Uncertainty and Predictability in Geophysics: Chaos and Multifractal Insights

Daniel Schertzer

CEREVE, Ecole Nationale des Ponts et Chaussées and Météo-France, Paris, France

Shaun Lovejoy

Physics Department, McGill University, Montreal, Canada

Uncertainty and error growth are crosscutting geophysical issues. Since the "chaos revolution" the dominant paradigm has been the "butterfly effect": the dependence on initial conditions is so sensitive that errors grow exponentially fast with characteristic times. This was the outcome of studying superficially simple caricatures of more involved systems. We critically analyze the physical relevance of these models and the mathematical generality of this effect. We emphasize that the atmosphere, oceans, rain etc., are spatially extended turbulent systems, with wide ranges of spatial scales. Turbulent phenomenology already shows that errors grow only slowly across these scales; they follow power laws, there are no characteristic times. An important recent realization is that in spite of strong anisotropies the dynamically significant range of scales is much larger than previously thought and that the role of intermittency is drastic and yields much more frequent extremes. The focus is now on time-space geophysical scaling behavior: their multifractality. It is found quite generally - not only for turbulent fields- that an infinite hierarchy of exponents is required to characterize the predictability decay from average to extreme events. Nevertheless, these laws are meaningful over the whole time range from short to long term; we give their explicit expression. This multifractal predictability suggests the advantages of stochastic rather than deterministic sub-grid parametrizations, and makes stochastic forecasting very attractive.

1. INTRODUCTION

Recently, there have been growing societal pressures to provide reliable predictions, in particular of extreme geophysical events (e.g. earthquakes, floods/droughts, cyclones, storms, etc.). Prediction time scales range from nearly zero for "nowcasting" to centuries for global change.

Prediction has several meanings, sometimes called "prediction of the first", "second" and "third kinds": these correspond to a pure initial value problem e.g. a Cauchy problem for differential equations, to a change of boundary value problem e.g. a Diriclet or von Neuman problem for partial differential equations, or to a mixture of both. Practical problems, such as global change, typically correspond to the third type. In spite of advances in prediction techniques,

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rather little has been done to determine their theoretical limits. The existence of limits are usually accepted at least in principle (e.g. the 'butterfly effect'), but one still hopes to extract some information in the midst of noise. This leads to a fundamental methodological problem, since without an estimate of the limits, it is difficult to evaluate and hence improve prediction skills.

This paper focuses on predictability limits and clarifies the crucial role of spatial scales. Two rather distinct approaches have been followed: one corresponding to the dynamical systems approach ('deterministic chaos') and the other one based on spatial complexity and scaling. Although both approaches share some common features, the types of predictability decay are quite different, corresponding to exponentials and power laws respectively. Although this dichotomy was first investigated by Lorenz in the 1960's, it is still not widely known. Since then, our knowledge of scaling processes has mushroomed and their close connection with intermittency has been clarified. Here we discuss the impact of these ideas showing that the loss of predictability is not smooth in time, but rather occurs by intermittent "puffs".

Although the predictability problem is ubiquitous in geophysics, most early developments and formalizations have occurred in the context of atmospheric or climate dynamics, where they have achieved their highest expression. This paper provides a general review of concepts together with applications to geophysics, e.g. hydrology (Sect. 2.3) atmospheric dynamics (Sect. 3.3). We hope that these examples will stimulate the development of similar applications in other geophysical domains.

2. DYNAMICAL SYSTEM APPROACH

2.1. Chaos Revolution?

The notion of "initial condition sensitivity" became well-known due to the work of [*Lorenz* 1963a] on his 3-component model (corresponding to the first three Fourier components of convection). By the 1980's such exponential error growth became the hallmark of the "deterministic chaos revolution" and it was widely viewed to be a generic property of nonlinear systems.

