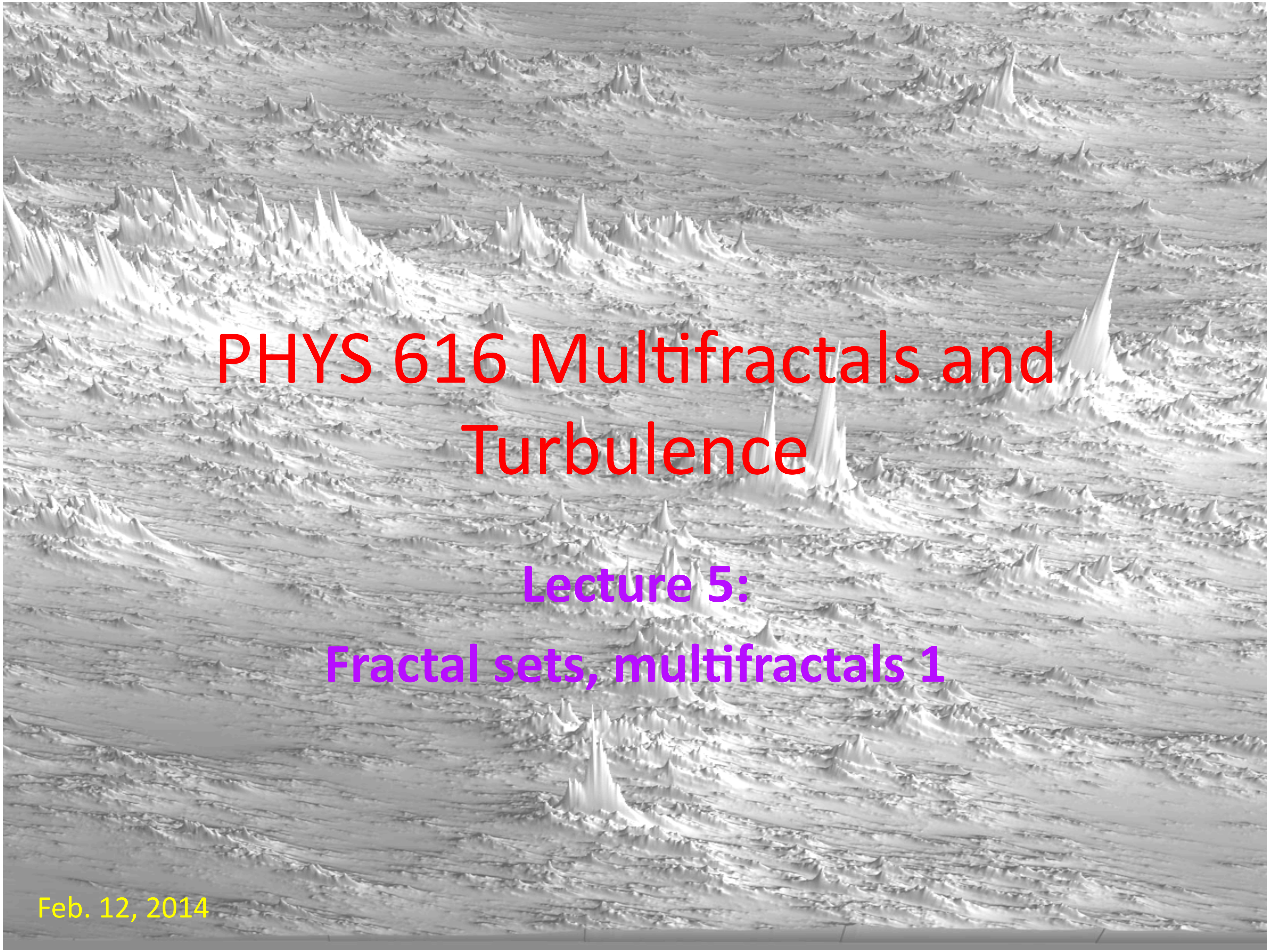


PHYS 616 Multifractals and Turbulence

Lecture 5:

Fractal sets, multifractals 1

Feb. 12, 2014

Multifractals and turbulence: Calendar 2014:

Jan. 15: Introduction our multifractal world part I

Jan. 22: Introduction our multifractal world part II

Jan. 29: Turbulence and spectra

Feb. 5: Fractal sets

Feb. 12: Fractal sets, multifractals part I

Feb. 14: Data analysis, wavelets

Feb. 19: Multifractals: part II

No classes the week of Feb. 24

Study break the week of March 3

March 12: Generalized Scale Invariance

March 19: Multifractal simulations

March 26: Causality, Fractional Integration, waves

April 2: Project presentations I

April 9: Project presentions II

April 16: : Project presentations III (if needed).

April 23: Deadline to hand in project reports.

Projects

Augstin Bussy:
Space-time scaling analysis of global precipitation from 1900 to present

Hossein Azizi & Gabriel Kocher
Spatio-temporal scaling of interface dynamics: applications to amorphous recrystallization and combustion synthesis

Martin Carrier-Vallieres
Fractal based imaging features for the early prediction of lung metastases in soft-tissue sarcoma cancer

Raphael Hébert:
GSI and Spectral Analysis of Atmospheric Wave Simulations with General Advection Velocity

Lenin del rio Amador
Haar wavelet analysis of a two-dimensional field defined on a fractal support. Application to global temperature measurements.

Thomas Van Himbeeck
Scaling properties of cloud radiances from 10 m to 100 km

$$\mu_{D,\delta}(A) \approx \inf\left[\sum_{\delta_i<\delta} \delta^D\right]$$

D dimensionsal Hausdorff measure of A, resolution δ

I.6 Box Counting Dimension and Fractal Dimension

I.6.1 Box Counting Dimension as an Approximation to Hausdorff Dimension

To understand the behaviour of Hausdorff measures approximate the Hausdorff measure by using disjoint boxes of size δ for the covering and take $w(t) \sim t^D$ and assume that using disjoint boxes produces a (near) optimum covering, *i.e.*, ignore the infimum. Then:

$$\mu_{D,\delta}(A) \sim \sum_{i=1}^{N(\delta)} \delta^D \sim N(\delta)\cdot\delta^D ,$$

where $N(\delta)$ is the total number of boxes required to cover set (for a set embedded in a three-dimensional space, use cubes; two-dimensional space, use squares, *etc.*). This covering can be thought of as defining the set A measured at resolution δ. Using the definition of Hausdorff dimension $D(A)$ of A :

$$\mu_{D(A)}(A) = \lim_{\delta\to 0} \mu_{D(A),\delta}(A) = \text{constant} ,$$

we obtain

$$N(\delta)\cdot\delta^{D(A)} \to \text{constant} \implies \boxed{N(\delta) \sim \delta^{-D(A)}}$$

Relation of box and Hausdorff dimensions

and so

$$D(A) = \lim_{\delta\to 0} \left\{\frac{-\log(N(\delta))}{\log(\delta)}\right\} .$$

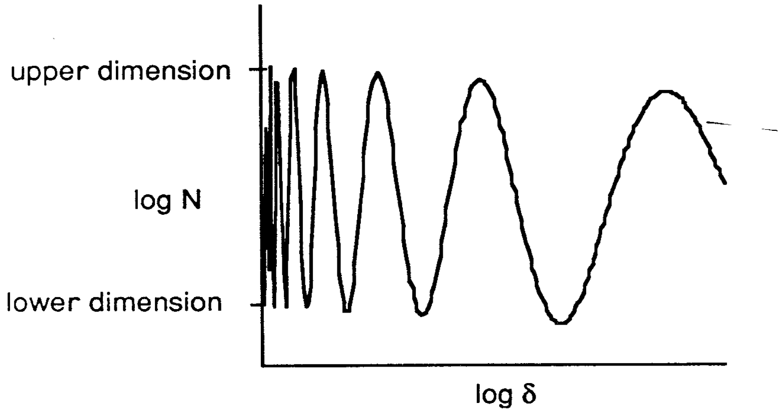
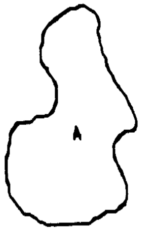
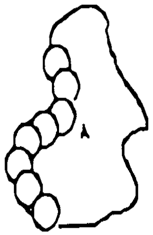


Figure I.13—The upper and lower limits of a function.

