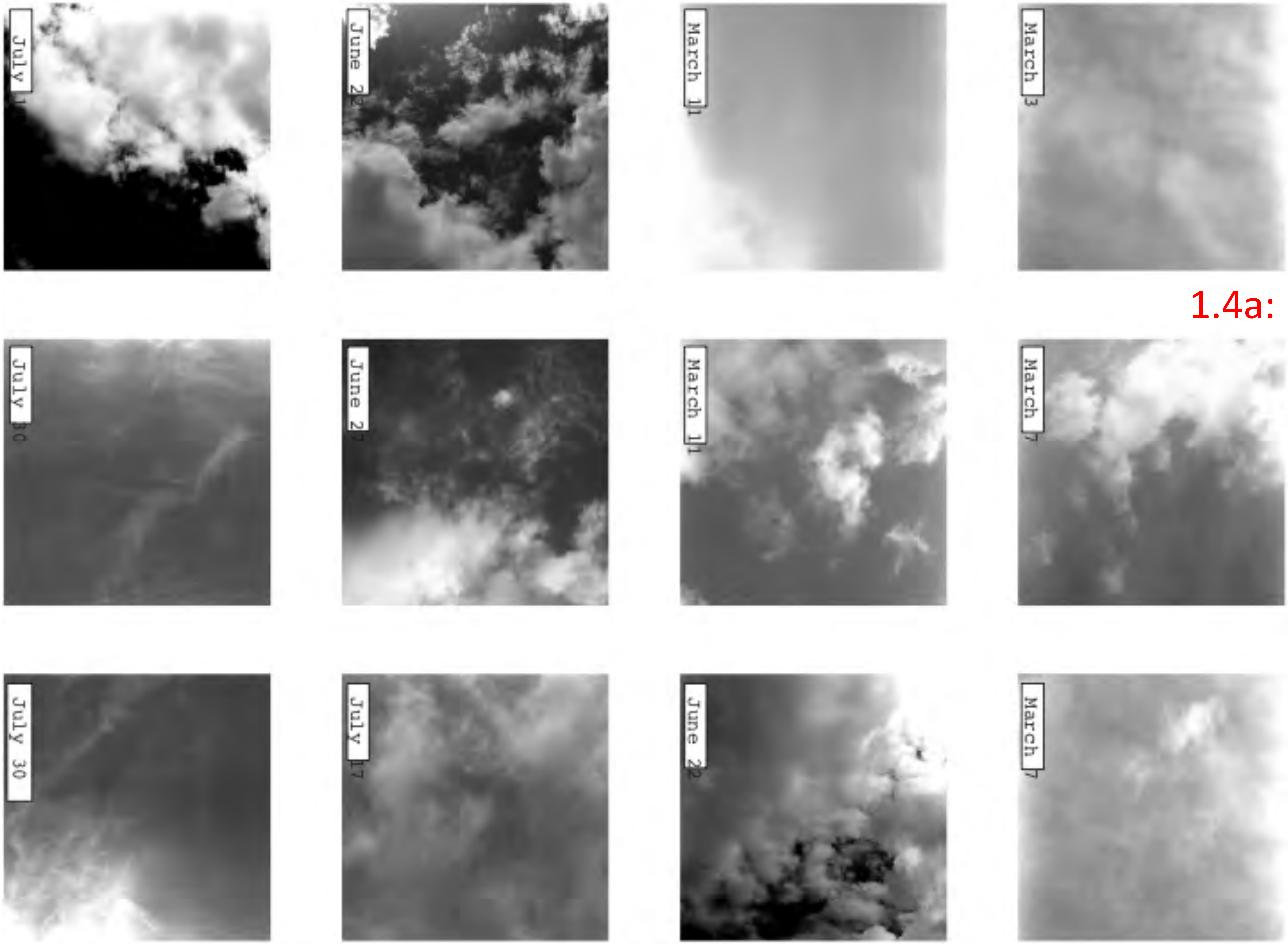

(1000 km)^-1
(1 km)^-1
-3
-2.5
-2
-1.5
-1
-0.5
Log 10 k
-2
-4
-6
Logθ
h
T
Aircraft
E(k) ≈k^-2
Log 10 E(k)

1.5a:



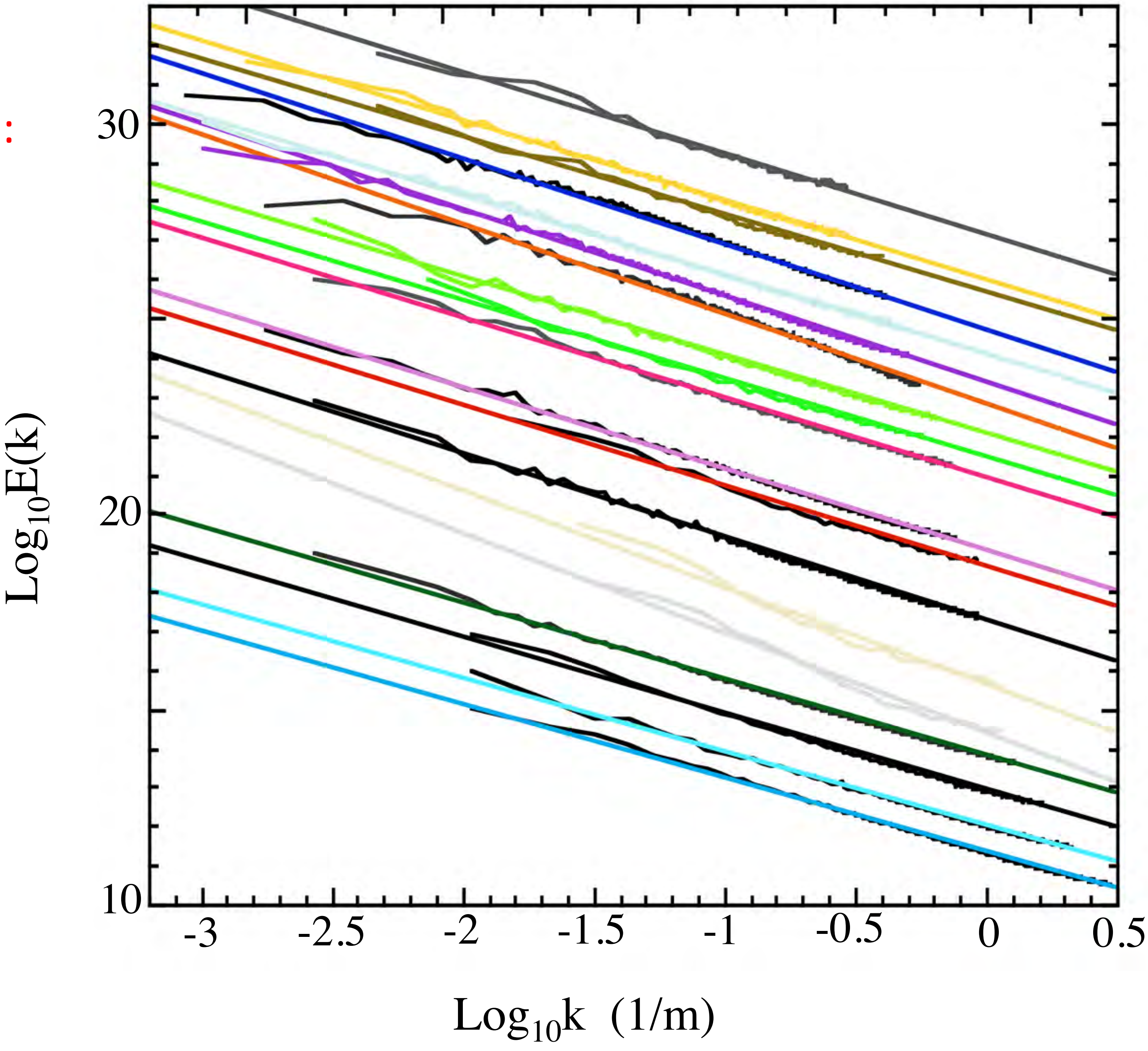

ECMWF interim reanalysis,
700 mb, ±45° latitude, All 0Z, 2006
Log 10 E(k)
6
4
2
- 2
- 4
Z (geopotential height)
-3
u (zonal wind)
T (temperature)
(10000 km)-1
w (vertical wind)
(1000 km)-1
Log 10 k
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Rain rate
-5/3
humidity