2.2. Exponential Error Growth

Exponential error growth emerged from the pioneering work of [*Lyapunov,* 1907], and was subsequently generalized into the elegant Multiplicative Ergodic Theorem (MET) [*Oseledets*, 1968], a cornerstone of chaos theory. A key assumption of the theorem is that temporal averages of a single sample of the process are the same as the average at one time over an ensemble of identical processes; i.e. that the process is "ergodic". Let us give some heuristics showing how this theorem follows from this common geophysical ergodicity assumption. Consider the simplest case, of a discrete (t = 0, 1, 2, 3…) nonlinear mapping G on real numbers. A well-known example concerns population dynamics [*May*, 1976] and the approach generally applies to finite difference approximations to scalar differential equations:

$$X(t+1) = G(X(t)) \tag{1}$$

The amplitude of the infinitesimal separation $\delta X(t)$ of a pair of points $\left(X^1(t), X^2(t) = X^1(t) + \delta X(t)\right)$ is *multiplicatively* modulated by the derivative of the map G at the point $X(t)$, i.e.:

$$\left|\delta X(t+1)\right| \approx \left|D_{X(t)}G\right| \; \left|\delta X(t)\right| \tag{2}$$

taking logarithms, one obtains:

$$Log\left[\,\left|\delta X(t)\right| / \left|\delta X(0)\right|\,\right] \approx \sum_{t'=0,\, t-1} Log\left(\left|D_{X(t')}G\right|\right) \tag{3}$$

If the process defined by Eq. (1) is ergodic, the right hand side of Eq. (3) is determined by replacing the time averages by ensemble averages (square brackets "⟨.⟩"):

$$\sum_{t'=0,t-1} Log\left(\left|D_{X(t')}G\right|\right) \approx t\left\langle Log\left(\left|D_x G\right|\right)\right\rangle \tag{4}$$

a result which yields an exponential error growth:

$$\left|\delta X(t)\right| \approx e^{\mu t}\left|\delta X(0)\right| \tag{5}$$

with a Lyapunov exponent:

$$\mu = < Log\left(\left|D_X G\right|\right) > \tag{6}$$

Equation (5) is valid as long as μ is finite, an assumption usually taken for granted. μ corresponds to the inverse of the characteristic time after which predictions are effectively impossible. Generalizations to finite d-dimensional systems proceed along the same lines. However, if we attempt to generalize this to evolving fields, i.e. to nonlinear partial differential equations (infinite dimensional (functional) spaces), we encounter severe difficulties. In fact, only a few limited extensions have been obtained [e.g. *Ruelle*, 1982]. Later on (Sect. 3.1), using physical phenomenology, we point out a key difficulty: the likely small scale divergence of μ, which violates the finiteness assumption of the mathematical derivation of the MET.

2.3. Low Dimensional Chaos in Geophysics?

2.3.1. Dimension estimates of geophysical attractors. The exponential growth of error implies that much of the original information is rapidly lost hence that only a few degrees of freedom might be required to specify the state of a system after long times. This raises the possibility that elaborate non-linear time series analysis techniques might be able to identify the few relevant parameters and greatly simplify the model. This explains the excitement generated, twenty years ago, when the embedding theorem and the correlation dimension algorithm were discovered. Together, they gave some credence to the idea that geophysical systems might not only have finite dimensions, but that the dimensions might be low.

This evolution in thinking started with [*Packard et al.*, 1980] and [*Takens*, 1980] who considered a scalar observable h of a (possibly vector-valued and continuous) time-series $\boldsymbol{x}(t)$ constrained on a (finite dimensional) strange attractor. They showed that in order to obtain a faithful image of this attractor (called a “reconstruction”), it was sufficient to use a (discrete) time series $X_n = h\{\boldsymbol{x}(n\Delta t)\}$. For instance, among the many components of the climate $\boldsymbol{x}(t)$, one may consider the temperature $h\{\boldsymbol{x}(t)\}$and reconstruct the climate attractor with the help of a discrete time series $h\{\boldsymbol{x}(n\Delta t)\}$. This may be achieved using “delay vectors” $\boldsymbol{Y}_n = (X_n, X_{n-\tau}, X_{n-2\tau}, ..., X_{n-(d-1)\tau})$, for any specified (integer) time delay τ, as soon as the dimension d of the resulting ‘embedding space’ E_d that they span is large enough. Defining the box dimension D_0 as the the scaling exponent of the number of attractor points N in a box of size $\ell : N(\ell) \approx \ell^D$, the precise requirement is $d > 2D_0$. In spite of its simple definition, the box dimension D_0 was numerically difficult to evaluate for large d.