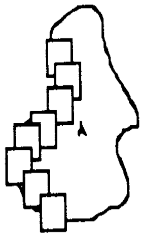
Different definitions equivalent to Box dimensions

See Falconer 1990



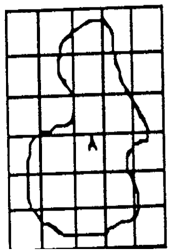


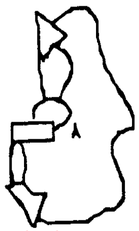
Least number of closed balls δ that cover A



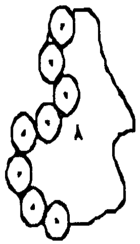
Least number of cubes of side δ that cover A

The number of mesh cubes that intersect A





The least number of sets of diameter at most δ that cover A



The greatest number of disjoint balls of radius δ with centres inside A

Fractal Geometry

Fractal Geometry (Mandelbrot, 1977, 1983) provides the simplest nontrivial example of scale invariance, and is useful for characterizing fractal sets. Unfortunately in geophysics we are usually much more interested in fields (with values at each point) and rarely interested in geometrical sets. However, over a wide range of scales fractal dimensions can still be useful in "counting the occurrences of a given phenomenon"—as long as this question can properly be posed. If this is the case and the phenomenon is scaling, then the number of occurrences ($N_A(l)$ at resolution scale l in space and/or time of a phenomenon occurring on a set A) follows a power law:

$$N_A(l) \sim \left(\frac{L}{l}\right)^{D_F}$$

D_F being the (unique) fractal dimension, generally not an integer, and L the (fixed) largest scale (here and below the sign ~ means equality within slowly varying and constant factors).

Fractal Codimensions: Geometric

The notion of fractal codimension C_F can be defined both statistically and geometrically. While the latter is much more popular, we will demonstrate that the former is much more useful and more general since it applies not only to deterministic but also to stochastic processes.

1) Definition 1: Geometric definition of a fractal codimension:

Let $A \subset E$ (the embedding space) with $\dim(E) = D$ and $\dim(A) = D_F(A)$. Then the codimension $C_F(A)$ is defined as:

$$C_F(A) = D - D_F(A)$$

Example, a line ($D_F(A)=1$) in E= three dimensional space dim(E)=D=3, hence: $C_F(A)=3-1=2$

This definition corresponds merely to an extension of the (integer) codimension definition for vector sub-spaces, i.e., E_1 and E_2 being in direct sum (i.e., $E_1 \cap E_2 = \varnothing$):

$$E = E_1 \oplus E_2 \quad \Rightarrow \quad \mathrm{codim}(E_1) = \dim(E_2)$$

e.g. E_1= line, E_2= plane, E= 3-D space

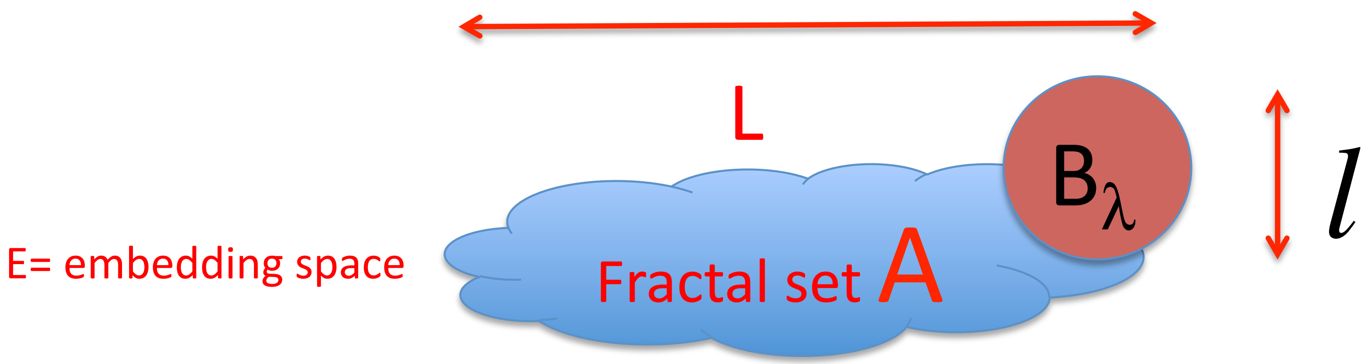
Fractal Codimensions: Probabilistic

2) Definition 2: Probabilistic definition of a fractal codimension:

In fact the codimension C_F can be considered to be more fundamental than the notion of fractal dimension D_F and should be introduced directly. Consider the (scaling) behaviour the probability ("*Pr*") that a ball B_λ (of size $\ell = L/\lambda$) intersects the set A is:

$$\Pr(B_\lambda \cap A) \sim \lambda^{-C_F(A)}$$

where B_λ = ball of size and $\ell = L/\lambda$ and C_F is thus directly defined as an exponent measure of the fraction of the space occupied by the fractal set A (size L) in an embedding space E which can even be an infinite dimensional space.



Geometric versus probabilistic

3) Relating the two definitions

Since the probability of the event $(B_\lambda \cap A)$ is defined as:

$$\Pr(B_\lambda \cap A) \sim \frac{N(B_\lambda \cap A)}{N(B_\lambda \cap E)} \sim \frac{\lambda^{D_F(A)}}{\lambda^{D(E)}}$$

Number of balls B_λ needed to cover A

Number of balls B_λ needed to cover E

where $N(B_\lambda \cap A)$ refers to for example the number of balls B_λ needed to cover the set A and $N(B_\lambda \cap E)$ is the corresponding number for the entire space. It is easy to check that when $C_F(A) < D = \dim(E) < \infty$ the two definitions are equivalent:

$$C_F(A) \le D < \infty, \ \{\text{definition 1} \equiv \text{definition 2}\} \ \forall \ D_F \ge 0$$

Latent dimension paradox

Dimension of A for given codimension $D_F(A) = D - C_F(A)$

Rather obviously the statistical definition does not imply any limitation on *C*. However, the equivalence between the two definitions does not hold any longer as soon as $C_F(A) > D$ since:

$$for\ C_F(A) > D,\ \{\text{both definition 1 and definition 2}\} \Rightarrow D_F(A) < 0$$

This is the so-called “latent” dimension “paradox” corresponding to the fact that a deterministic geometric definition is no longer possible: indeed there is no possible definition of a negative Hausdorff dimension! This is not surprising since the definition 2 overcomes many limitations of the Hausdorff dimension which is defined for compact sets (hence bounded sets): the codimension measures the *relative sparseness* of a phenomenon (the relative frequency of its occurence), whereas the dimension measures it *absolute sparseness* (the absolute frequency of its occurence). Obviously, we don't need to know the latter in order to be able to determine the former. However, it turns out historically that the (fractal) dimension was introduced first.

Paradox solved with “sampling dimension”

The Intersection theorem

$$\Pr\left(E_1 \cap E_1\right) = \Pr\left(E_1\right)\Pr\left(E_2\right)$$

The sets E_1, E_2 are assumed to be statistically independent

$$\lambda^{-C\left(E_1 \cap E_2\right)} = \lambda^{-C\left(E_1\right)}\lambda^{-C\left(E_2\right)}$$

Probabilities In terms of codimensions

$$C(E_1 \cap E_2) = C(E_1) + C(E_2)$$

Addition of codimensions ("intersection theorem")

See next slide for example

Meteorological measuring network

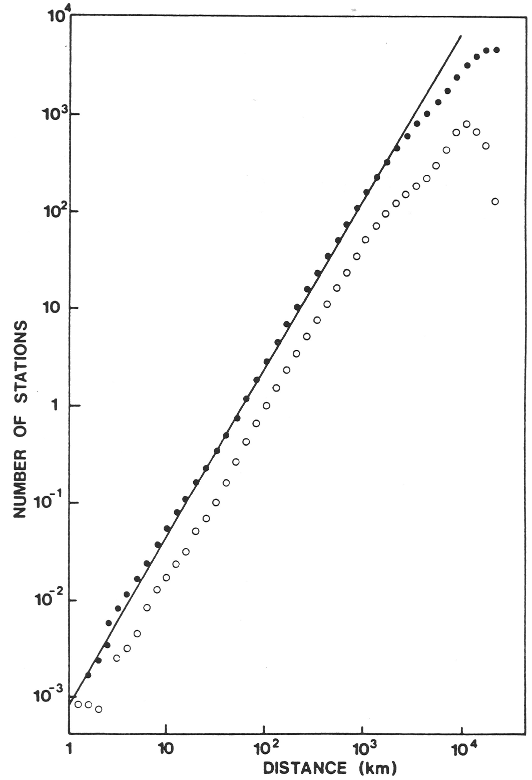
Fractal set:
each point
is a station

9962 stations (WMO)

