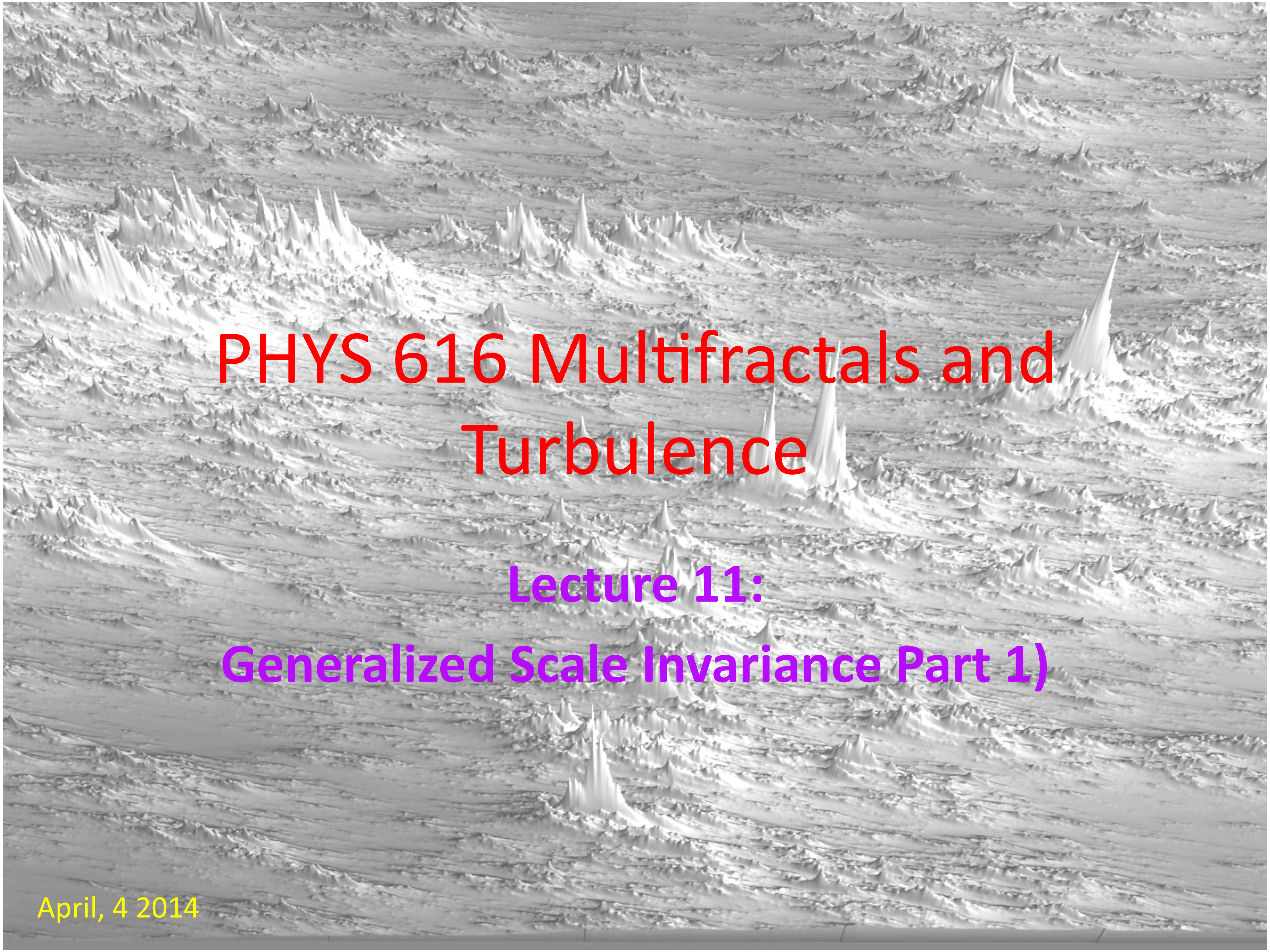


PHYS 616 Multifractals and Turbulence

Lecture 11:

Generalized Scale Invariance Part 1)

April, 4 2014

Schedule

- April 9: Finish last lecture (1 hour)
- Presentations (Gabriel and Hussein, 30 minutes)
- April 18: (10-12) Remaining presentations
- April 21: Hand in Reports

The Obukhov and Bolgiano buoyancy subrange 3D isotropic turbulence

Let us start with the thermodynamic energy equation for an ideal gas:

$$\frac{D\log\theta}{Dt}=\frac{\kappa}{T}\nabla^2 T+\frac{\dot{Q}}{c_v T}$$

Gas constant

Specific heat at constant volume

Potential temperature

$$\theta=T\left(\frac{p}{p_0}\right)^{R/c_v}$$

Reference pressure

e.g. (Lesieur, 1987) where θ is the potential temperature, κ is the molecular heat diffusivity and T the absolute temperature and $D/Dt\equiv\partial/\partial t+\underline{v}\cdot\nabla$ is the advective derivative, $\dot{Q}$ is the rate of heat input per unit mass and c_v is the specific heat at constant volume. We will assume a quasi-steady state where heat sources $\dot{Q}$ create fluctuations/structures at large scales which are transferred by the nonlinear terms to smaller scales where they are eventually smoothed out by the dissipation term. As in our discussion of the energy flux cascade, we argue that at scales much larger than the dissipation scales that the right hand side of the above equation is ≈ 0 so that log θ is an advected scalar:

$$\frac{D\log\theta}{Dt}\approx 0$$

Boussinesq approximation (1)

Let us now introduce the Boussinesq approximation.
Consider small fluctuations θ' with respect to a time averaged state θ_0: and express the potential temperature in terms of these:

$$\theta(\underline{r},t)=\theta_0(\underline{r})+\theta'(\underline{r},t)$$

Note that for this decomposition to be useful, there must exist a scale separation between "fast" and "slow" processes. The full Boussinesq approximation for the potential temperature fluctuations yields:

$$\frac{D\theta'}{Dt}+\underline{v}\cdot\nabla\theta_0\approx\kappa\nabla^2\theta'+\frac{\dot{Q}}{c_v}$$

Recall: $\frac{\partial\rho}{\partial t}=-\underline{v}\cdot\nabla\rho+\kappa\nabla^2\rho+f_\rho$

We can now consider a vertically stratified fluid in which $\theta_0(\underline{r})=\theta_0(z)$ hence $\underline{v}\cdot\nabla\theta_0\approx w\frac{d\theta_0}{dz}$ where $w = v_z$. If w is small enough, then we have:

$$\frac{D\theta'}{Dt}\approx\kappa\nabla^2\theta'+\frac{\dot{Q}}{c_v}$$

Hence, neglecting sources and dissipation: $\frac{D\theta'}{Dt}\approx 0$ Scalar advection equation

Hence the following flux is conserved $\chi_\theta=\frac{\partial\theta'^2}{\partial t}$

Boussinesq approximation (2)

Again, assuming that the small scale dissipation just balances the forcing, we have an equation of scalar advection, only this time directly for the fluctuation θ'. Note that in this model since the stratification is accounted for in the $\theta_0(z)$ function, the fluctuation θ' is considered isotropic even when the overall fluid is strongly stratified; this is an example of "locally isotropic turbulence" (Kolmogorov, 1941). It has a new term due to the buoyancy force, as does the velocity equation:

$$\frac{D\underline{v}}{Dt} = -\frac{1}{\rho_0}\nabla p' - \boxed{g\frac{\theta'}{\theta_0}\hat{z}} - 2\underline{\Omega}\times\underline{v} + \nu\nabla^2\underline{v}$$

All Buoyancy effects

where $\widehat{k}$ is a unit vertical vector and p' is the pressure fluctuation analogous to θ': $p(\underline{r},t) = p_0(\underline{r}) + p'(\underline{r},t)$, and $\rho_0(z)$ is the corresponding mean density function. The direct effects of gravity are thus confined to the single term Δf in the equation for the vertical component:

$$\Delta f = g\frac{\theta'}{\theta_0}; \quad \boxed{f = g\log\theta}$$

where the fluctuations Δf are thus responsible for the buoyancy effects.

First way of accounting for buoyancy (the classical isotropic "buoyancy subrange")

First consider the original classical approach based on potential temperature variance flux:

$$\boxed{\chi_\theta = \frac{\partial \theta'^2}{\partial t}}$$

which is taken as a fundamental cascade quantity along with ε. In order to obtain the scaling laws for the velocity field in a fluid dominated by buoyancy forces (i.e. in a hypothetical isotropic "buoyancy subrange" where the energy flux ε can be neglected) we then argue that only χ_θ (with units K^2/s) and the coupling constant g/θ_0 (with units $m^2/K/s$) between the fluctuation θ' and the velocity field are dimensionally relevant. Dimensional analysis on χ_θ and g/θ_0 then yields the unique scaling "Bolgiano-Obukhov" (BO) law:

$$\Delta v(\underline{\Delta r}) \approx \chi_\theta^{1/5} \left(\frac{g}{\theta_0}\right)^{2/5} |\underline{\Delta r}|^{3/5}$$

corresponding to a $k^{-11/5}$ spectrum (neglecting intermittency; i.e. using $\beta = 1+2H$, section 2.5). In the context of the Boussinesq approximation, this isotropic law applies to the fluctuations in the velocity about a totally stratified anisotropic mean state $\theta_0(z)$.

The isotropic BO law and BO length scale

Staying within this classical framework for isotropic fluctuations, we may now inquire as to over what scale range this new BO law should apply given that it is in competition with the usual energy flux dominated regime. In other words, what happens when we apply a full dimensional analysis to ε, χ_θ, and g/θ_0? The answer is that there is a unique "Bolgiano-Obukhov" length scale L_{BO}:

$$L_{BO} = \frac{\varepsilon^{5/4}}{\chi_\theta^{3/4}\left(g/\theta_0\right)^{3/2}}$$

Order of 1 m

According to this classical theory, we see that as the effect of gravity is reduced ($g \to 0$), $L_{BO} \to \infty$, so that the stratification disappears and we recover the usual isotropic 3D Kolmogorov law (i.e. dominated by ε). Therefore, we interpret the scale L_{BO}

Isotropic Kolmogorov turbulence: $L<L_{BO}$

Isotropic BO turbulence for scales $L>L_{BO}$.