The correlation dimension algorithm [*Grassberger*, 1983] overcame this difficulty [e.g. *Schuster*, 1988; *Tsonis*, 1992]: only the distance between pairs of delay vectors is required, not the embedding space. The correlation dimension D_2 of a set of points is defined as the scaling exponent of the average number of points in a sphere of radius r centered at one of them, i.e.:

$$< N(d,r) > \propto r^{D_2(d)} \tag{7}$$

The advantage of D_2 is that very large embedding dimensions d can easily be explored numerically. Initially, strange attractors were thought to have a unique (fractal) dimension, therefore the box-dimension and the correlation dimension were considered as equivalent. If this property, as well as the other assumptions of the embedding theorem, is satisfied, then for large d, the estimates $D_2(d)$ should converge towards the theoretical value D_2,—at least over a range of r. This is illustrated with the help of Figure 1-a, which displays $Log < N(d,r) >$ vs. $Log(r)$ for a synthetic rainfall time series Figure 1-b. As d increases, the slopes converge to $D_2 \approx 2.7$.

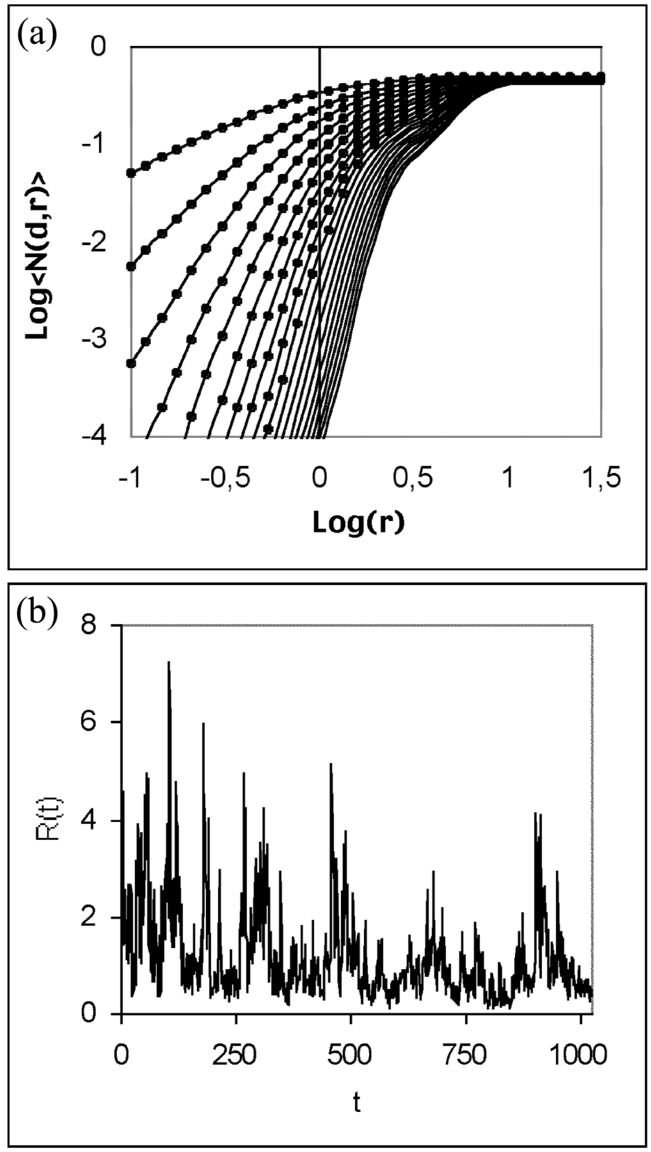

Figure 1: The correlation dimension (D_2) algorithm (a) applied to estimating the dimension D_2 of a synthetic rainfall series (b). (a) shows *Log<N(d, r)>* vs. *Log(r)*. As d increases, the slopes on the lower left reach an apparently constant value, the correlation dimension $D_2 \approx 2.7$. (b) the time series produced by a lognormal cascade with 12 cascade steps, each of factor 2 in scale, i.e. with $N = 4096$ data points.

Loosely speaking an attractor dimension measures how its points are concentrated in the phase space. A low dimension corresponds to a very small fraction of the available phase space, implying very constrained dynamics. This explains the excitement following [*Nicolis and Nicolis*, 1984], who analyzed the isotope record of a deep-sea core and estimated that $D_2 \approx 3.1$ for the climate and concluded that 4 ODE's might be sufficient to model climate evolution. Similar analyses have been performed on many geophysical data sets leading to numerous claims of very low dimensionality ($D_2 \approx 5$–7), especially for rain [*Essex et al.*, 1987; *Fraedrich*, 1986; *Hense*, 1986; *Jayawardena and Lai*, 1994; *Rodriguez-Iturbe et al.*, 1989; *Tsonis et al.*, 1993]. Although the proliferation of low dimension estimates gave some credence to the applicability of the butterfly effect to geophysics, many questions emerged [*Marzocchi*, 1997] that we are going to discuss now.