Number

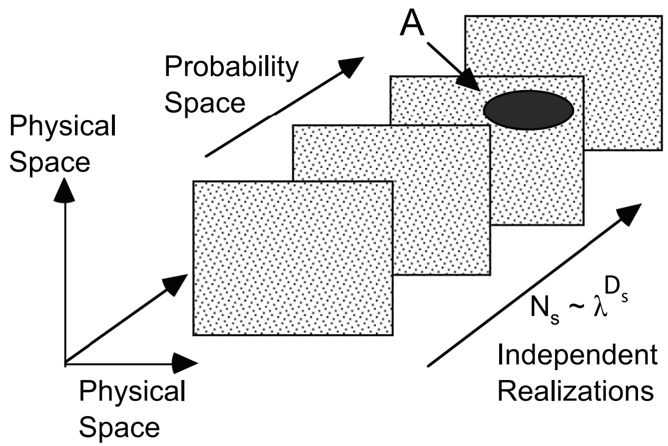
$$n(L) \propto L^D$$

Density $\rho(L) = n(L)L^{-2} \propto L^{-C}; \quad C = d - D; \quad d = 2$



The fractal dimension
of the network= 1.75

Sampling dimensions



Sampling Dimension

$$D_s = \frac{\log N_s}{\log \lambda}$$

Effective dimension of space with N_s samples

$$\Delta_s = d + D_s$$

Illustration showing how in random processes the effective dimension of space can be augmented by considering many independent realizations N_s. As, $N_s \to \infty$ the entire (infinite dimensional) probability space is explored. When the process is observed on a low dimensional cut of dimension d (such as the d = 2 dimensional sketch shown on a single sample (picture) $N_s = 1$, $D_s = 0$, as long as $d > c$, we may introduce the (positive) dimension $D = d - c$, which is then the geometrical dimension of the set with singularities. However, structures with $D<0$ will be too sparse to be observed (they will almost surely not be present on a given realization/picture). In order to observe them we must increase the number of samples N_s or equivalently the sampling dimension D_s to reach them.

Monofractal sets

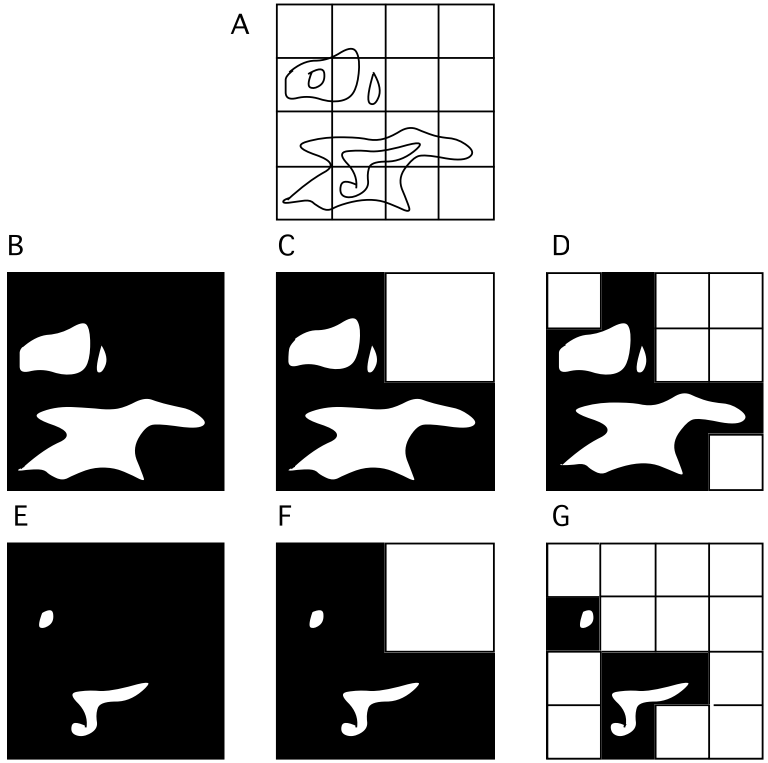
(singular) multifractal fields....

Multifractality and Functional Box Counting

$$N_T(L) \approx L^{-D(T)}$$

Box counting low threshold (large D)

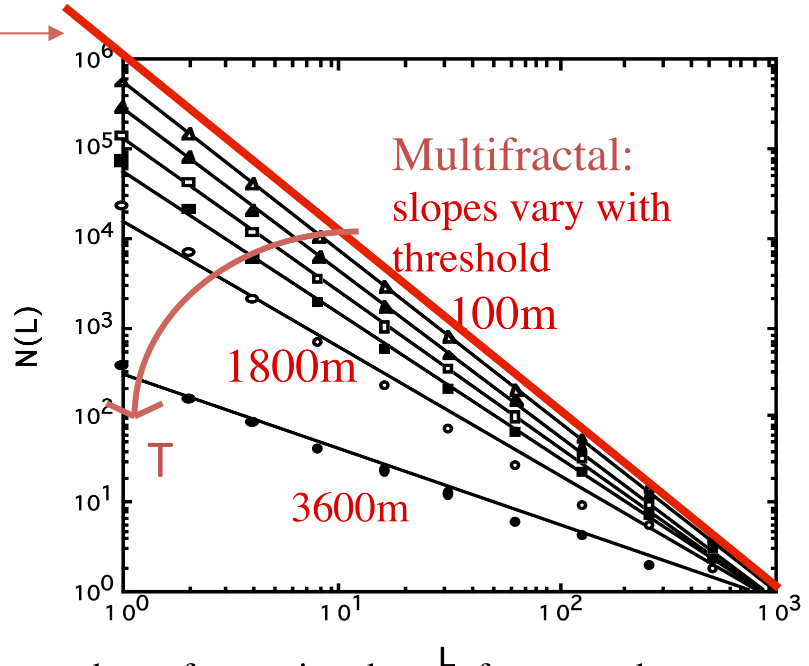
Box counting high threshold (low D)



-Monofractal: D(T) <2 , constant

-Multifractal: D(T)<2, decreasing

Functional box counting on French topography: 1 -1000km



$N_T(L) \approx L^{-D(T)}$

Systematic resolution dependence

N(L) = number of covering boxes for exceedance sets at various altitudes.

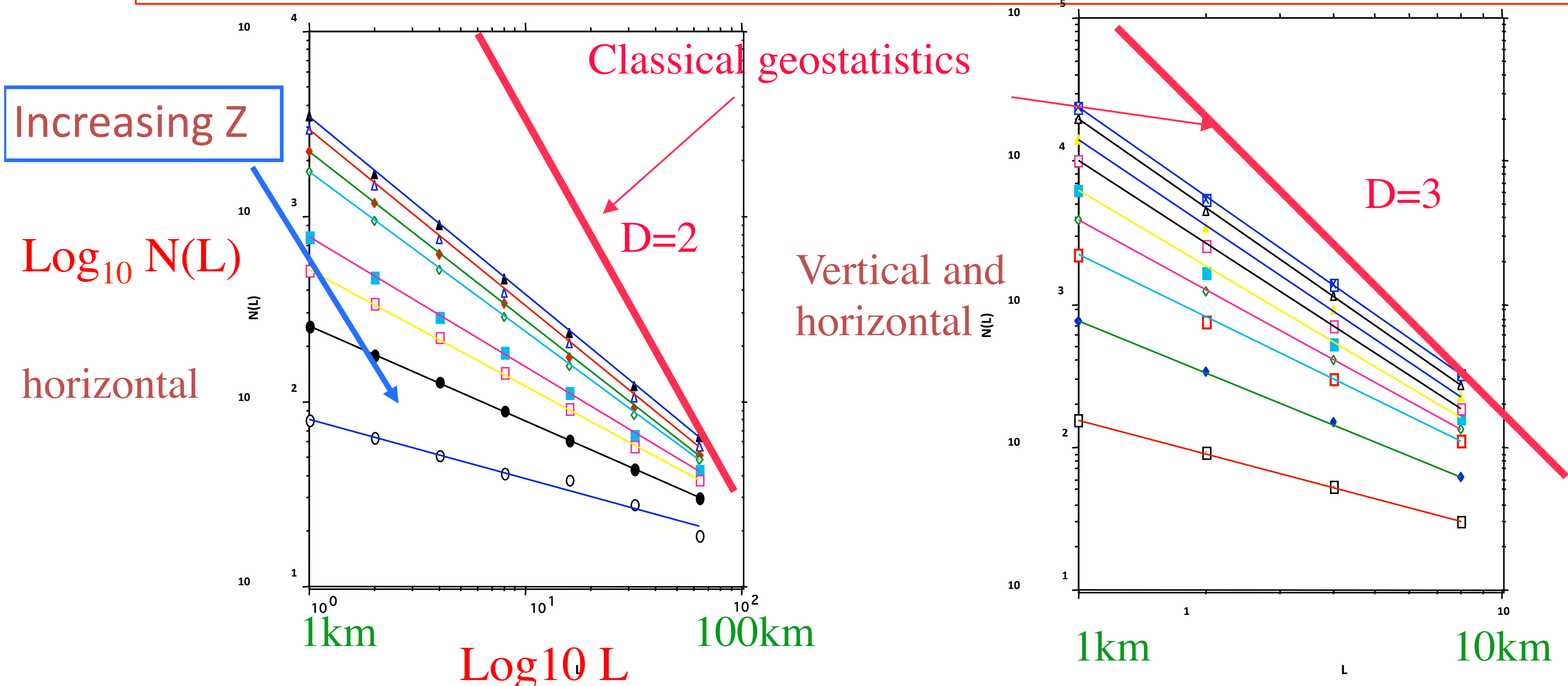
The dimensions d increase from 0.84 (3600m) to 1.92 (at 100m).