Second way of accounting for buoyancy:

The Anisotropic scaling theory: the 23/9D Kolmogorov-Bolgiano-Obukhov model

A second way to approach buoyancy driven turbulence is to make a more physically based argument (which essentially avoids the Boussinesq and other approximations), noting that the v and θ fields are only coupled by the Δf buoyancy force term $\underline{(f = g\log\theta)}$ so f is the fundamental physical and dimensional quantity rather than θ.

$$\frac{D\log\theta}{Dt} = \frac{\kappa}{T}\nabla^2 T + \frac{\dot{Q}}{c_v T}$$

Hence, neglecting sources and dissipation:

$$Df\,/\,Dt \approx 0$$

Scalar advection equation

f obeys a passive scalar advection equation and therefore the corresponding buoyancy force variance flux:

Hence the following flux is conserved

$$\varphi = \frac{\partial f^2}{\partial t}$$

Compare with: $\chi_\theta = \dfrac{\partial \theta'^2}{\partial t}$

is conserved by the nonlinear terms. In this case, the only quantities available for dimensional analysis are ε (units m^2/s^3) and φ (units m^2/s^5), not ε, χ_θ and g/θ_0. In this approach, there is no separation between a stratified "background" state and a possibly isotropic fluctuation field so that there is no rationale for assuming that the φ cascade is associated with any isotropic regime.

Stratified scaling turbulence

The two basic turbulent fluxes ε, φ can co-exist and cascade over a single wide range regime with the former dominating in the horizontal, the latter in the vertical:

$$\Delta v(\Delta x)=\phi_h \Delta x^{H_h};\quad \phi_h=\varepsilon^{1/3};\quad H_h=1/3$$

$$\Delta v(\Delta z)=\phi_v \Delta z^{H_v};\quad \phi_v=\varphi^{1/5};\quad H_v=3/5$$

where Δx is a horizontal and Δz a vertical lag (for the moment we ignore the other horizontal coordinate y). Again, the fluxes ε, φ dimensionally define a unique length scale l_s:

$$l_s=\left(\frac{\phi_h}{\phi_v}\right)^{1/(H_v-H_h)}=\frac{\varepsilon^{5/4}}{\varphi^{3/4}}$$

The above laws can be expressed as: $$\Delta v(\Delta x,\Delta z)\overset{s}{=}\phi_h \left\|(\Delta x,\Delta z)\right\|^{H_h}$$

With “canonical” scale function: $$\left\|(\Delta x,\Delta z)\right\|=l_s\left(\left(\frac{\Delta x}{l_s}\right)^2+\left(\frac{\Delta z}{l_s}\right)^{2/H_z}\right)^{1/2}$$

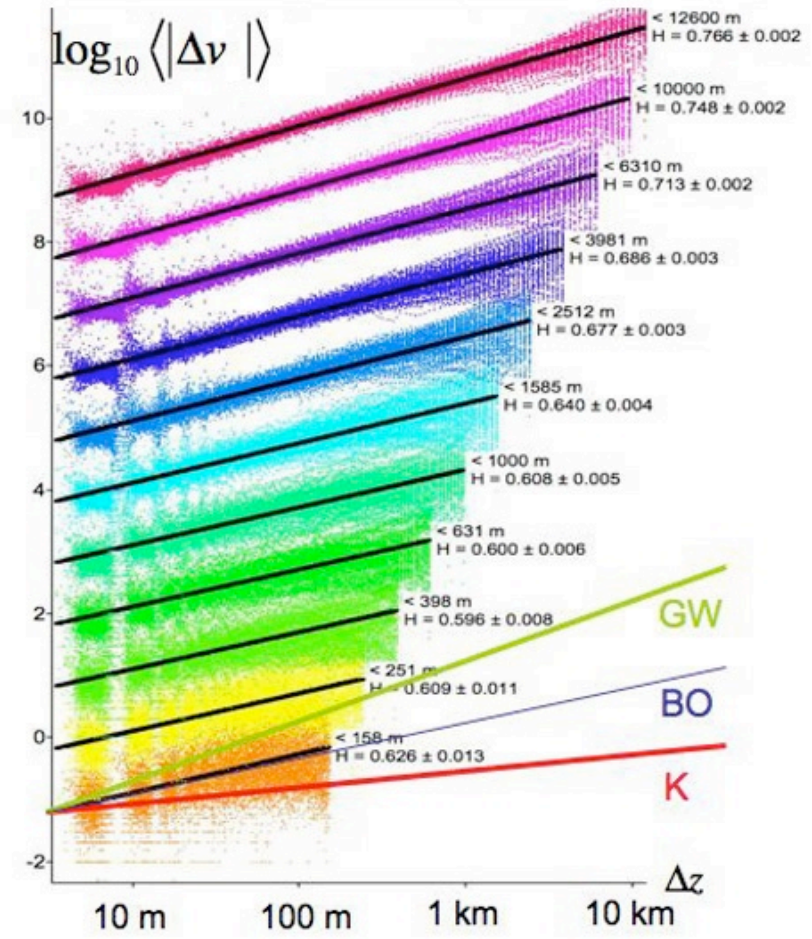
$$H_z=H_h/H_v=(1/3)/(3/5)=5/9$$

Check:

$$\Delta v(\Delta x)=\Delta v(\Delta x,0)\overset{s}{=}\phi_h l_s^{H_h}\left(\frac{\Delta x}{l_s}\right)^{H_h}=\varepsilon^{1/3}\Delta x^{1/3}$$

$$\Delta v(\Delta z)=\Delta v(0,\Delta z)\overset{s}{=}\phi_h\left\|(0,\Delta z)\right\|^{H_h}=\phi_h l_s^{H-H/H_z}\Delta z^{H_h/H_z}=\varphi^{1/5}\Delta z^{3/5}$$

The empirical status of the 23/9D model



Mean absolute vertical gradients of horizontal wind (first order structure functions) for layers of thickness increasing logarithmically, with (black) regression lines added. The (coloured) reference lines have slopes H_v=1/3 (Kolmogorov), H_v=3/5 (Bolgiano-Obukhov), H_v=1 (gravity waves). The regression H_v estimates are given next to the lines. The data for each level are offset by one order of magnitude for clarity, units *m/s*.

Local, trivial and scaling anisotropy

Local Isotropy:

The isotropic BO law assumes “local isotropy”= isotropy of fluctuations around a totally stratified “background” state

$$\left\|\left(\Delta x,\Delta z\right)\right\|=\left(\left(a\Delta z\right)^{2}+\left(\left|b\left(\Delta x,\Delta z\right)\right|^{2}\right)^{H}\right)^{1/2}$$

Local isotropy assumes a scale break and features two different scaling regimes.

Trivial anisotropy

$$\left\|\left(\Delta x,\Delta z\right)\right\|=\left(\left(a\Delta z\right)^{2}+\left(b\left|\left(\Delta x,\Delta z\right)\right|^{2}\right)\right)^{1/2}$$

Ellipses with constant eccentricity

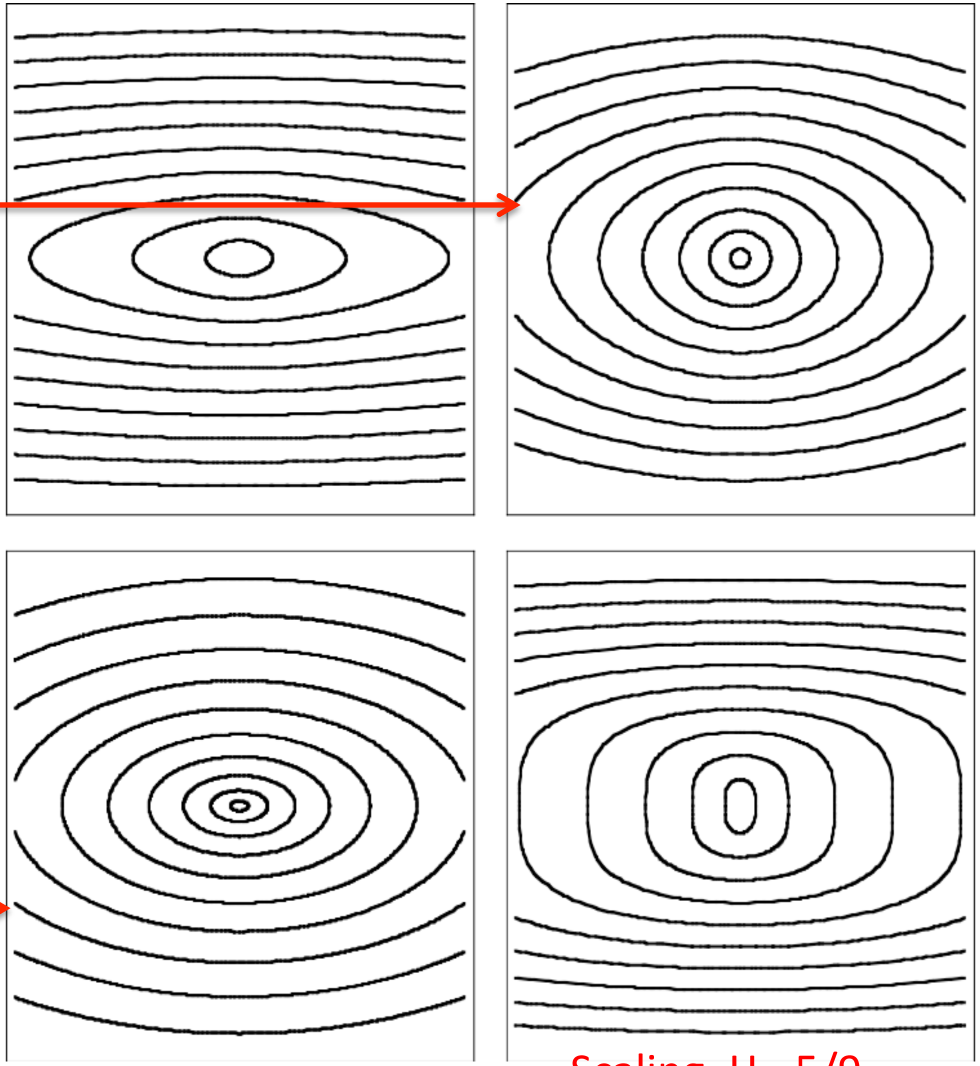
Scaling anisotropy

(Nondimensional scale functions)

$$\left\|\left(\Delta x,\Delta z\right)\right\|=\left(\left(\frac{\Delta x}{l_{s}}\right)^{2}+\left(\frac{\Delta z}{l_{s}}\right)^{2/H_{z}}\right)^{1/2}$$