2.3.2. Limitations Due to Sample Size

Although the embedding space is only implicit in the correlation dimension algorithm, an essential problem remains: empirical analyses are performed on finite samples! This leads to an artificial confinement of empirical points to a small fraction of the embedding space and spurious low estimates of correlation dimensions $D_2(d)$, which do not reflect the dynamics. To avoid this problem, a minimal range of scales is required [*Ruelle*, 1990]: a decade yields the celebrated rule of thumb that the minimum number of points for D is: $N_D \approx 10^D$ [*Nerenberg and Essex*, 1990] and *Essex, 1991]*. [*Grassberger*, 1986] pointed out that the results of [*Nicolis and Nicolis*, 1984] were based on only 184 measurements, implying a maximum reliable dimension $D_2 \approx 2.3$, i.e. less than their estimate of $D_2 \approx 3.1$.

2.3.3. More Questions on the Estimates

The correlation dimension D_2 and the box dimension D_0, required by the embedding theorem are two special cases of the infinite hierarchy of dimensions (Sect. 3.4.1) that characterize the multifractal behavior of a strange attractor [*Grassberger*, 1983; *Hentschel and Procaccia*, 1983]. Physically, this means that systems are not homogeneously constrained. For example, the clustering of point pairs characterized by D_2 is generally less extreme than the clustering of triplets (D_3), quartets (D_4) etc. Perhaps the fundamental point is that the embedding theorem *hypothesizes* that the dynamics are deterministic. One therefore cannot draw any conclusion from a low D_0: this is neither a requisite nor an indication of chaos (see Sect. 2.4). In particular, nonlinear time series analysis techniques are inherently incapable of distinguishing between low-dimensional deterministic systems and high dimensional stochastic systems. The classical example is the reconstruction of the stochastic process known as "Brownian motion" [*Osborne and Provenzale*, 1989; *Theiler*, 1991], which is *linear* and has a box-counting dimension $D_0(d) = 2$ for any $d \geq 2$.

Similar results are obtained for a *nonlinear* stochastic model—a multiplicative cascade—which has been often invoked for rain (Sect. 3.4.2) and which displays an estimate $D_2 \approx 2.7$ Figure 1 a, b). According to the sample size criterion ($N = 4096$ and $\log_{10}(N) \approx 3.6 > 2.7$), this estimate is reliable but nevertheless does not imply a small number of degrees of freedom!