Lovejoy and Schertzer 1990

Implications for geostatistics:

If we now consider the areas A_T exceeding a given threshold then we find that they systematically decrease as the resolution becomes finer (decreasing *L*): $A_T = L^d N_T = L^{C(T)}$; with $C(T) = d - D(T)$. We see that contrary to standard assumptions (including those of classical geostatistics, that unless $C(T) = 0$, the areas depend on the subjective resolution *L*; the reference lines indicate that for the topography, all the regions defined by the thresholds have $C(T) = d\text{-}D(T) > 0$ so that they have systematic resolution dependencies.

Functional Box counting on 3D radar rain scans

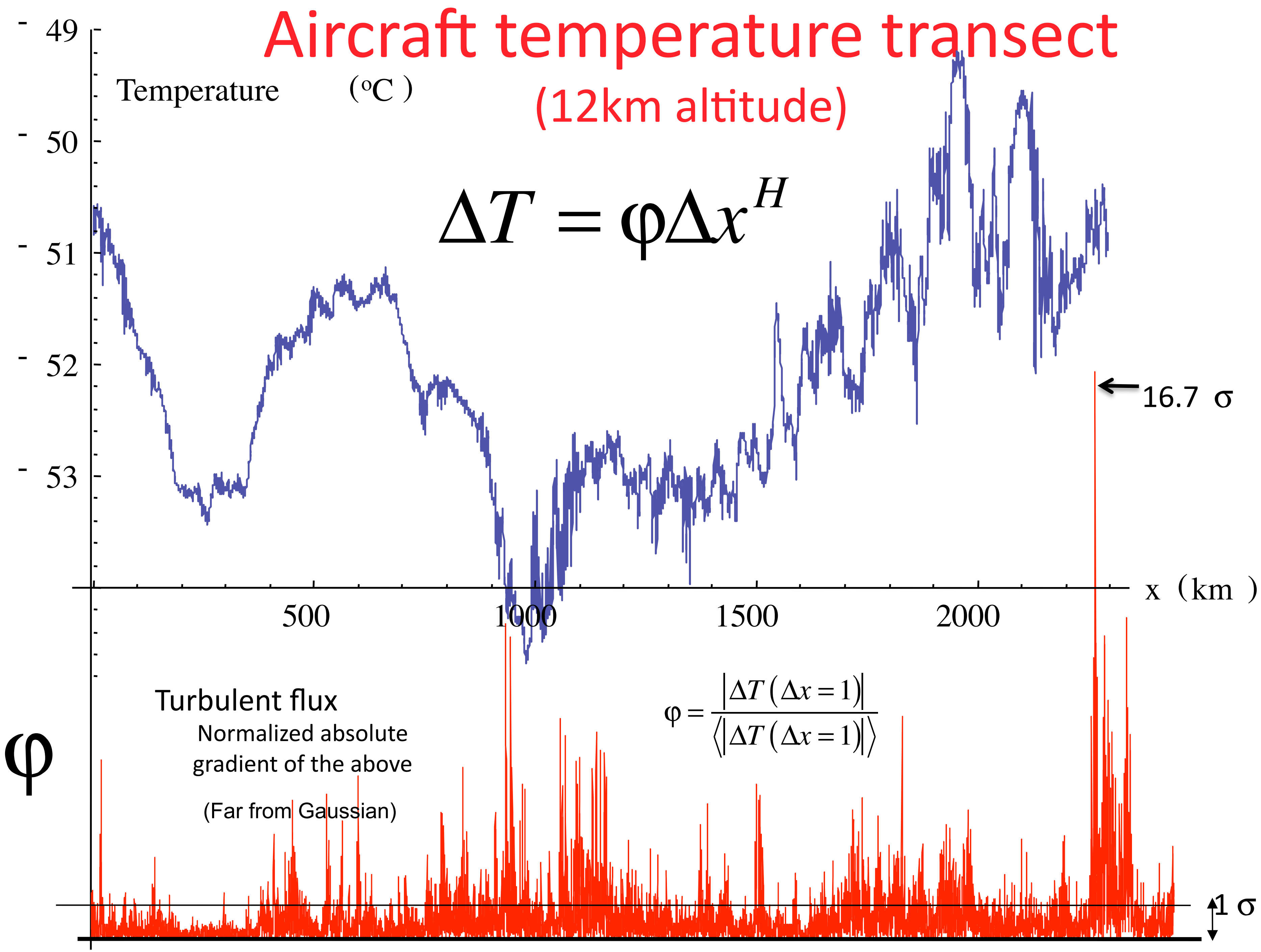


$$N_T(L) \approx L^{-D(T)}$$

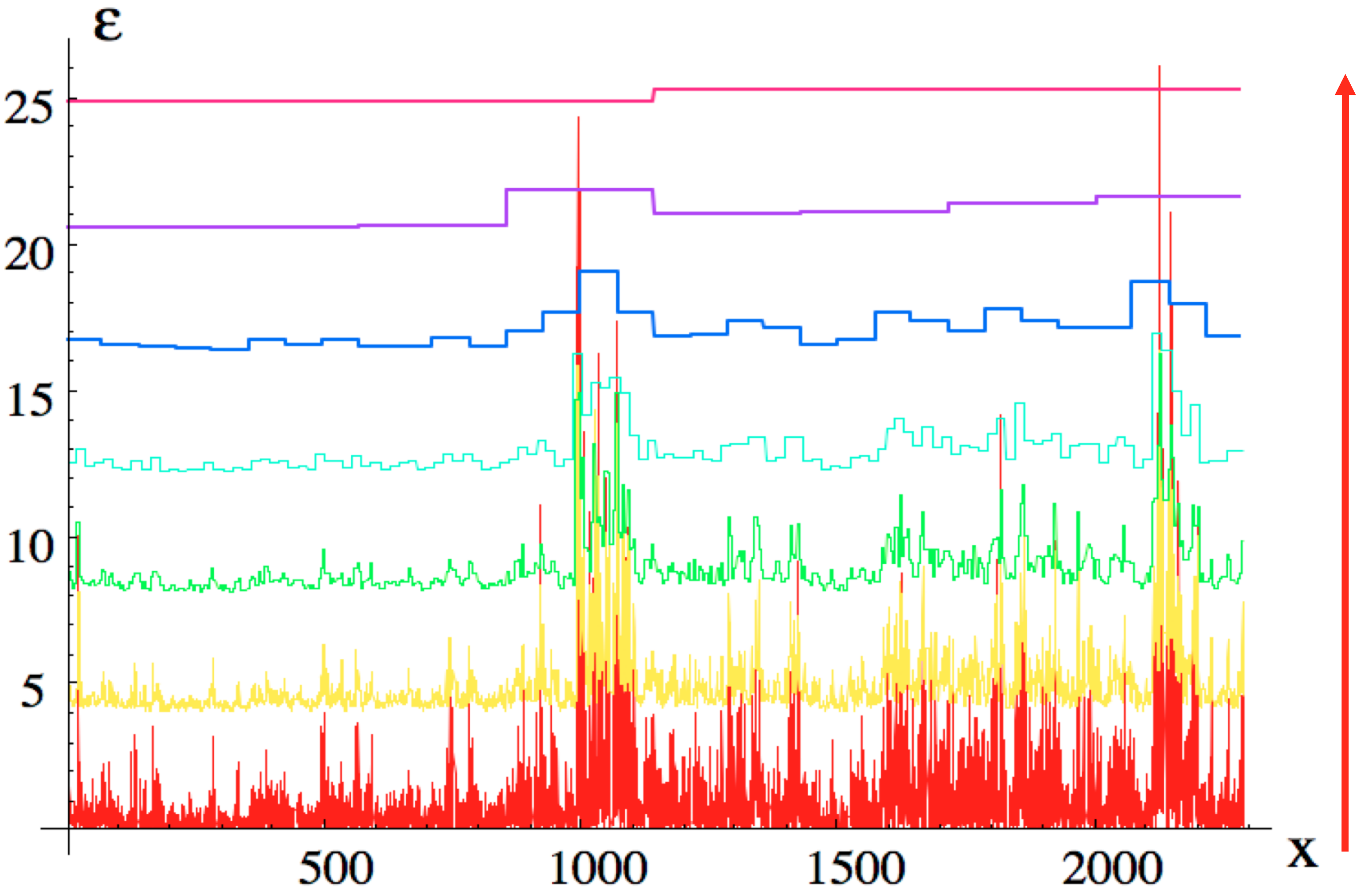
reflectivity thresholds increasing (top to bottom) by factors of 2.5 (dat from Montreal).