1.4a:



July 1
June 22
March 11
March 3
July 30
June 27
March 11
March 7
July 30
July 17
June 22
March 7

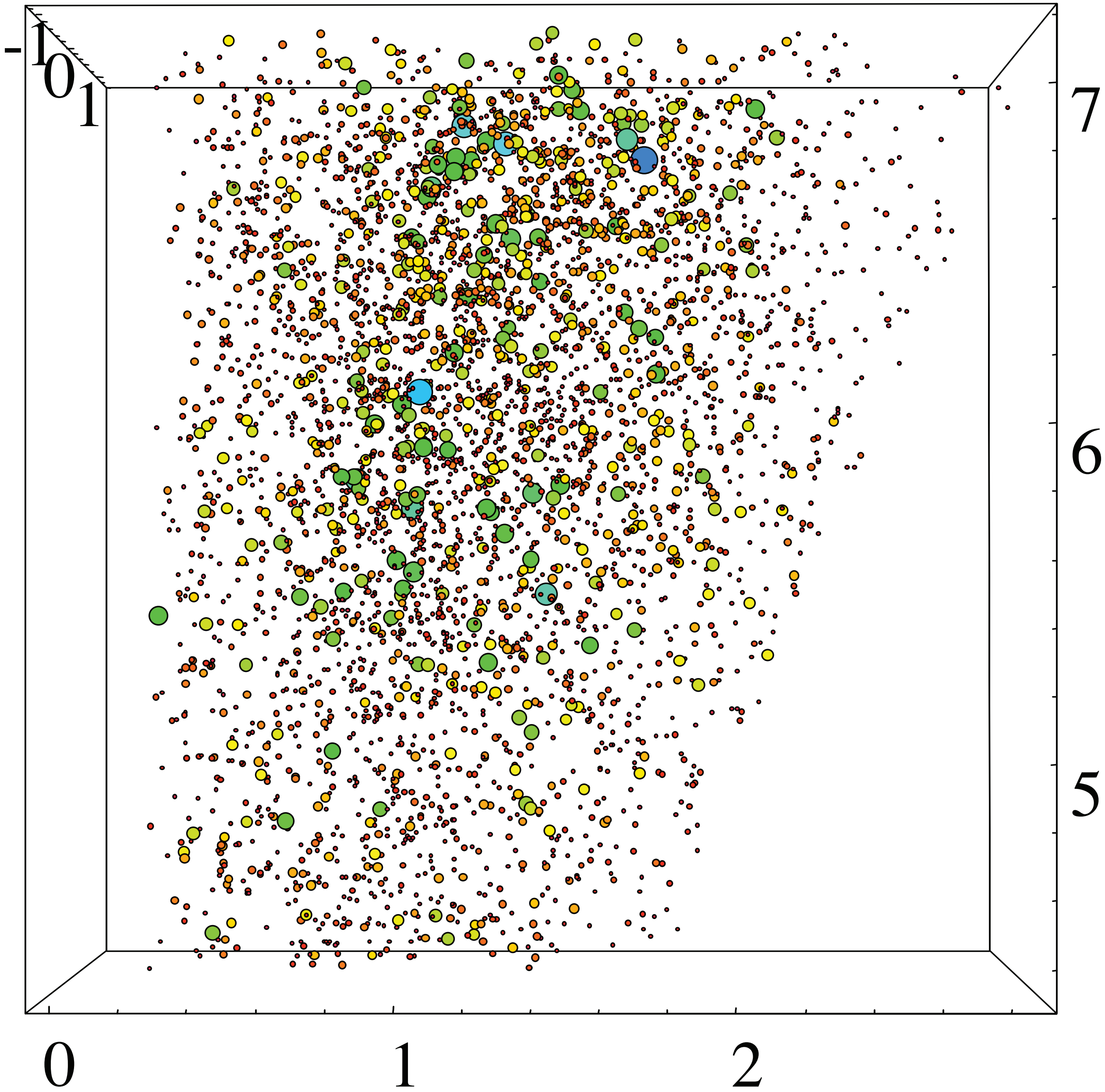
1.4b:



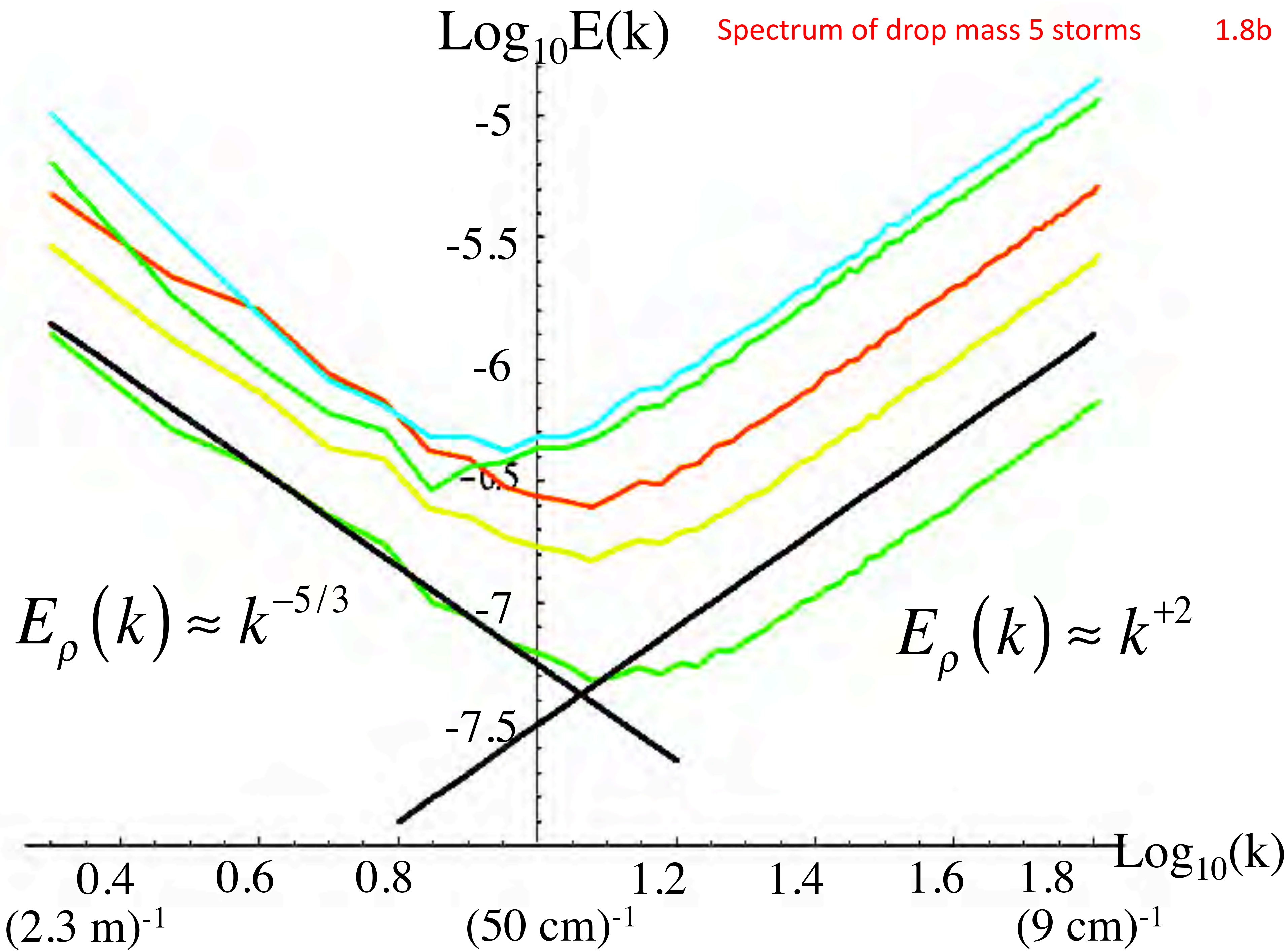
The atmosphere:
2) The inner scale… in rain