2.4. High Dimensional Chaos?

How to deal with geophysical systems if they do not have low finite dimensions? A radical change that has occurred in weather forecasting during the last decade illustrates the issue. Whereas attention had long been focused on large-scale deterministic modeling, it became clear that small-scale uncertainty must be considered as a first order problem. Deterministic modeling has progressively been replaced by Ensemble Prediction Systems (EPS) [*Molteni et al.*, 1996; *Palmer*, 2000; *Toth and Kalnay*, 1993]. EPS involves a probabilistic study of the trajectories of an ensemble of solutions of a deterministic numerical model started from different initial conditions or from (slightly) different models: e.g. 50 perturbed trajectories are routinely run at the European Center for Medium-Range Weather Forecasts.

Whereas theorists have primarily considered fields as elements of infinite dimensional functional spaces [e.g. *Eckmann and Ruelle*, 1985], meteorologists are interested in the large but finite dimensional projection of meteorological fields onto the phase space (the 'resolved scales') of their numerical models (typically 10^6–10^7 grid-points). There was agreement on the need to dynamically obtain relevant statistics by following the time-evolution of the density of points in phase space. For finite dimensional phase spaces, the equation for this density is the "Liouville equation" [*Liouville*, 1838] and has attracted attention in meteorology [*Ehrendorfer*, 1994; *Epstein*, 1969]. Consider a well-posed finite d-dimensional differential systems:

$$\dot{\underline{X}}(t) = \frac{d}{dt}\underline{X} = \underline{F}(\underline{X}, t) \tag{8}$$

The probability density $\rho(\underline{X}, t)$ (with respect to the volume measure $dX_1, dX_{,2} \ldots dX_d$ of the phase state spanned by $X_1, X_2, \ldots X_d$) satisfies a phase space continuity equation [e.g. *Nicolis*, 1995], the Liouville equation:

$$\frac{\partial}{\partial t}\rho(\underline{X}, t) + \sum_{i=1}^{d} \frac{\partial}{\partial X_i}\left[\dot{X}_i(t)\rho(\underline{X}, t)\right] = 0 \tag{9}$$

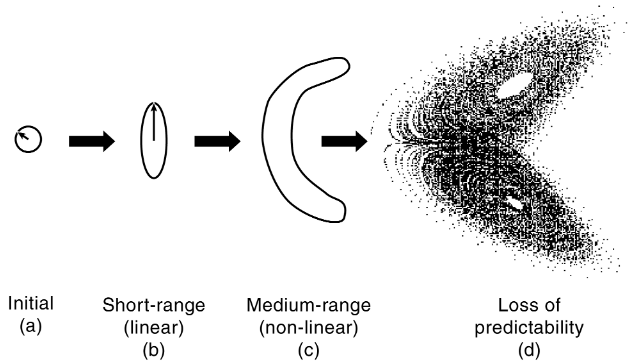

Figure 2: Scheme of the evolution of the empirical pdf in an Ensemble Prediction System according to [*Palmer*, 1999]: from the phase space region occupied by the initial ensemble (a), to (b) linear growth phase, to (c) nonlinear growth phase, to (d) loss of predictability.

Figure 2, reproduced from [*Palmer*, 1999], displays the expected time evolution of the empirical solution of the Liouville equation in an EPS phase space (represented for simplicity as being two dimensional, i.e. spanned by a pair of variables (X_1, X_2)). After an early linear (b) and later nonlinear (c) period of dispersion, the probability converges quickly (exponentially) to an invariant measure (d) of a strange attractor. This is the standard schematic used to illustrate the domain of numerical weather forecasts (b,c) and the loss of predictability (d). Below, we question the physical relevance of this scheme (Sect. 3.4.3).