Ellipses with eccentricity changing with scale

Local, trivial and scaling anisotropy



local

X25

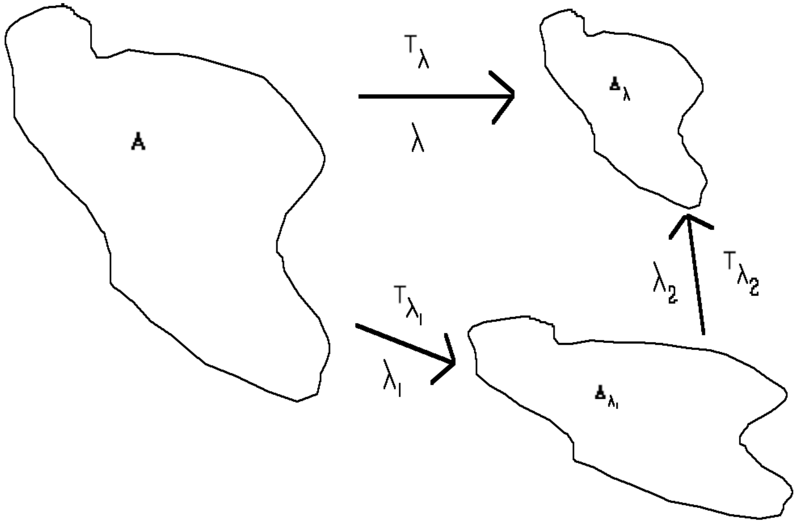
Trivial
(constant eccentricity)

Scaling, H_z=5/9

Upper left, local isotropy with broken symmetry, (the upper right contours are the same but with a factor 25 blow-up showing small scale “local” isotropy). The bottom left contours show trivial anisotropy with a = 1.7, b =1, the bottom right contours showing scaling anisotropy with H_z =5/9 with l_s = 1. The range displayed is $-5<\Delta x<5$, $-5<\Delta z<5$ (except for the 25x blow-up, upper right, the range is 25 times smaller).

Generalized Scale Invariance

The scale changing operator T_λ which transforms the scale of vectors by scale ratio λ



T_λ is the rule relating the statistical properties at one scale to another and involves only the scale ratio. This implies that T_λ has certain properties. In particular, if and only if $\lambda_1\lambda_2 = \lambda$, then:

$$B_\lambda = T_\lambda B_1 = T_{\lambda_1\lambda_2} B_1 = T_{\lambda_1} B_{\lambda_2} = T_{\lambda_2} B_{\lambda_1}$$

it is also commutative $$T_\lambda = T_{\lambda_2} T_{\lambda_1} = T_{\lambda_1} T_{\lambda_2}$$

This implies that T_λ is a one parameter multiplicative group with parameter λ: $$\boxed{T_\lambda = \lambda^{-G}}$$

One parameter Lie group, G= generator

The Elements of GSI

T_λ is a generalized contraction on a vector space E, it is a one-parameter (semi-) group for the positive real scale ratio λ ($\lambda \geq 1$ for a semi-group), i.e.:

$$\forall \lambda, \lambda' \in R^+ : T_{\lambda'} \circ T_\lambda = T_{\lambda'\lambda}$$

and admits a generalized scale denoted $\|\underline{r}\|$ (double lines to distinguish it from the usual Euclidean metric $|\underline{r}|$), which in addition to being nonnegative, satisfies the following three properties:

i) *Nondegeneracy*: $$\|\underline{r}\| = 0 \Leftrightarrow \underline{r} = \underline{0}$$

ii) *Linearity* with the contraction parameter 1/λ:

$$\forall \underline{x} \in E, \forall \lambda \in R^+ : \tilde{T}_\lambda \|\underline{r}\| \equiv \left\| \underline{\underline{T}}_\lambda \, \underline{r} \right\| = \lambda^{-1} \|\underline{r}\|$$

iii) *Strictly decreasing balls*: the balls defined by this scale

$$B_\ell = \{ \underline{r} \,|\, \|\underline{r}\| \leq \ell \}$$

Anisotropic Hausdorff measures

must be strictly decreasing with the contraction: $$\forall L \in R^+, \forall \lambda > 1 : B_{L/\lambda} \equiv T_\lambda(B_L) \subset B_L$$

and therefore: $$\forall L \in R^+, \forall \lambda' \geq \lambda \geq 1 : B_{L/\lambda'} \subset B_{L/\lambda}$$

Isotropic special case

The usual Euclidean norm $|\underline{r}|$ of a metric space is the scale associated with the isotropic contraction $T_\lambda \underline{r} = \underline{r} / \lambda$. Properties *i*, *ii* are rather identical to those of a norm, whereas the last one is weaker than the triangle inequality, which is required for a norm. As for norms, unicity of generalized scale is not expected for a given T_λ.

The effect of $T_\lambda=\lambda^{-G}$

(Linear GSI, matrix G)

If *G* is an *nxn* matrix and $\underline{r}$ is an *n*-dimensional vector and (the usual case) that *G* is diagonalizable with transformation matrix Ω:

$$G' = \Omega^{-1} G \Omega$$

so that *G*' is diagonal (with eigenvalues Λ_i, with *i* = 1…, *n*), then we obtain:

$$\lambda^{-G} = \Omega \lambda^{-G'} \Omega^{-1} = \Omega \begin{pmatrix} \lambda^{-\Lambda_1} & 0 & \ldots & 0 \\ 0 & \lambda^{-\Lambda_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \lambda^{-\Lambda_n} \end{pmatrix} \Omega^{-1}$$

Hint: use $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ To show $\lambda^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^b \end{pmatrix}$

True for all diagonal matrices

Example of anisotropic "Blow down"

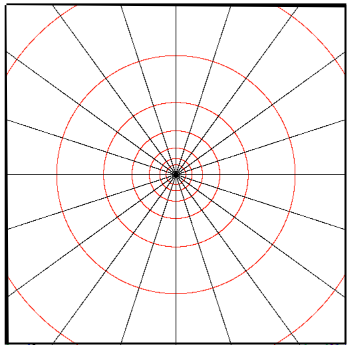
$$T_\lambda = \lambda^{-G}$$

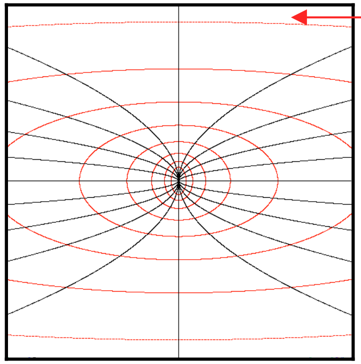
A generalized blow-down with increasing of the acronym "NVAG". If *G* = *I*, we would have obtained a standard reduction, with all the copies uniformly reduced converging to the centre of the reduction. Here the parameters are $G = \begin{pmatrix} 1.3 & -1.3 \\ 0.3 & 0.7 \end{pmatrix}$

and each successive reduction is by 28%.

Scale functions in linear GSI
(position independent)

Isotropic
(self similar)





Scale isolines in red

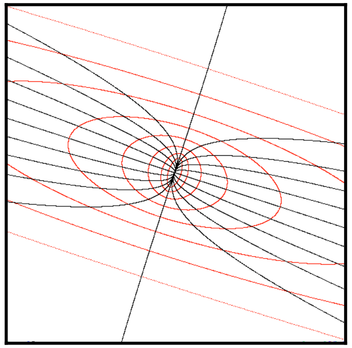
Self-affine

$$T_\lambda = \lambda^{-G}$$

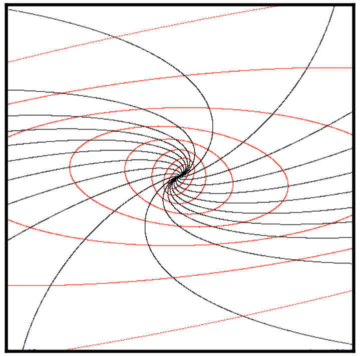
$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & 0 \\ 0 & 0.65 \end{pmatrix}$$

Stratification
dominant (real
eigenvalues)



$$G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}$$



Rotation
dominant
(complex
eigenvalues)

$$G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}$$

Some properties of linear GSI

First, it is convenient to decompose *G* into “pseudo-quaternions” (or equivalently, Pauli matrices):

$$\mathrm{G} = \mathrm{d}\mathbf{1} + \mathrm{e}\mathbf{I} + \mathrm{f}\mathbf{J} + \mathrm{c}\mathbf{K} \longleftrightarrow G = \begin{pmatrix} d-c & f-e \\ f+e & d+c \end{pmatrix}$$

Where:

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

These matrices satisfy the following anticommutation relations:

$$\{\mathbf{I},\mathbf{J}\} = 0, \quad \{\mathbf{I},\mathbf{K}\} = 0, \quad \{\mathbf{J},\mathbf{K}\} = 0$$

(Important for Lie algebra)

Equation for the eigenvalues Λ $|G-\Lambda 1| = \begin{pmatrix} d-c-\Lambda & f-e \\ f+e & d+c-\Lambda \end{pmatrix} = 0$ $\qquad (d-\Lambda)^2 = c^2 + f^2 - e^2 = a^2$

hence $\Lambda_x = d + a; \quad \Lambda_y = d - a$ Eigenvectors: $(G-\Lambda 1)\underline{r} = 0$

are real or complex, for clarity we have used the notation $\Lambda_x = \Lambda_1$, $\Lambda_y = \Lambda_2$.