Science: Lovejoy, Schertzer and Tsonis 1987

Cascades and Multifractals


Aircraft temperature transect
(12km altitude)
Temperature (°C)
-49
-50
-51
-52
-53
$\Delta T = \varphi \Delta x^{H}$
16.7 σ
x (km)
500
1000
1500
2000
Turbulent flux
Normalized absolute
gradient of the above
(Far from Gaussian)
φ
$\varphi = \frac{|\Delta T(\Delta x = 1)|}{\langle|\Delta T(\Delta x = 1)|\rangle}$
1 σ

Degrading the resolution

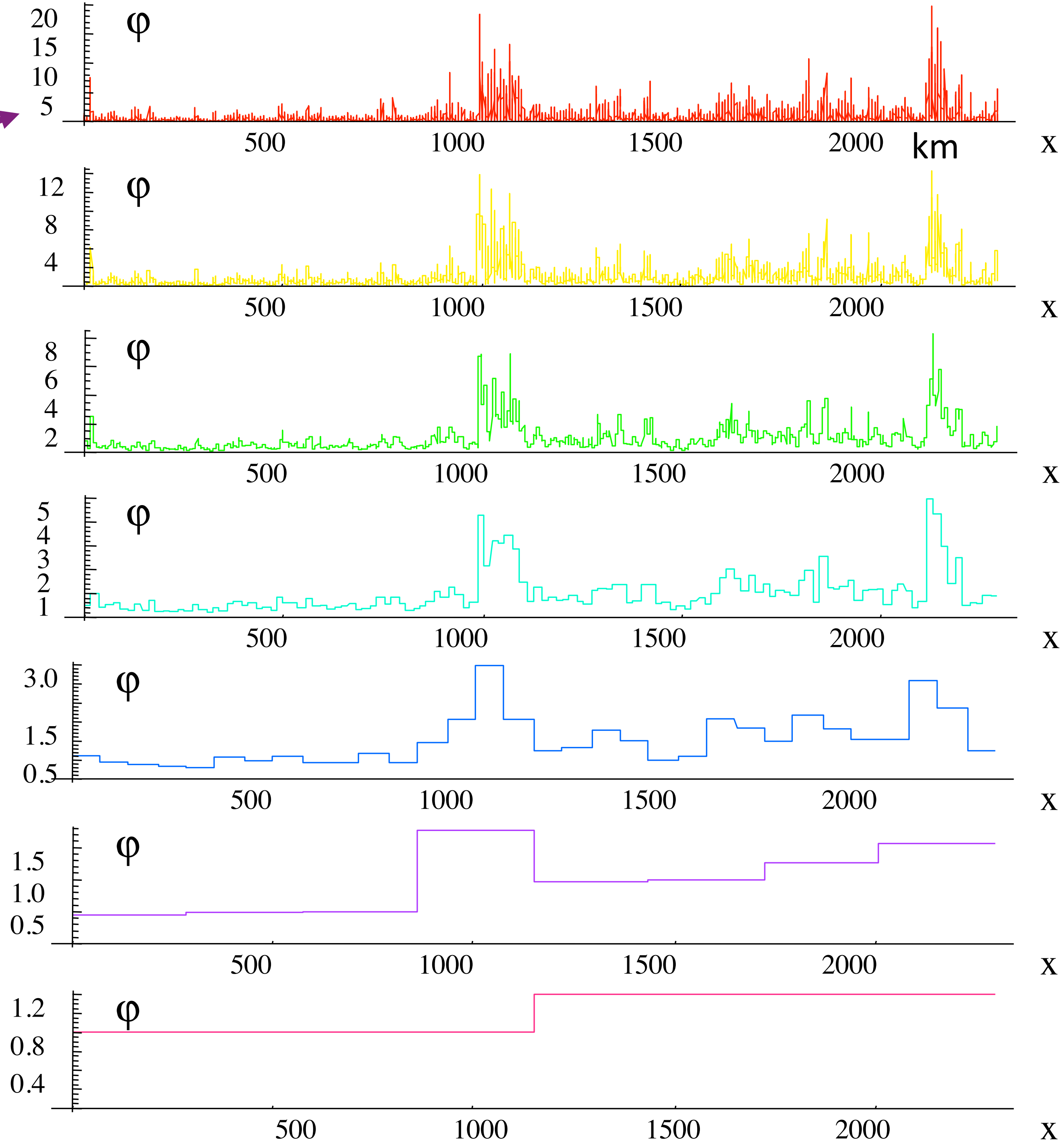


From high to low resolution by averaging

Same…

Temperature turbulent flux φ at 280m resolution

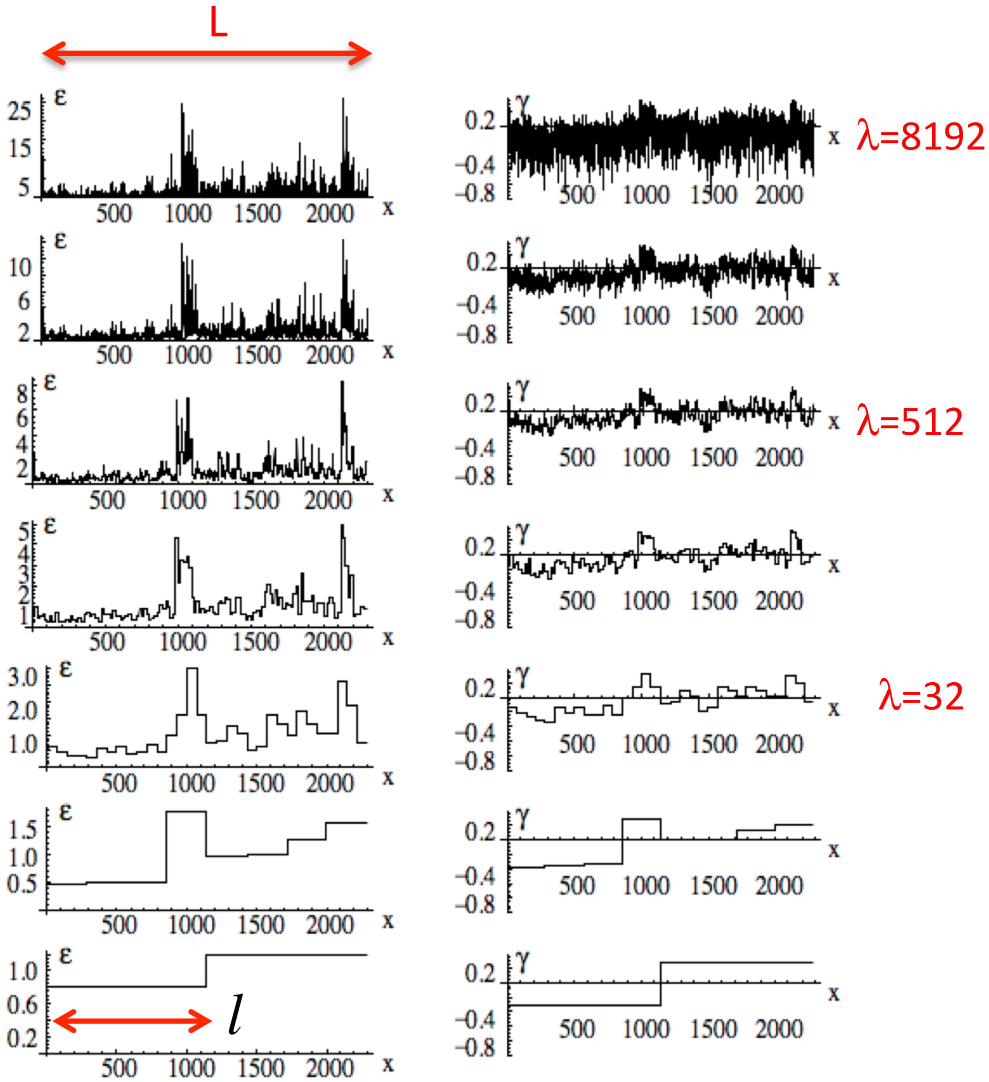
High to low Resolution: degrading by factors of 4