1.8a

Stereophotography of rain drops (the 10% largest), roof of the physics building

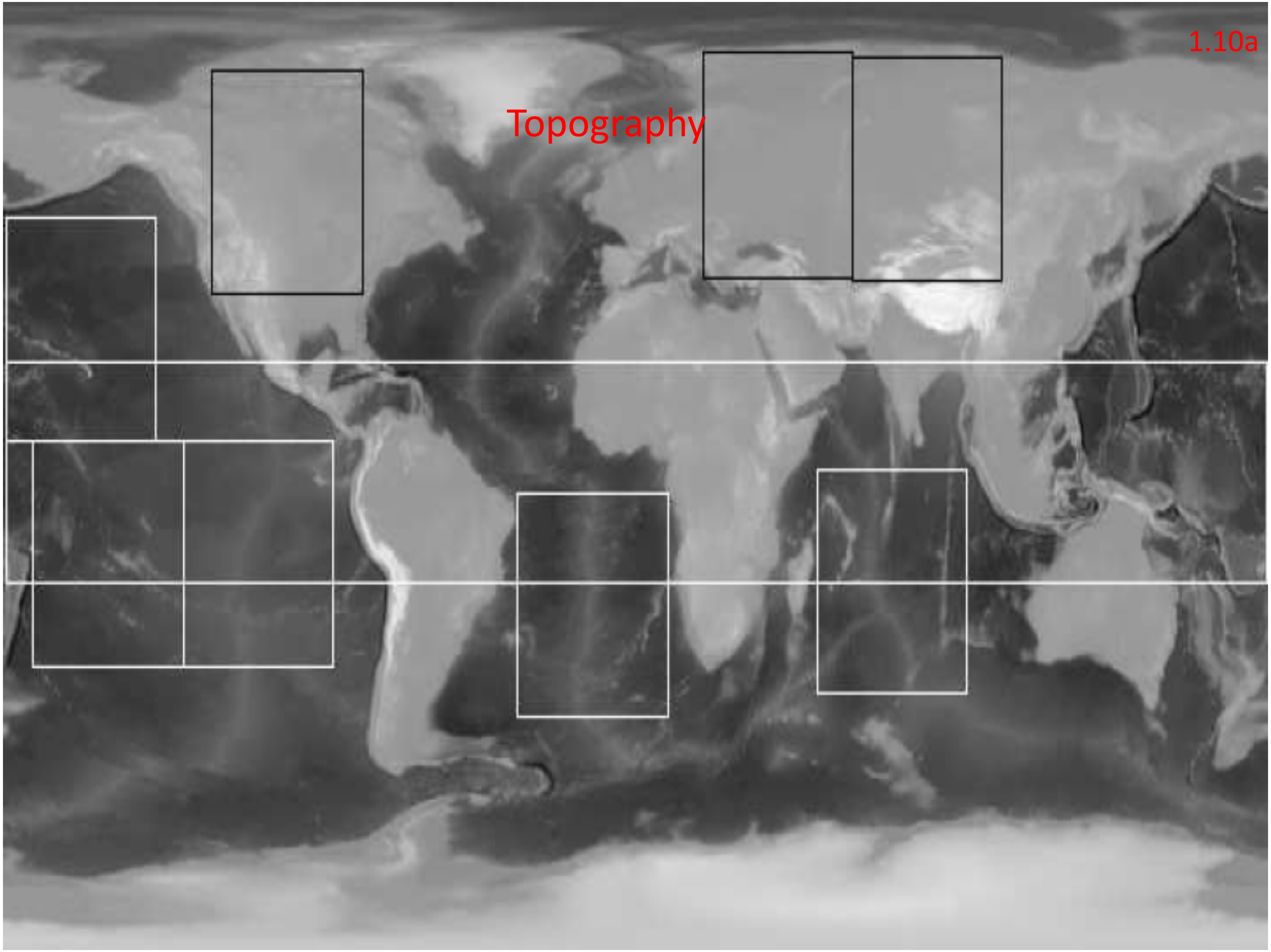


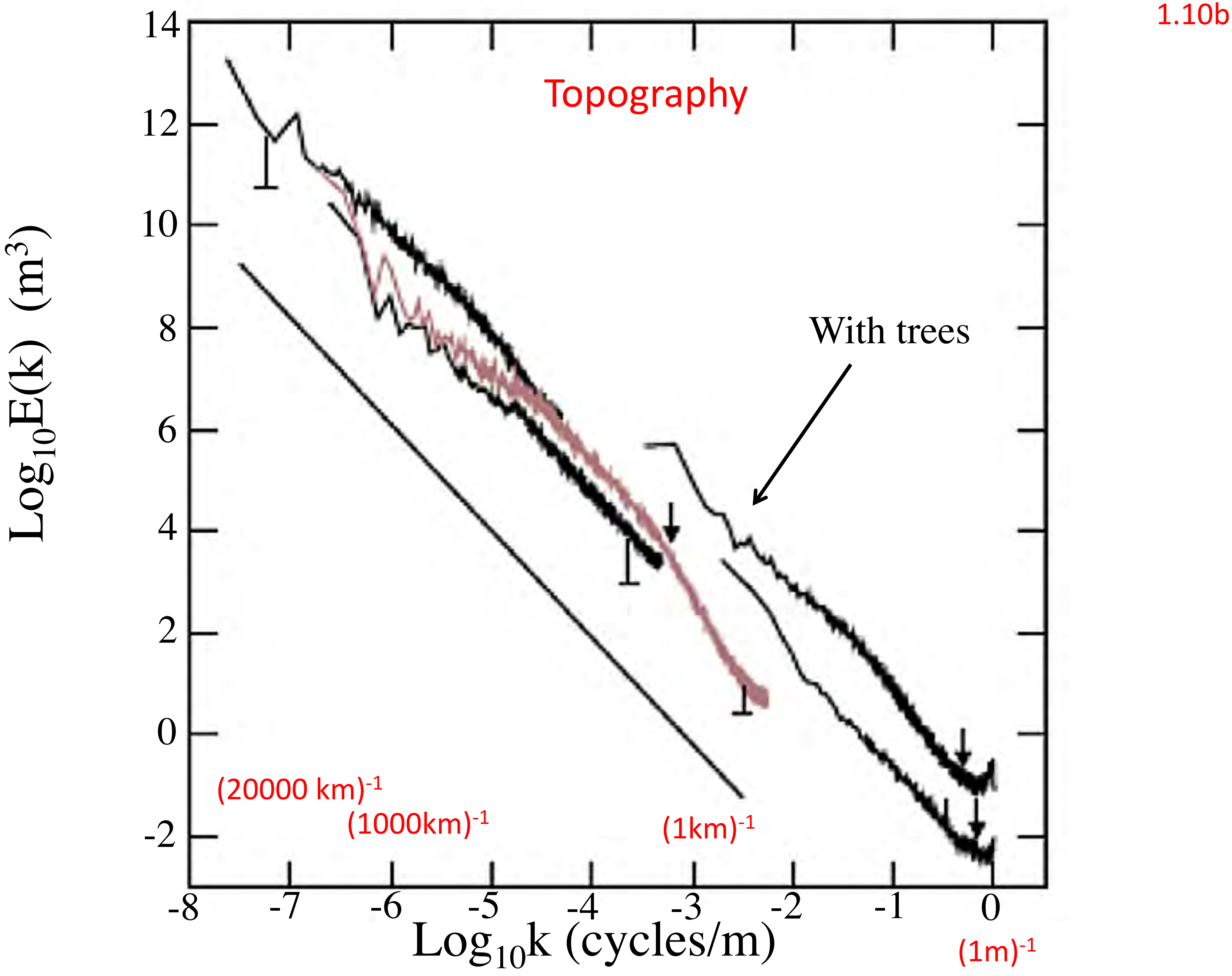
Desaulniers-Soucy et al 1999



Spectrum of drop mass 5 storms
1.8b
$\mathrm{Log}_{10}E(k)$
-5
-5.5
-6
-6.5
-7
-7.5
$E_\rho(k) \approx k^{-5/3}$
$E_\rho(k) \approx k^{+2}$
0.4
0.6
0.8
1.2
1.4
1.6
1.8
$\mathrm{Log}_{10}(k)$
$(2.3\ \mathrm{m})^{-1}$
$(50\ \mathrm{cm})^{-1}$
$(9\ \mathrm{cm})^{-1}$