Or, in terms of the trace and determinant:

$$\det G = d^2 - a^2 \qquad TrG = 2d \qquad a^2 = (Tr(G)/2)^2 - Det(G) = c^2 + f^2 - e^2$$

The condition of decrease of the balls for *G* is merely:

$$Tr(G) > 0 \; and \; \det(G) > 0 \Leftrightarrow d > 0 \; and \; d^2 > a^2$$

Check: $T_\lambda = \lambda^{-G} = \Omega^{-1} \begin{pmatrix} \lambda^{-(d+a)} & 0 \\ - & \lambda^{-(d-a)} \end{pmatrix} \Omega$

To be always decreasing, need d>0, $d| > a|$

Defining the balls by an implicit equation

From the above, we can see that if we define a unit scale (the vectors such that $\|\underline{r}\| = 1$), this defines the borders of a unit ball B_1 and using the scale changing operator, this defines the other (nonunit) scales and balls. In general, the unit ball B_1 will be defined by an implicit equation:

$$B_1 = \{\underline{r} : f_1(\underline{r}) < 1\};$$
$$\partial B_1 = \{\underline{r} : f_1(\underline{r}) = 1\}$$

Unit ball and “frontier”

where ∂B_1 is the "frontier" of the unit ball, and f_1 is a function of position ($\underline{r}$) (the use of open balls has the advantage that they generate a topology of the space. Comparing, we see that $f_1(\underline{r}) = \|\underline{r}\|$.

A technical consideration in constructing a viable GSI system is that the corresponding balls B_λ must be decreasing ($B_{\lambda_1} \subset B_{\lambda_2};\quad \lambda_1 > \lambda_2$): this is necessary to insure that the vectors $\underline{r}_\lambda$ are unique. Let us consider this question in more detail by introducing the function f_λ to define B_λ. From eqs. 3, 7 we can define B_λ from the function f_λ defined as:

$$\forall \underline{r} : f_\lambda(\underline{r}) = f_1\left(T_\lambda^{-1}\underline{r}\right),$$

$$B_\lambda = \{\underline{r} : f_\lambda(\underline{r}) < 1\}$$
$$\partial B_\lambda = \{\underline{r} : f_\lambda(\underline{r}) = 1\}$$

The other balls

Alternatively, in terms of the scale function, we have: $f_\lambda(\underline{r}) = \left\|T_\lambda^{-1}\underline{r}\right\|$. In order to ensure that the frontiers of the balls do not cross and that the scale is thus uniquely defined, consider the balls defined by λ and $\lambda+d\lambda$; it is easy to see that since a crossing point satisfies $\partial f_\lambda / \partial\lambda = 0$ we must have:

$$\frac{\partial f_\lambda}{\partial \lambda} > 0$$

← No crossing (uniqueness of scale)

for all $\underline{r}$. This positivity (rather than negativity) requirement is necessary to ensure that T_λ corresponds to a scale reduction rather than enlargement.

The condition that the balls do not cross (uniqueness of scale)

Make the transformation of variables:

$$u = \log \lambda \qquad \frac{\partial}{\partial \lambda} = \frac{\partial u}{\partial \lambda}\frac{\partial}{\partial u} = e^{-u}\frac{\partial}{\partial u}$$

The implicit equations for two neighbouring balls is thus:

$$f_{u+du}(\underline{r}) = f_u(\underline{r}) + \frac{\partial f_u}{\partial u} du$$

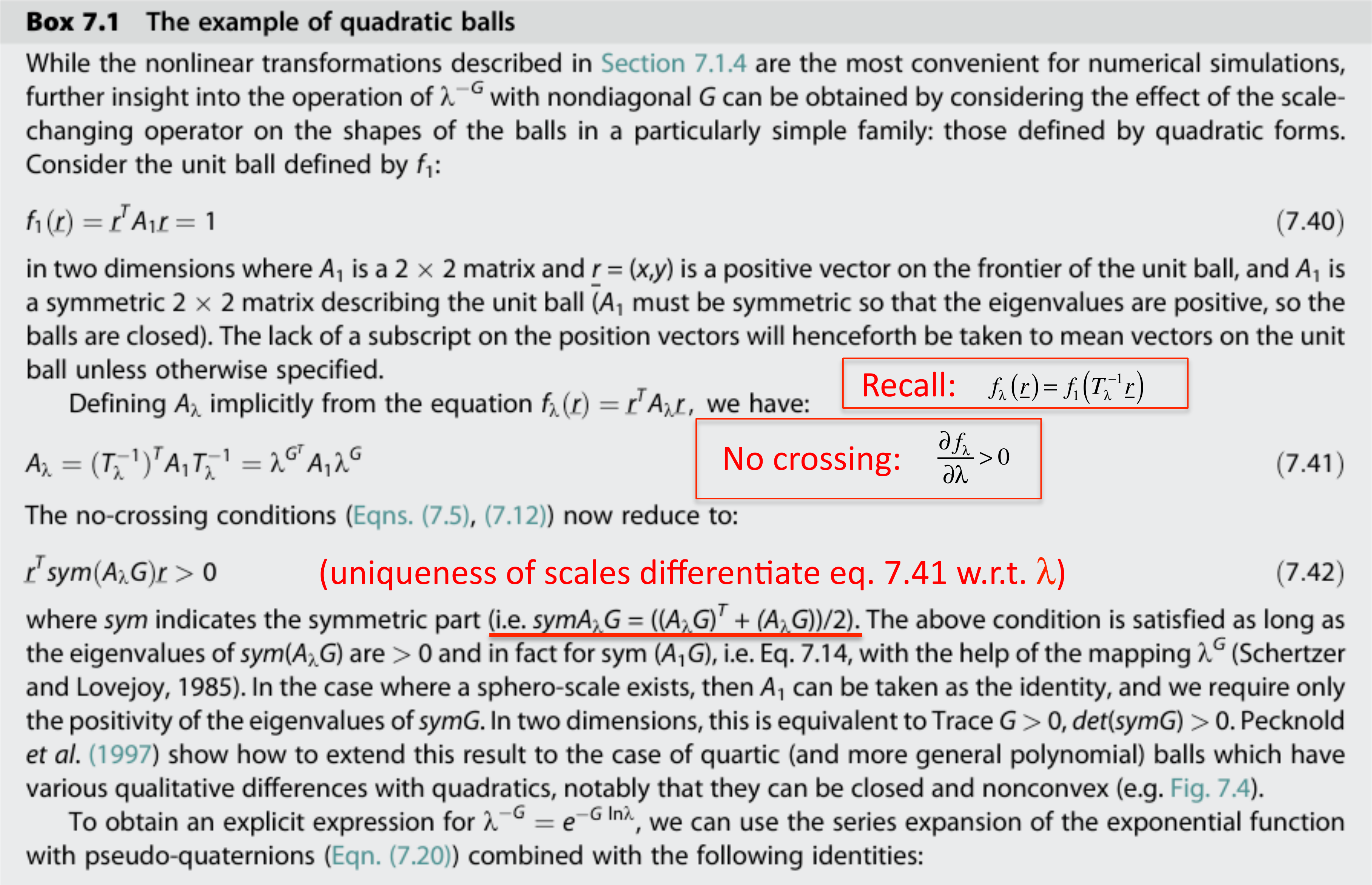
If a vector r exists that satisfies both

$$f_u(\underline{r}) = 1; \quad f_{u+du}(\underline{r}) = 1$$

Then the balls cross at that point.
Therefore, the balls will not cross if for all r̲, u: $\frac{\partial f_u}{\partial u} > 0$ Hence: $e^u \frac{\partial f_u}{\partial u} > 0$

But: $e^u \frac{\partial f_u}{\partial u} = \frac{\partial f_u}{\partial \lambda} = \frac{\partial f_\lambda}{\partial \lambda}$ Hence: $\frac{\partial f_\lambda}{\partial \lambda} > 0$

Quadratic Balls (1)



Box 7.1 The example of quadratic balls

While the nonlinear transformations described in Section 7.1.4 are the most convenient for numerical simulations, further insight into the operation of λ^{-G} with nondiagonal G can be obtained by considering the effect of the scale-changing operator on the shapes of the balls in a particularly simple family: those defined by quadratic forms. Consider the unit ball defined by f_1:

$$f_1(\underline{r}) = \underline{r}^T A_1 \underline{r} = 1 \tag{7.40}$$

in two dimensions where A_1 is a 2×2 matrix and $\underline{r} = (x,y)$ is a positive vector on the frontier of the unit ball, and A_1 is a symmetric 2×2 matrix describing the unit ball (A_1 must be symmetric so that the eigenvalues are positive, so the balls are closed). The lack of a subscript on the position vectors will henceforth be taken to mean vectors on the unit ball unless otherwise specified.

Defining A_λ implicitly from the equation $f_\lambda(\underline{r}) = \underline{r}^T A_\lambda \underline{r}$, we have:

Recall: $f_\lambda(\underline{r}) = f_1\left(T_\lambda^{-1}\underline{r}\right)$

$$A_\lambda = (T_\lambda^{-1})^T A_1 T_\lambda^{-1} = \lambda^{G^T} A_1 \lambda^G \tag{7.41}$$

No crossing: $\frac{\partial f_\lambda}{\partial \lambda} > 0$

The no-crossing conditions (Eqns. (7.5), (7.12)) now reduce to:

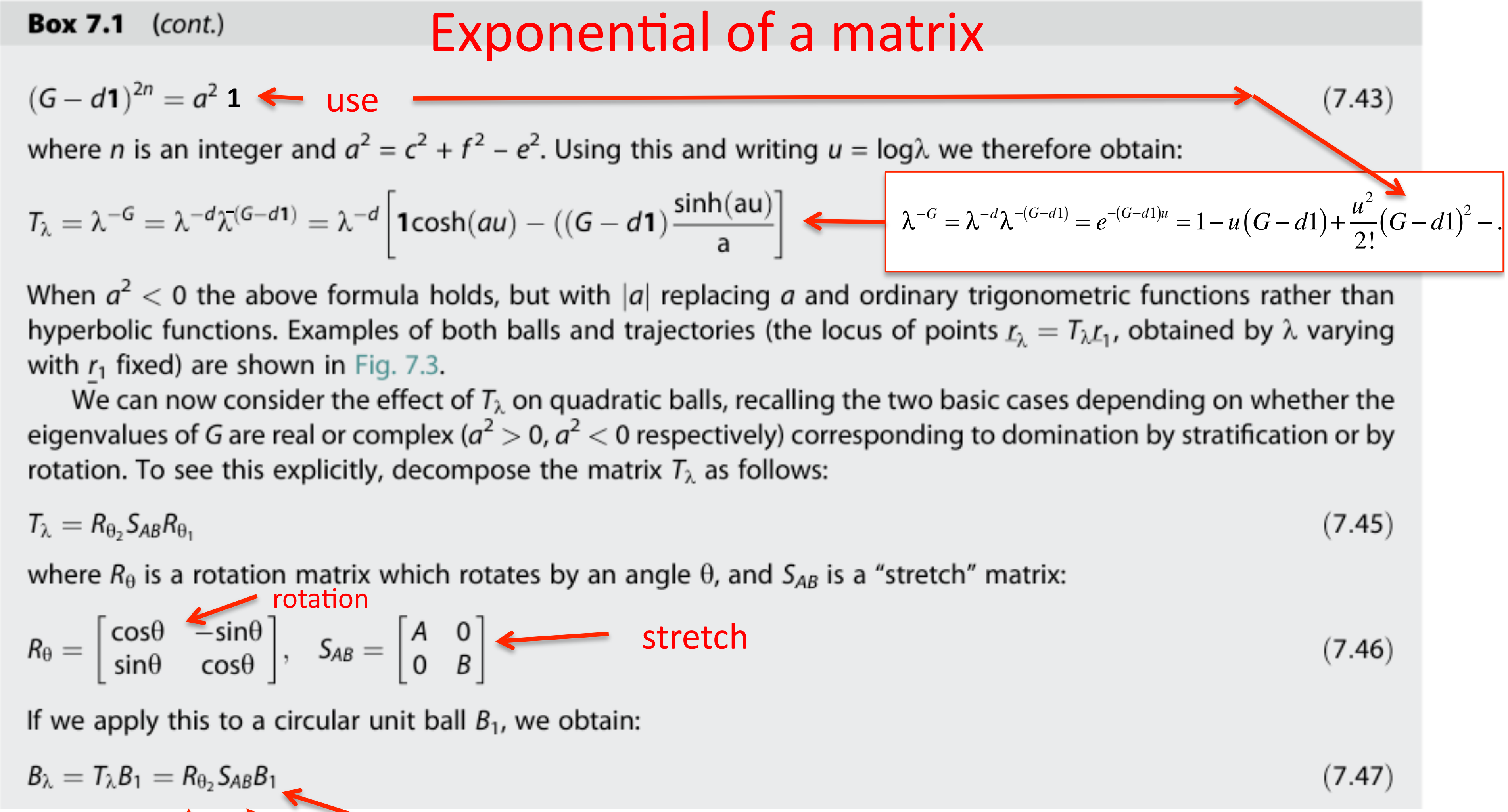
$$\underline{r}^T sym(A_\lambda G)\underline{r} > 0 \tag{7.42}$$

(uniqueness of scales differentiate eq. 7.41 w.r.t. λ)

where *sym* indicates the symmetric part (i.e. $symA_\lambda G = ((A_\lambda G)^T + (A_\lambda G))/2$). The above condition is satisfied as long as the eigenvalues of $sym(A_\lambda G)$ are > 0 and in fact for sym $(A_1 G)$, i.e. Eq. 7.14, with the help of the mapping λ^G (Schertzer and Lovejoy, 1985). In the case where a sphero-scale exists, then A_1 can be taken as the identity, and we require only the positivity of the eigenvalues of *symG*. In two dimensions, this is equivalent to Trace $G > 0$, $det(symG) > 0$. Pecknold *et al.* (1997) show how to extend this result to the case of quartic (and more general polynomial) balls which have various qualitative differences with quadratics, notably that they can be closed and nonconvex (e.g. Fig. 7.4).

To obtain an explicit expression for $\lambda^{-G} = e^{-G\ln\lambda}$, we can use the series expansion of the exponential function with pseudo-quaternions (Eqn. (7.20)) combined with the following identities:

Quadratic Balls (2)



Box 7.1 *(cont.)*

Exponential of a matrix

$$(G - d\mathbf{1})^{2n} = a^2\,\mathbf{1} \qquad (7.43)$$

use

where n is an integer and $a^2 = c^2 + f^2 - e^2$. Using this and writing $u = \log\lambda$ we therefore obtain:

$$T_\lambda = \lambda^{-G} = \lambda^{-d}\lambda^{-(G-d\mathbf{1})} = \lambda^{-d}\left[\mathbf{1}\cosh(au) - ((G - d\mathbf{1})\frac{\sinh(au)}{a}\right]$$

$$\lambda^{-G} = \lambda^{-d}\lambda^{-(G-d1)} = e^{-(G-d1)u} = 1 - u(G-d1) + \frac{u^2}{2!}(G-d1)^2 - .$$

When $a^2 < 0$ the above formula holds, but with $|a|$ replacing a and ordinary trigonometric functions rather than hyperbolic functions. Examples of both balls and trajectories (the locus of points $\underline{r}_\lambda = T_\lambda \underline{r}_1$, obtained by λ varying with $\underline{r}_1$ fixed) are shown in Fig. 7.3.

We can now consider the effect of T_λ on quadratic balls, recalling the two basic cases depending on whether the eigenvalues of G are real or complex ($a^2 > 0$, $a^2 < 0$ respectively) corresponding to domination by stratification or by rotation. To see this explicitly, decompose the matrix T_λ as follows:

$$T_\lambda = R_{\theta_2} S_{AB} R_{\theta_1} \qquad (7.45)$$

where R_θ is a rotation matrix which rotates by an angle θ, and S_{AB} is a "stretch" matrix:

rotation

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad S_{AB} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad (7.46)$$

stretch

If we apply this to a circular unit ball B_1, we obtain:

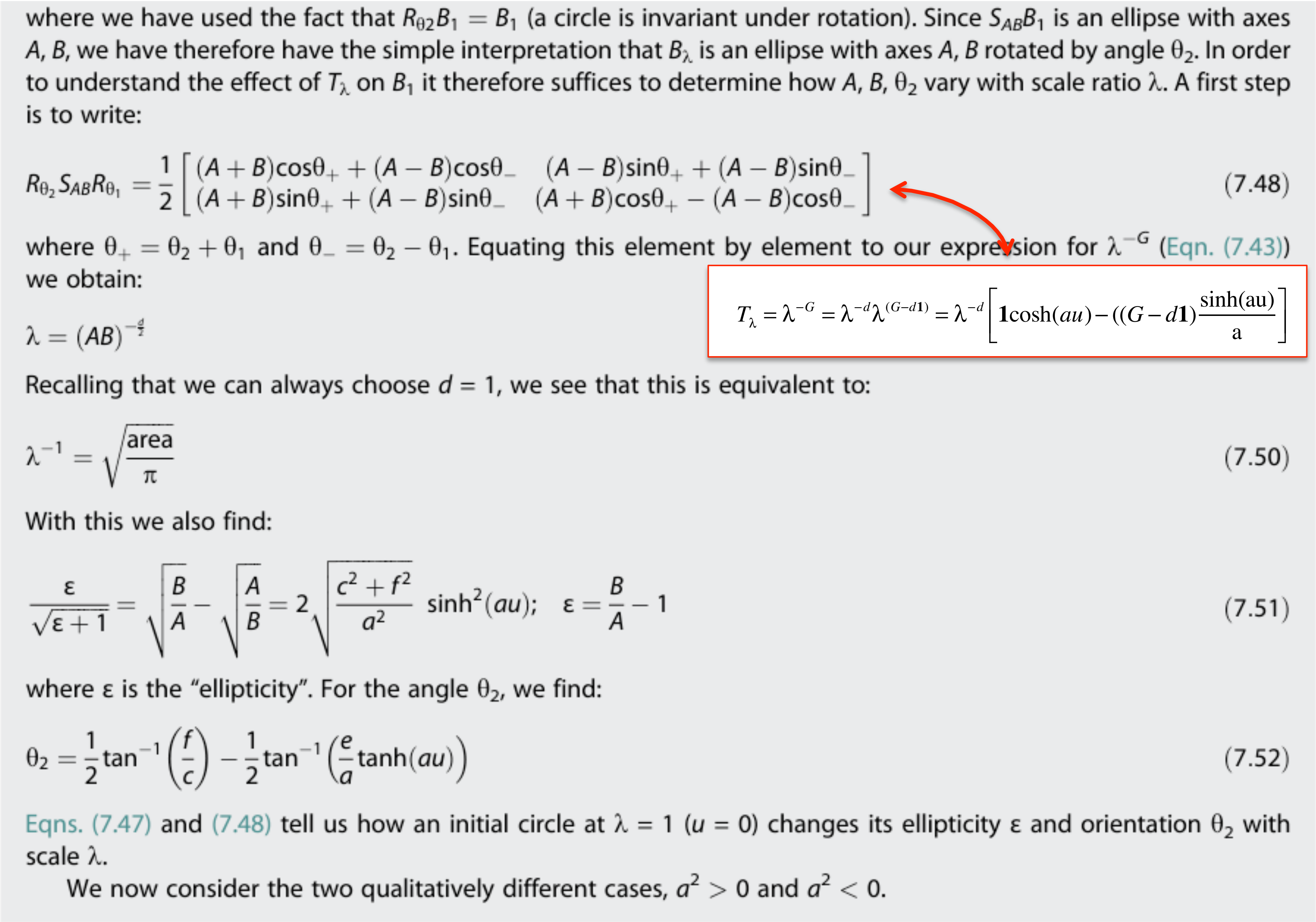
$$B_\lambda = T_\lambda B_1 = R_{\theta_2} S_{AB} B_1 \qquad (7.47)$$

circle

Creates an ellipse

Creates a rotated ellipse

Quadratic Balls (3)



where we have used the fact that $R_{\theta 2}B_1 = B_1$ (a circle is invariant under rotation). Since $S_{AB}B_1$ is an ellipse with axes A, B, we have therefore have the simple interpretation that B_λ is an ellipse with axes A, B rotated by angle θ_2. In order to understand the effect of T_λ on B_1 it therefore suffices to determine how A, B, θ_2 vary with scale ratio λ. A first step is to write:

$$R_{\theta_2}S_{AB}R_{\theta_1} = \frac{1}{2}\begin{bmatrix} (A+B)\cos\theta_+ + (A-B)\cos\theta_- & (A-B)\sin\theta_+ + (A-B)\sin\theta_- \\ (A+B)\sin\theta_+ + (A-B)\sin\theta_- & (A+B)\cos\theta_+ - (A-B)\cos\theta_- \end{bmatrix} \qquad (7.48)$$

where $\theta_+ = \theta_2 + \theta_1$ and $\theta_- = \theta_2 - \theta_1$. Equating this element by element to our expression for λ^{-G} (Eqn. (7.43)) we obtain:

$$T_\lambda = \lambda^{-G} = \lambda^{-d}\lambda^{(G-d\mathbf{1})} = \lambda^{-d}\left[\mathbf{1}\cosh(au) - ((G-d\mathbf{1})\frac{\sinh(\mathrm{au})}{\mathrm{a}}\right]$$

$$\lambda = (AB)^{-\frac{d}{2}}$$

Recalling that we can always choose $d = 1$, we see that this is equivalent to:

$$\lambda^{-1} = \sqrt{\frac{\text{area}}{\pi}} \qquad (7.50)$$

With this we also find:

$$\frac{\varepsilon}{\sqrt{\varepsilon+1}} = \sqrt{\frac{B}{A}} - \sqrt{\frac{A}{B}} = 2\sqrt{\frac{c^2+f^2}{a^2}}\sinh^2(au); \quad \varepsilon = \frac{B}{A} - 1 \qquad (7.51)$$

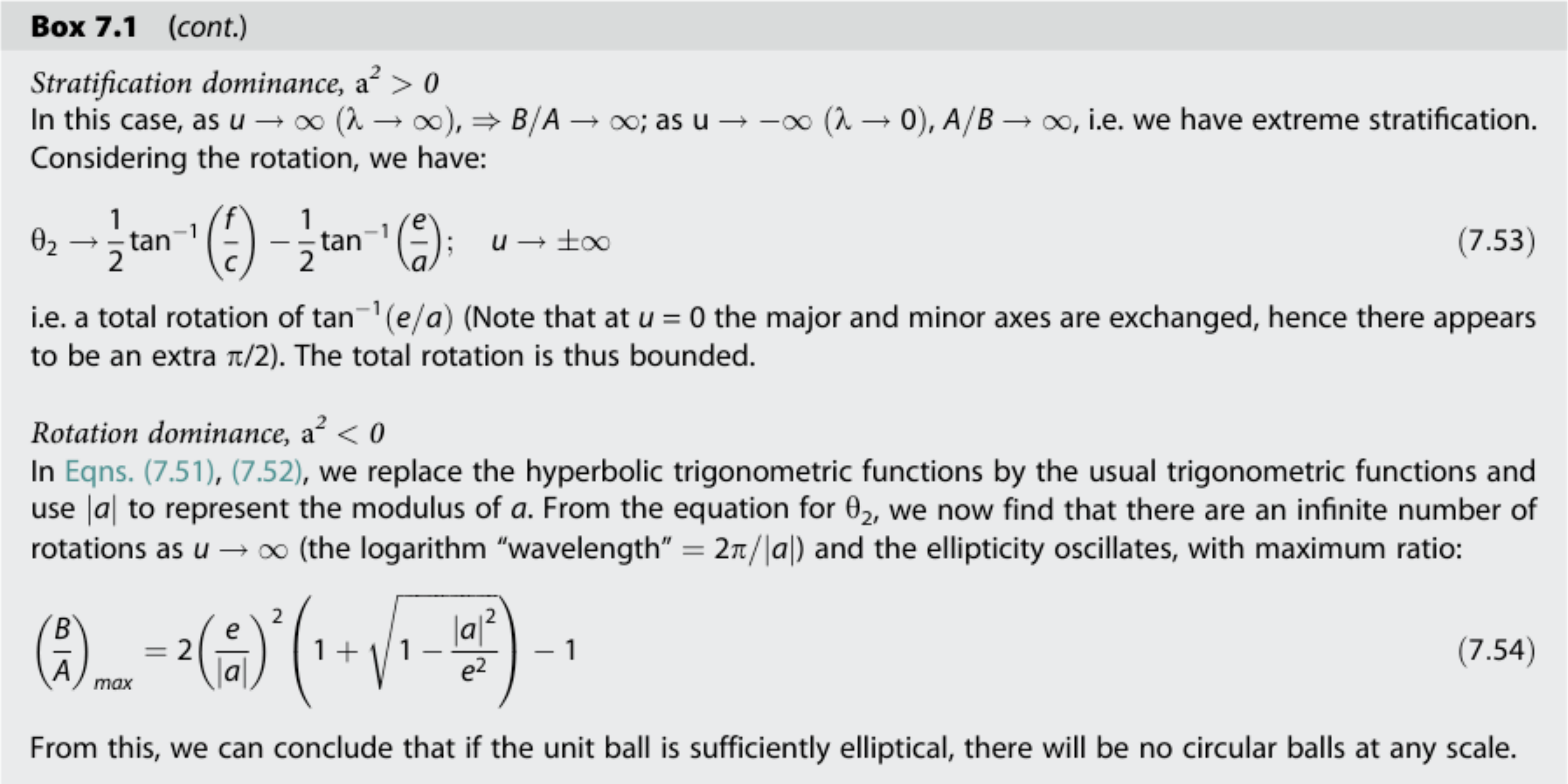
where ε is the "ellipticity". For the angle θ_2, we find:

$$\theta_2 = \frac{1}{2}\tan^{-1}\left(\frac{f}{c}\right) - \frac{1}{2}\tan^{-1}\left(\frac{e}{a}\tanh(au)\right) \qquad (7.52)$$

Eqns. (7.47) and (7.48) tell us how an initial circle at $\lambda = 1$ ($u = 0$) changes its ellipticity ε and orientation θ_2 with scale λ.

We now consider the two qualitatively different cases, $a^2 > 0$ and $a^2 < 0$.

Quadratic Balls (4)



Box 7.1 *(cont.)*

Stratification dominance, $a^2 > 0$

In this case, as $u \to \infty$ ($\lambda \to \infty$), $\Rightarrow B/A \to \infty$; as $u \to -\infty$ ($\lambda \to 0$), $A/B \to \infty$, i.e. we have extreme stratification. Considering the rotation, we have:

$$\theta_2 \to \frac{1}{2}\tan^{-1}\left(\frac{f}{c}\right) - \frac{1}{2}\tan^{-1}\left(\frac{e}{a}\right); \quad u \to \pm\infty \tag{7.53}$$

i.e. a total rotation of $\tan^{-1}(e/a)$ (Note that at $u = 0$ the major and minor axes are exchanged, hence there appears to be an extra $\pi/2$). The total rotation is thus bounded.

Rotation dominance, $a^2 < 0$

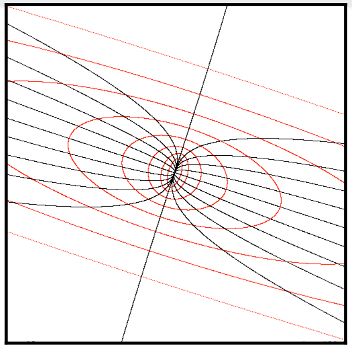
In Eqns. (7.51), (7.52), we replace the hyperbolic trigonometric functions by the usual trigonometric functions and use $|a|$ to represent the modulus of a. From the equation for θ_2, we now find that there are an infinite number of rotations as $u \to \infty$ (the logarithm "wavelength" $= 2\pi/|a|$) and the ellipticity oscillates, with maximum ratio:

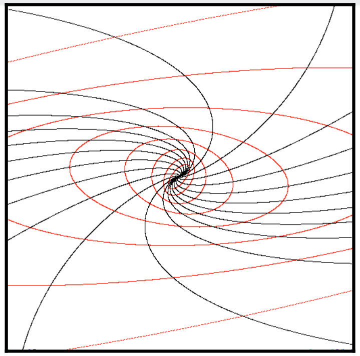
$$\left(\frac{B}{A}\right)_{max} = 2\left(\frac{e}{|a|}\right)^2\left(1 + \sqrt{1 - \frac{|a|^2}{e^2}}\right) - 1 \tag{7.54}$$

From this, we can conclude that if the unit ball is sufficiently elliptical, there will be no circular balls at any scale.

Stratification dominant
(real eigenvalues)

$$G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}$$





Rotation dominant
(complex
eigenvalues)

$$G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}$$

GSI morphologies (1): Rotational invariants

Clearly parameters which simply rotate structures or which give them isotropic dilations do not change the morphologies so that of the four parameters *c,d,e,f* therefore only two need to be considered. For example, we are interested in characteristics of *G* which are independent of absolute orientation. Therefore, consider the primed coordinate system rotated by angle θ:

$$G' = R^{-1}GR$$
$$R = 1\cos\theta + I\sin\theta$$

where R is the rotation matrix. Clearly, the 1 and I components of G commute with R so that they are unaffected by the rotation; hence d, e are rotationally invariant. In addition, both the trace (=2d) and determinant (=d^2-a^2) are rotational invariants (alternatively, and equivalently, the eigenvalues $d\pm a$ are invariant) so that we have d, e, a as rotational invariants. We therefore may conclude that since $a^2+e^2 = f^2+c^2$ that the latter is also invariant; let us define $r = \sqrt{c^2 + f^2}$.