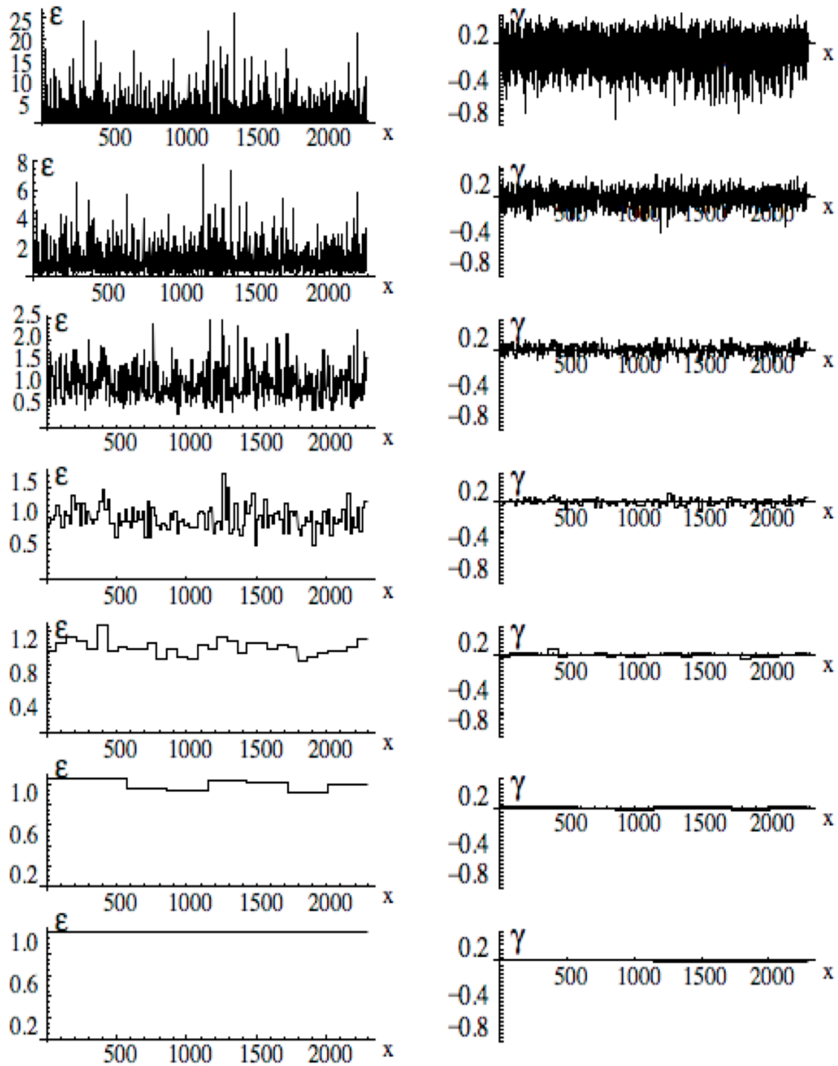
Flux versus singularities

$$\varepsilon' = \lambda^{\gamma}; \quad \varepsilon' = \frac{\varepsilon}{\langle \varepsilon \rangle}; \quad \lambda = \frac{L}{l}$$

$$\gamma = \frac{Log\,\varepsilon'}{Log\,\lambda}$$

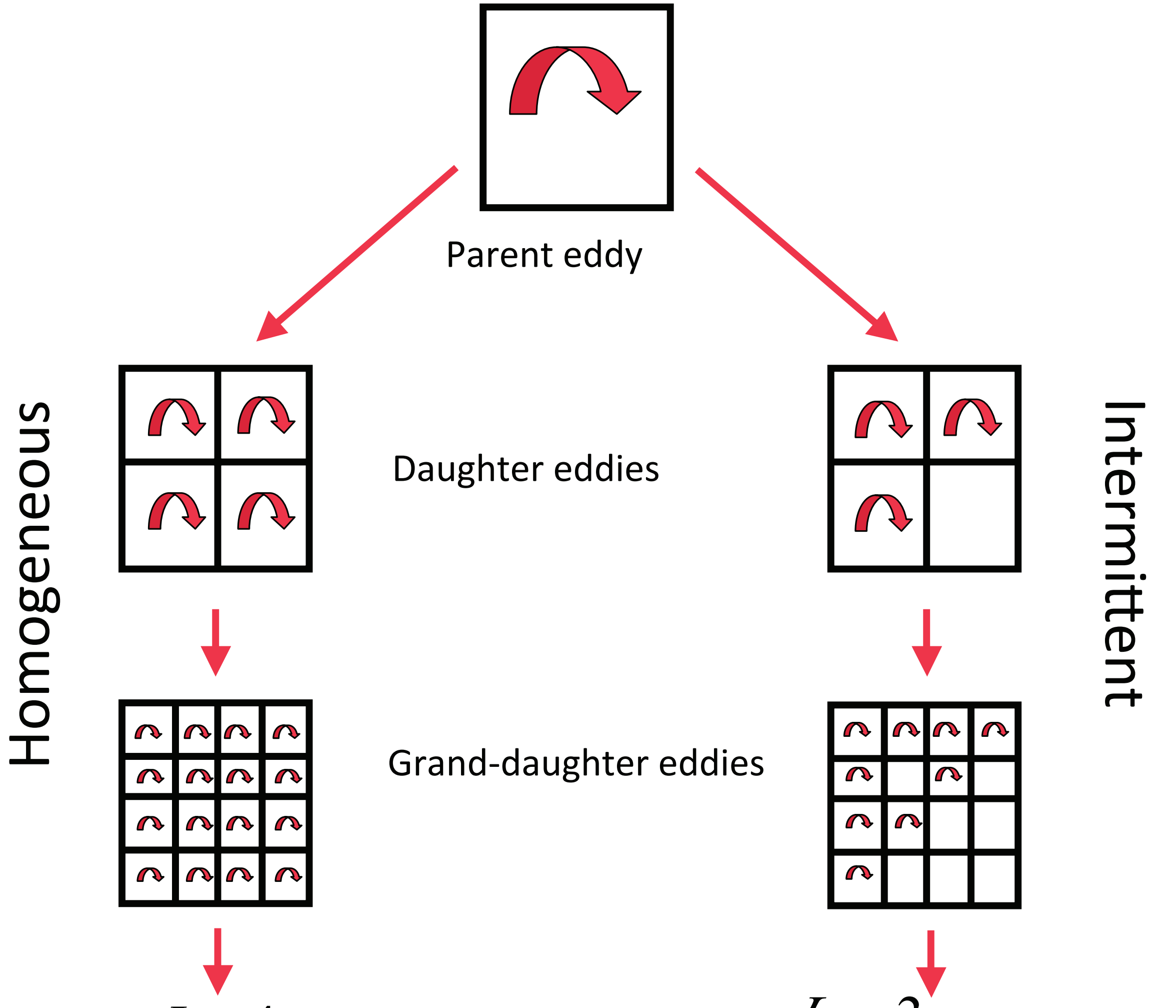


Real fluxes



Shuffled fluxes

Cascades



$$D = \frac{Log4}{Log2} = 2$$

$$D = \frac{Log3}{Log2} = 1.58...$$

Beta model

An initial attempt to handle intermittency reduces it to the simple notion of "on/off" intermittency, i.e. a cascade with the simple alternative alive/dead of the offspring.

This leads to a confinement of the turbulence to a tiny support; a very small subregion of the flow. The right hand side of the figure shows the result of such a stochastic cascade obtained by randomly multiplying the energy flux of a "mother" eddy to obtain that of the "daughter" eddies either by 0 (dead sub-eddy) or by a positive value λ_0^c

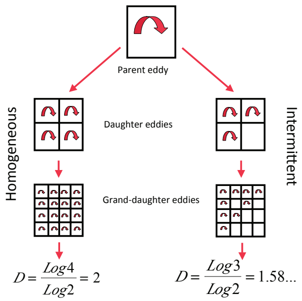
(corresponding to an active sub-eddy, with fixed probability λ_0^{-c}

In this model, we divide the spatial scales by λ_0 (here λ_0 = 2) and then flip coins to determine the on or off state; more precisely:

Each step:

$$\Pr\left(\mu\varepsilon = \lambda_0^c\right) = \lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon = 0\right) = 1 - \lambda_0^{-c}$$