The atmosphere:
3) Atmospheric Boundary Conditions

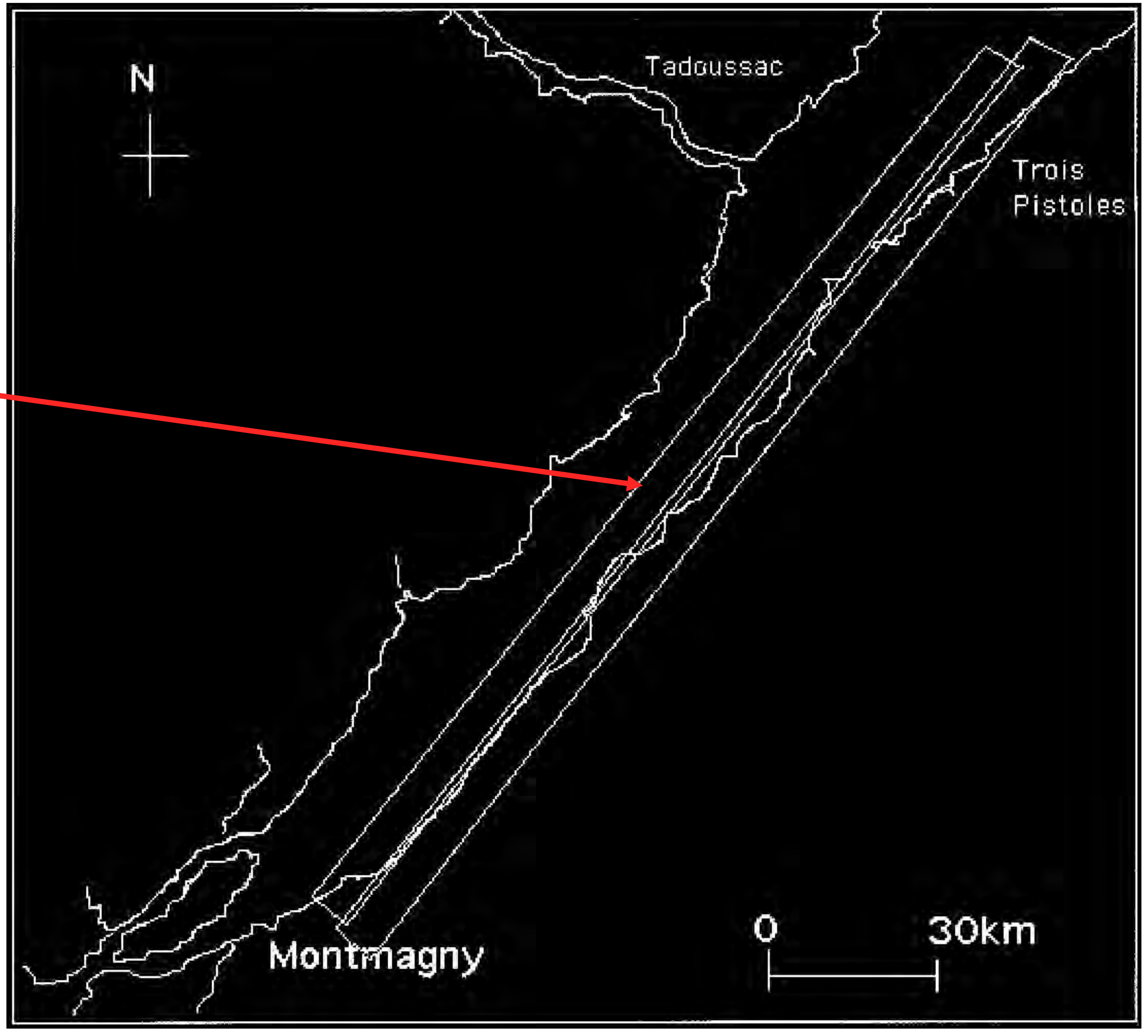

1.10a
Topography



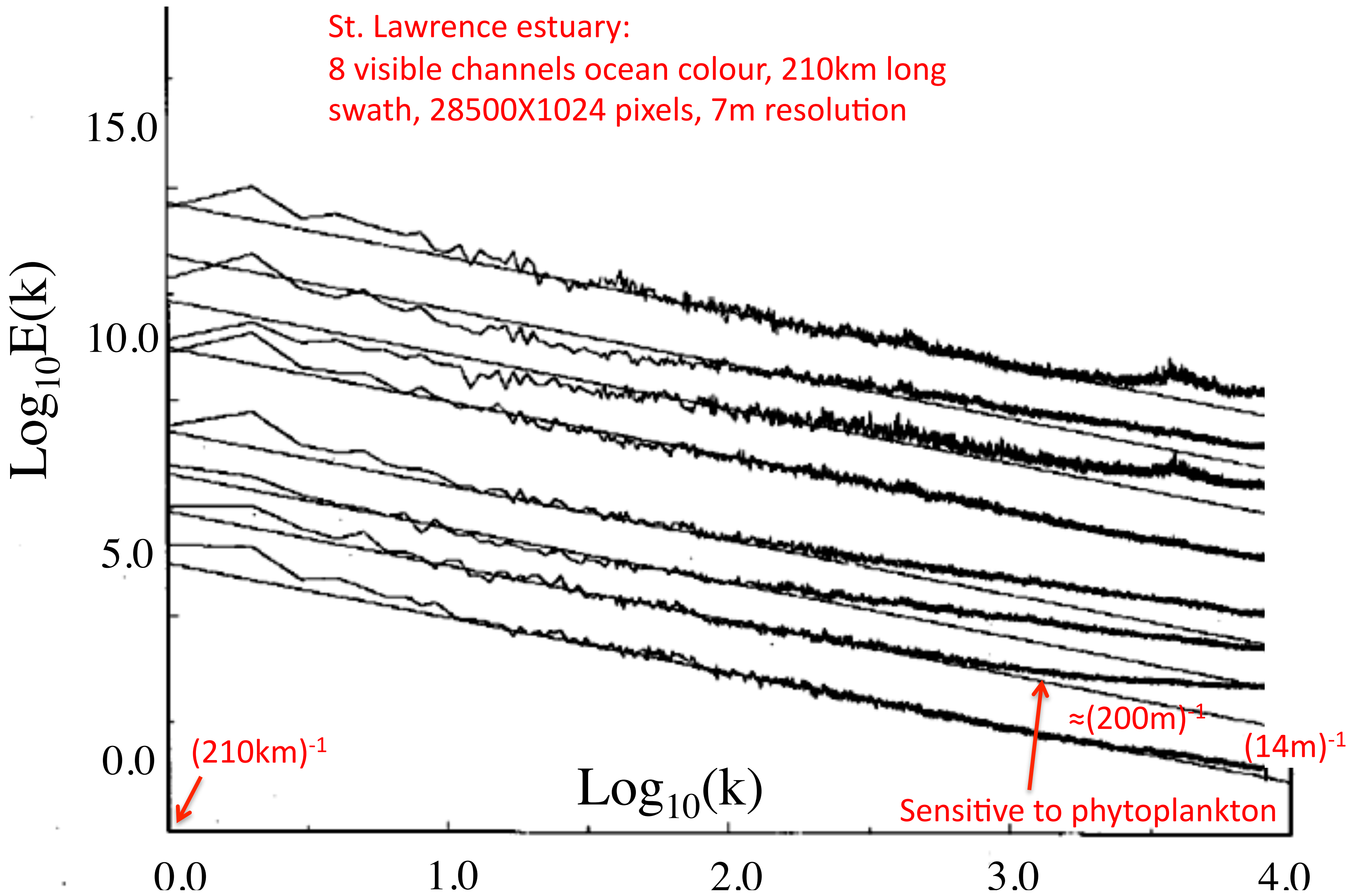
1.10b
Topography
With trees
Log10E(k) (m3)
Log10k (cycles/m)
14
12
10
8
6
4
2
0
-2
-8
-7
-6
-5
-4
-3
-2
-1
0
(20000 km)-1
(1000km)-1
(1km)-1
(1m)-1

Ocean Colour:
Mies sensor, experimental region

210km long swath,
28500X1024 pixels,
7m resolution,
(8 visible channels)



Ocean surface



Vegetation and soil moisture indices

Spectra of six bands of MODIS radiances over a 512x512 pixel region of Spain (at 250 *m* resolution; k=1 corresponds to 128 *km*):