Trace and determinant Rotational invariants:

Any matrices A,B,C: $Tr(ABC) = Tr(CAB)$ hence $TrG' = Tr\left(R^{-1}GR\right) = Tr\left(RR^{-1}G\right) = Tr(G)$

Also $\det(ABC) = \det(A)\det(B)\det(C)$ hence $\det(G') = \det\left(R^{-1}GR\right) = \det\left(R^{-1}\right)\det(G)\det(R) = \det\left(R^{-1}R\right)\det(G) = \det(G)$

GSI morphologies (2): Interpretation in term of eigenvectors

The eigenvalues of *G* are *d*±*a* and the (unnormalized) eigenvectors are: $\left(\frac{c\pm a}{f+e},1\right)$

Difference in angle ($\Delta\theta$) between the eigenvectors is given by: $\cos(\Delta\theta)=\cos(\theta_+-\theta_-)=\frac{e}{r}$ (rotationally invariant). When *e*<*r* and as *e* approaches *r*, the eigenvalues are real and the eigenvectors become more and more parallel; for *e*>*r*, they become complex.

Average angle: $\bar{\theta}$

$$\cos(2\bar{\theta})=\cos(\theta_++\theta_-)=\frac{-f}{\sqrt{c^2+f^2}}=\frac{-f}{\sqrt{a^2+e^2}}$$

c, *f* thus determine the absolute orientation of the balls but not the shapes. This means that we can consider only the case *r* = *f*, *c* = 0 without loss of generality. This means we need only consider the following matrix:

$$G=\begin{pmatrix} d & r-e \\ r+e & d \end{pmatrix}$$

A further restriction on the parameter space is a consequence of the fact that interchanging the *x* and *y* axes (i.e. a reflection of structures about the bisectrix, the line *x* = *y*) is equivalent to changing the sign of *e* (this follows since *e*->-*e* implies *G*->G^T (the transpose). Therefore one need only consider *e*>0.

GSI morphologies (3): nonuniqueness of G

Finally, overall “blowups” don’t change the morphologies of structures. For example, if we normalize *G* by (half) its trace:

$$G' = G / d$$

then a scale function $\|\underline{r}\|'$ satisfying the scale equation for *G*’:

$$\left\|\lambda^{-G'}\underline{r}\right\|' = \lambda^{-1}\|\underline{r}\|'$$

is obtained from the original scale function (satisfying $\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\|\underline{r}\|$) by:

$$\|\underline{r}\|' = \|\underline{r}\|^{d}$$

so that in 2D we can always take Trace $G = 2$ (i.e. $d = 1$).

Proof:

$$\left\|\left(\lambda^{d}\right)^{-G'}\underline{r}\right\| = \lambda^{-1}\|\underline{r}\|$$

$$\left(\left\|\left(\lambda^{d}\right)^{-G'}\underline{r}\right\|^{d}\right)^{1/d} = \left(\left(\lambda^{d}\right)^{-1}\|\underline{r}\|^{d}\right)^{1/d}$$

$$\left\|\lambda'^{-G'}\underline{r}\right\|' = \lambda'^{-1}\|\underline{r}\|'$$

Therefore, if we are only interested in exploring the various morphologies in 2D linear GSI, it suffices to consider *d* =1, *r* = *f*, *c* = 0, i.e. to only consider the matrix:

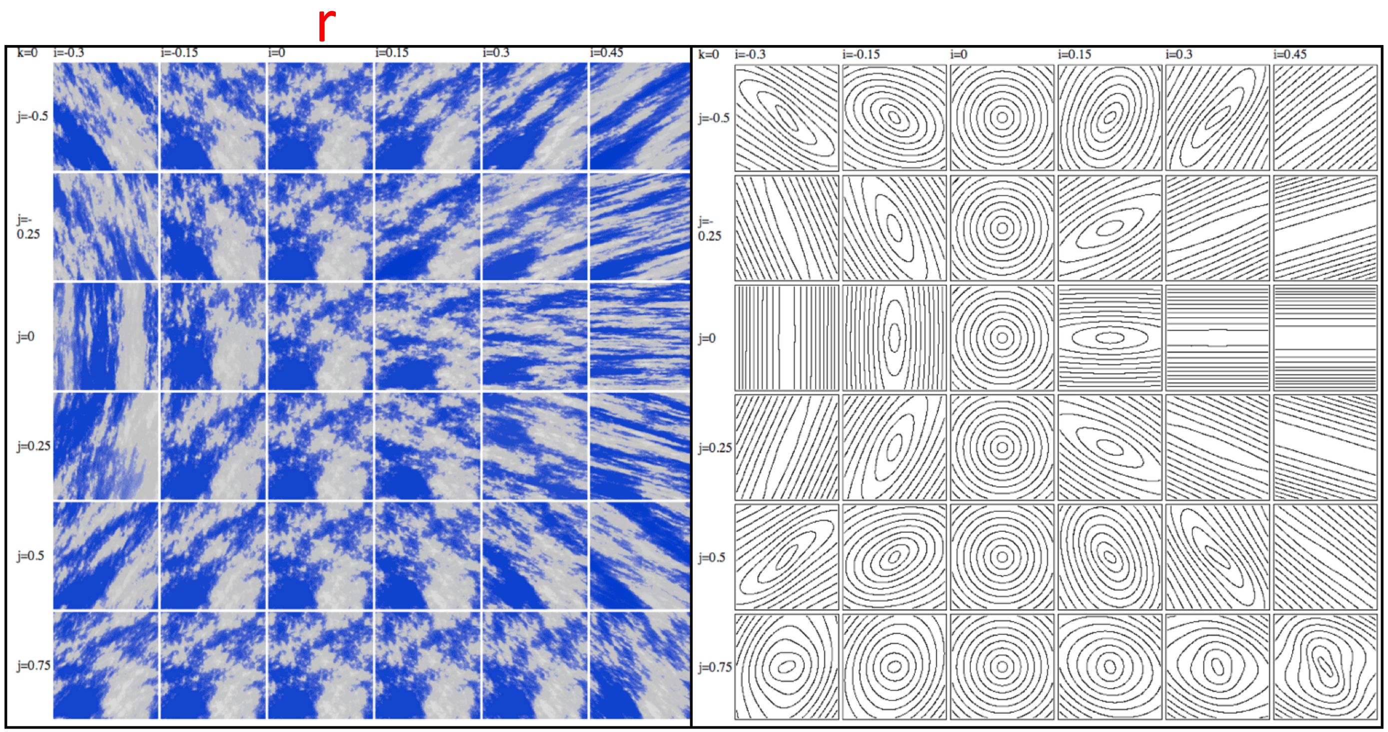
$$\boxed{G = \begin{pmatrix} 1 & r-e \\ r+e & 1 \end{pmatrix}}$$

Roundish unit ball

k =0: we vary *r* (denoted *i*) from -0.3, -0.15, …0.45 left to right and *e* (denoted *j*) from -0.5, -0.25, …0.75 top to bottom. On the right we show the contours of the corresponding scale functions.

$$G=\begin{pmatrix}1 & r-e\\ r+e & 1\end{pmatrix}$$

e



Highly anisotropic unit ball: *k* =10

Polar coordinate scale function for unit ball

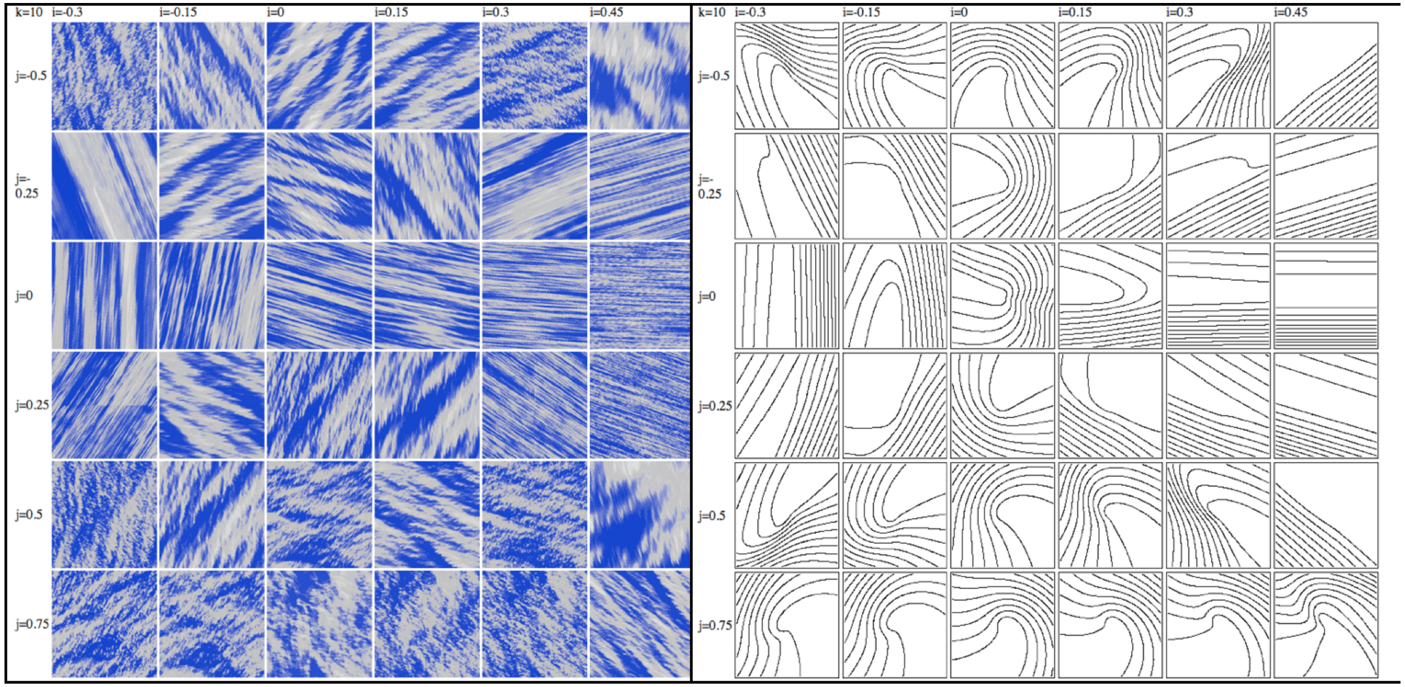
$$\|\underline{r}\| = r\Theta(\theta'')=1$$

with

$$\Theta(\theta'')=1+\frac{1-2^{-k}}{1+2^{-k}}\cos\theta''$$

Hence:

$$\max(\Theta(\theta''))/\min(\Theta(\theta''))=2^k$$

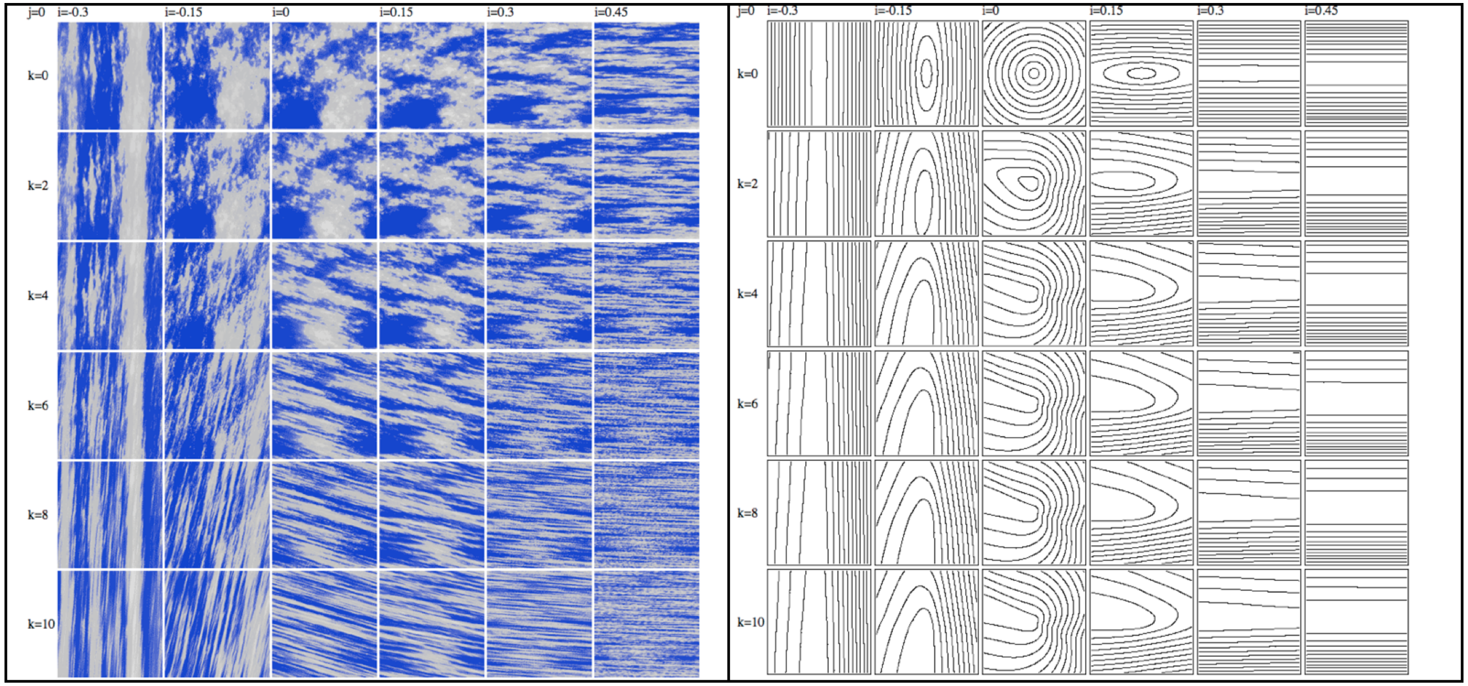


e = 0

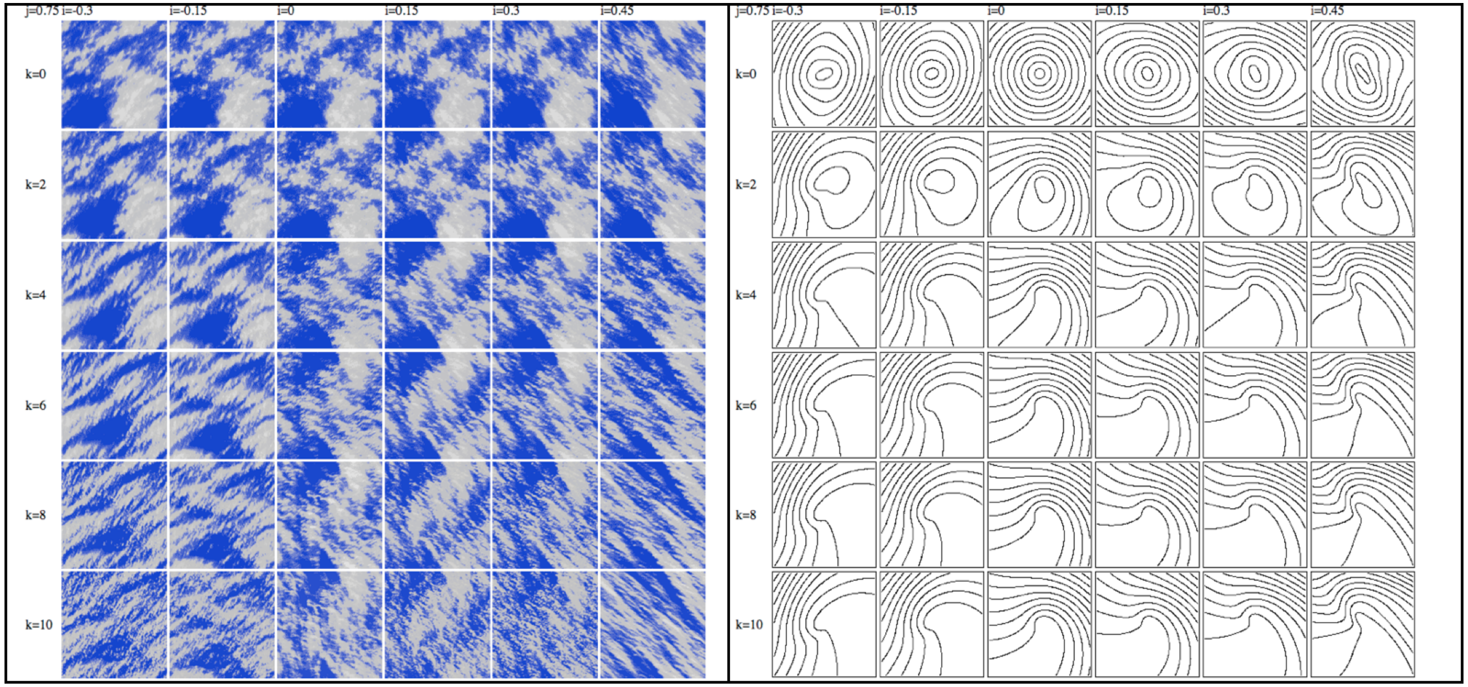
r is increased from -0.3, -0.15, …0.45 left to right, from top to bottom, *k* is increased from 0, 2, 4,..10.

r

k



e = 0.75

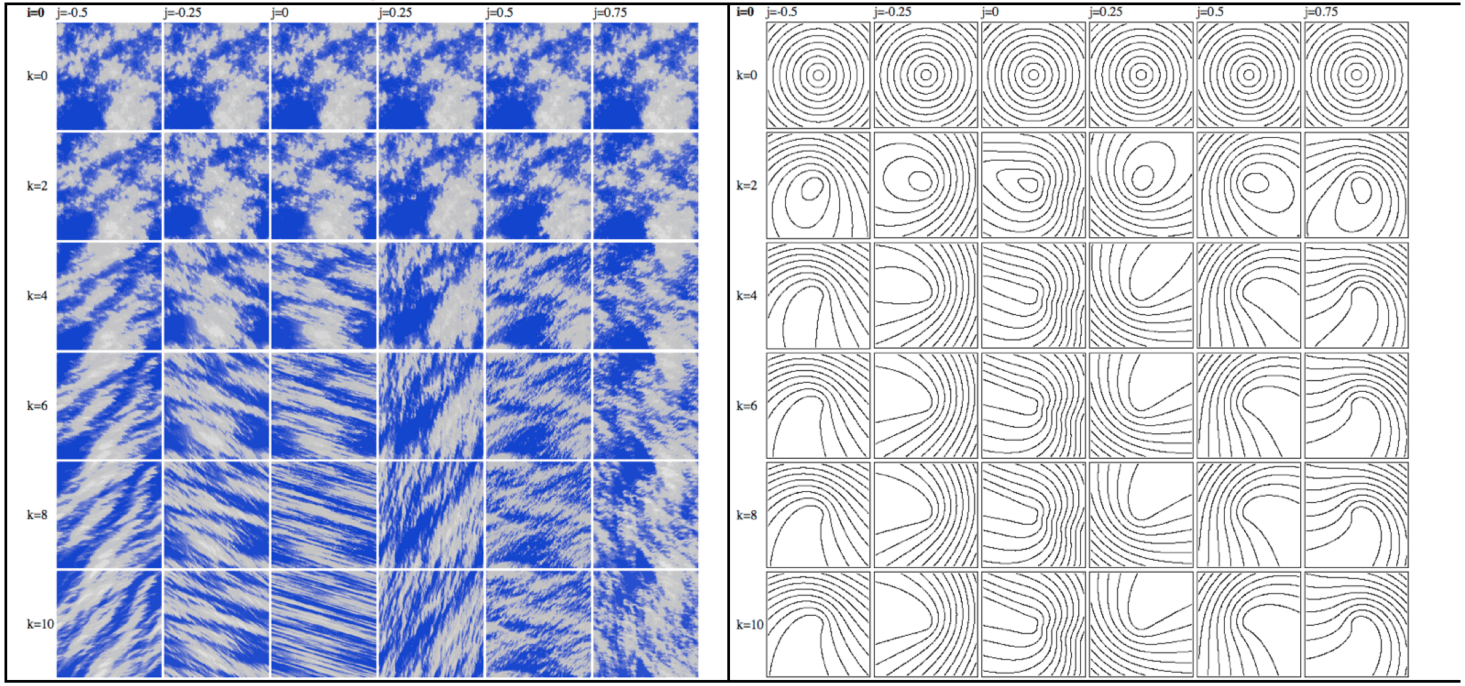


e

r = 0

e left to right is: -0.5, -0.25, …0.75.

k



r = 0.15

In all rows, from top to bottom, *k* is increased (0, 2, 4,..10),