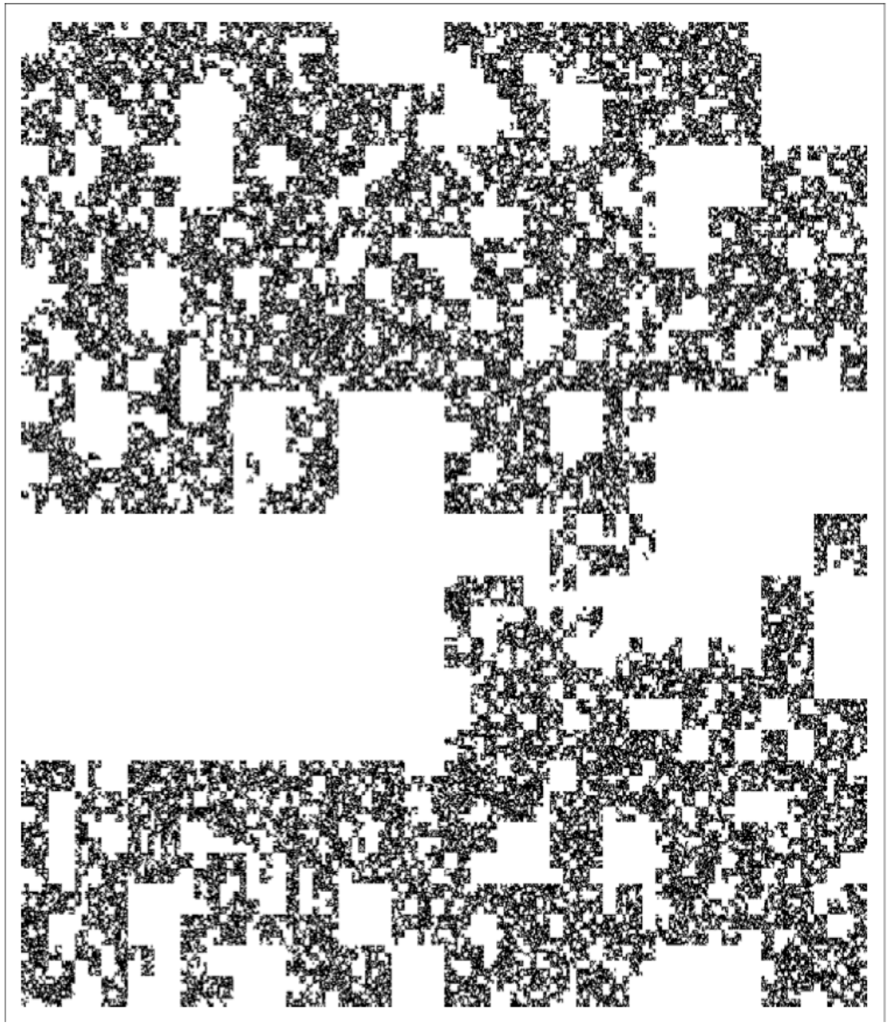


After n steps: $\varepsilon_n = \prod_{j=1}^{n} \mu\varepsilon_j$

("*Pr*" indicates "probability"). The nonzero value is taken as $\mu\varepsilon = \lambda_0^c$ so that the mean $\langle \mu\varepsilon \rangle = 1$; this implies a scale by scale conservation of the flux ε.

After n steps: $\lambda = \lambda_0^n$ $\quad \Pr(alive) = \left(\lambda_0^{-c}\right)^n = \lambda^{-c}$ Relation to dimension: $N_{alive} = N_{tot}\Pr = \lambda^d \lambda^{-c} = \lambda^D; \quad D = d - c$

Beta model



In this example, the probability that an eddy will remain alive is λ_0^{-C} =0.87 (using the scale ratio at each step λ_0 = 4 here and the codimension C = 0.2).

Alpha model

To see how the α model works, consider constructing it on the unit interval. At first, the interval is uniform so that the initial energy flux density $\varepsilon_0 = 1$. As in the β model, the cascade proceeds by dividing the unit interval successively into λ_0 subintervals (λ_0 is an integer = 2 in the figure) and multiplying the flux density by independent identically distributed random factors $\mu\varepsilon$ (the notation "μ" indicating "multiplicative increment"; it is analogous to the use of the "Δ" to denote an additive increment). Therefore after n (discrete) cascade steps, the smallest scale is λ_0^{-n}, the value of the energy flux density at a point $0 \le x \le 1$ is the product:

$$\varepsilon_n = \prod_{j=1}^{n} \mu\varepsilon_j$$

In order for the flux to be conserved from scale to scale, we constrain the weights $\mu\varepsilon$ so that $\langle\mu\varepsilon\rangle=1$ implying $\langle\varepsilon_n\rangle = 1$.

Alpha model

The α model is a two state (binomial) process with $\mu\varepsilon$ = either $\lambda_0^{\gamma+}$ or $\lambda_0^{\gamma-}$ where $\gamma_+>0$ corresponds to a boost ($\mu\varepsilon>1$) and γ_- to a decrease ($\mu\varepsilon<1$) . As in the β model, the corresponding probabilities can be written λ_0^{-c} and 1-λ_0^{-c} respectively where $c>0$ is a parameter (it corresponds to the maximum codimension of the process. Formally:

$$\Pr\left(\mu\varepsilon = \lambda_0^{\gamma_+}\right) = \lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon = \lambda_0^{\gamma_-}\right) = 1 - \lambda_0^{-c}$$

Although the α model apparently involves three parameters (γ_+, γ_-, c), due to the conservation constraint:

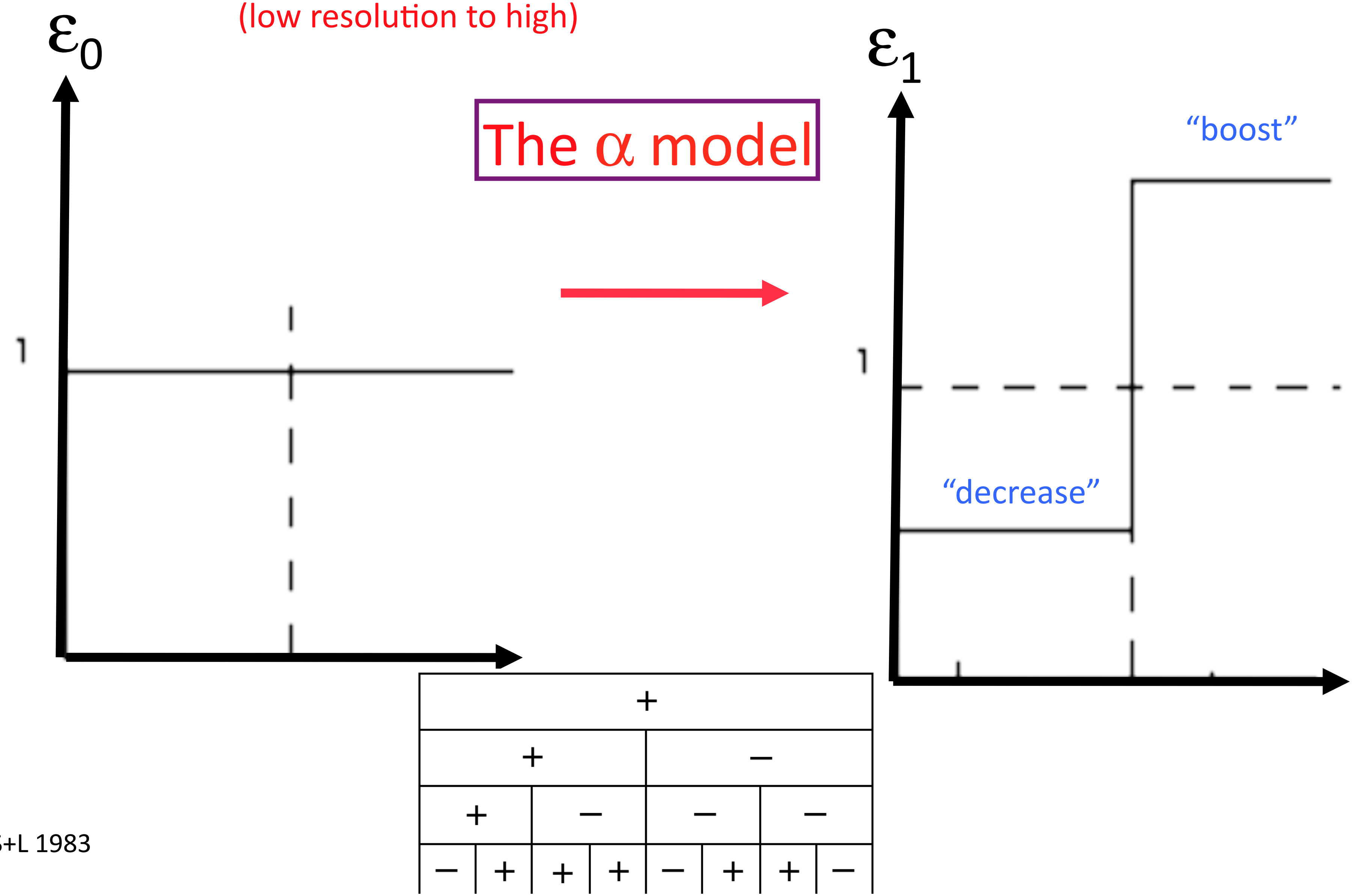
$$\langle\mu\varepsilon\rangle = \lambda_0^{-c}\lambda_0^{\gamma_+} + \left(1-\lambda_0^{-c}\right)\lambda_0^{\gamma_-} = 1$$

only two can be freely chosen.

We can see that the β model is recovered in the limit $\gamma_+ \to c$ which is the same as $\gamma_- \to -\infty$

Cascades and Multifractals

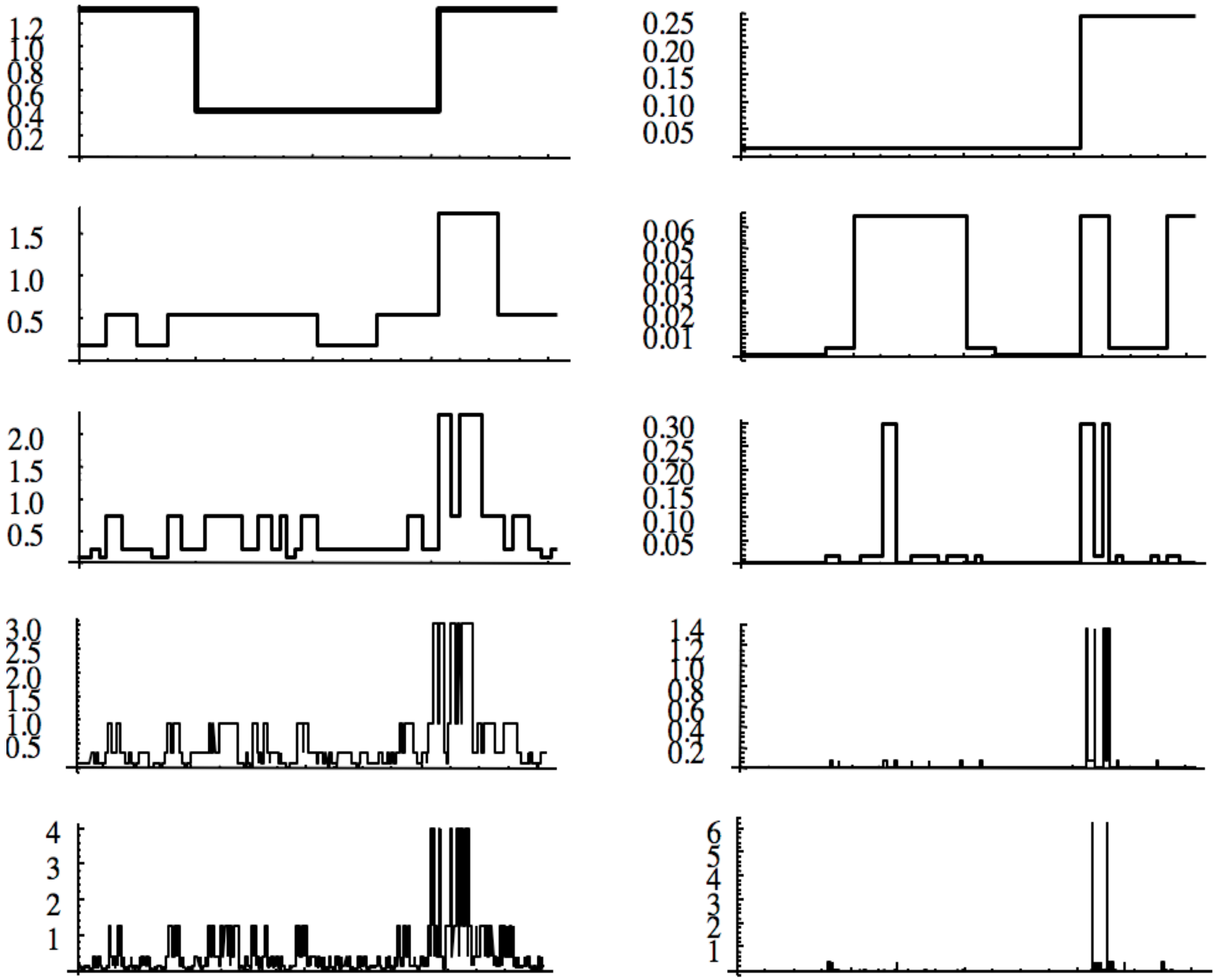
Simulations: **multiplicative** introduction of small scale details (low resolution to high)



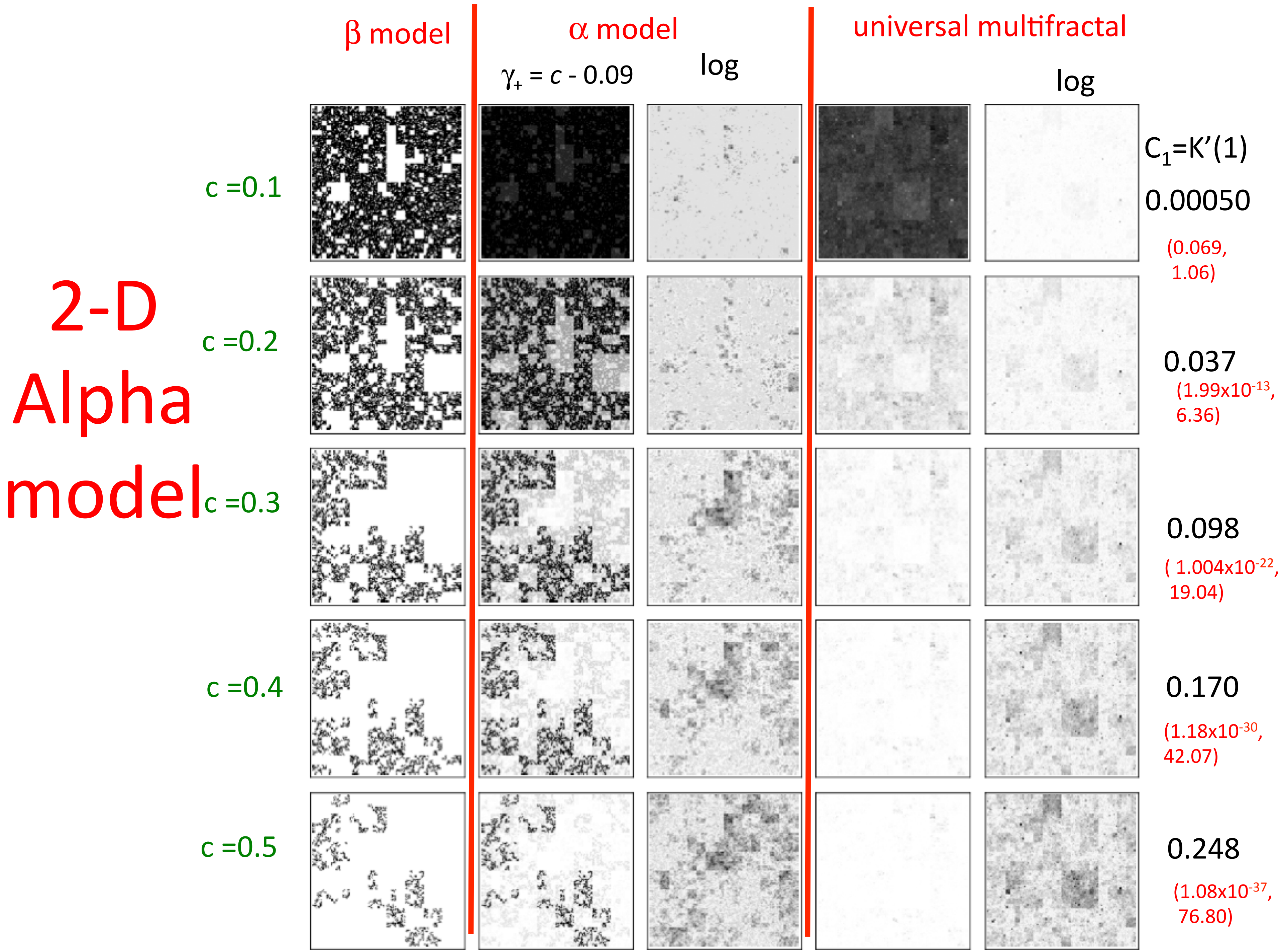
S+L 1983

Alpha model

$\gamma_+ = 0.2$, $c = 0.3$ ($C_1 = 0.087$) $\gamma_+ = 1.1$, $c = 1.2$ ($C_1 = 0.82$)



From top to bottom every second cascade step is shown (a factor of λ_0^2) is shown, 10 steps in all, the total range of scales is $2^{10} = 1024$). Notice the changing vertical scales


2-D
Alpha
model
β model
α model
γ+ = c - 0.09
log
universal multifractal
log
c =0.1
c =0.2
c =0.3
c =0.4
c =0.5
C1=K'(1)
0.00050
(0.069,
1.06)
0.037
(1.99x10-13,
6.36)
0.098
(1.004x10-22,
19.04)
0.170
(1.18x10-30,
42.07)
0.248
(1.08x10-37,
76.80)