Due to the scale truncation of the numerical models, the issue of noisy perturbations arising from subgrid processes is fundamental. If these perturbations were gaussian white noises $f(t)$ of intensity ε, then the state of the system would evolve as:

$$\frac{d}{dt}\underline{X}=\underline{F}(\underline{X},t)+\underline{f}(t);\left\langle f_i(t)f_j(t')\right\rangle=\varepsilon\delta_{i,j}\delta(t-t') \quad (10)$$

and the Liouville equation for the probability density would generalize into the Fokker-Planck equation [e.g. *Gardiner*, 1990]:

$$\frac{\partial}{\partial t}\rho(\underline{X},t)+\sum_{i=1}^{d}\frac{\partial}{\partial X_i}\left[\dot{X}_i(t)\rho(\underline{X},t)\right]-\varepsilon\Delta_X\rho(\underline{X},t)=0 \quad (11)$$

where Δ_x is the Laplacian diffusion operator (in the phase space). However, these perturbations may be strongly non-gaussian and it is better to consider a "fractional" generalization of the Fokker-Planck equation, which involves fractional derivatives (e.g. [*Schertzer et al.*, 2001] and references therein), in particular, fractional powers of the Laplacian. Furthermore, the noises may be colored rather than white [e.g. *Hasselmann*, 1976]. We will now consider the possibility of taking into account all the scales together.

3. SCALING AND MULTIFRACTAL APPROACHES

3.1. Phenomenology of High Dimensional Systems and Scale Symmetries

3.1.1. Scale symmetry of generating equations. Geophysically relevant equations are nonlinear. They can respect (more or less formal) scale symmetries. This has classically been known as "self-similarity" [e.g. Sedov, 1972], but with unnecessary limitations. Consider the Navier-Stokes equations that are used in many geophysical domains:

$$\frac{\partial \underline{u}}{\partial t}+(\underline{u}\cdot\nabla)\underline{u}=-\frac{\nabla p}{\rho}+\nu\nabla^2\underline{u}+\underline{f};\frac{\partial}{\partial t}\rho+\nabla(\underline{u}\rho)=0 \quad (12)$$

with $\underline{u}$ = velocity, t = time, p = pressure, ρ = fluid density[1], ν = viscosity, $\underline{f}$ = forcing density (external forcing, gravity), as well as the associated advection-diffusion equations for a scalar field θ (f_0 = forcing density for the scalar, κ = diffusivity):

$$\frac{\partial\theta}{\partial t}+(\underline{u}\cdot\nabla)\theta=\kappa\nabla^2\theta+f_\theta \quad (13)$$

In geophysics, active scalar fields θ are as important as the velocity field $\underline{u}$, e.g. convection, where ρ sensitively depends on θ, either the temperature (atmosphere or lithosphere) or on the salinity (oceans), contrary to the (academic) passive case (ρ, $\underline{f}$, independent of θ). Although the basic mathematical properties of their solutions (e.g. existence and uniqueness) are unsolved "Hilbert problems" [*Hilbert*, 1902] in both cases, these equations remain formally invariant under any (affine) contraction of the time-space (of scale ratios λ, λ^{1-H}):

$$\underline{x}\to\underline{x}/\lambda \qquad t\to t/\lambda^{1-H} \quad (14)$$

as long as the dependent variables are suitably renormalized:

$$\begin{aligned}&\underline{u}\to\underline{u}\big/\lambda^{H};\theta\to\theta/\lambda^{H'};\rho\to\rho/\lambda^{H''}\\&\nu\to\nu\big/\lambda^{1+H};\kappa\to\kappa\big/\lambda^{1+H''};p\to p\big/\lambda^{H''-1}\\&\underline{f}\to\underline{f}\big/\lambda^{2H-1};f_\theta\to f_\theta/\lambda^{H'+H''-1}\end{aligned} \quad (15)$$

[1] Hereafter ρ no longer denotes a probability density, as in a previous section.

One may either consider the asymptotic case of fully developed turbulence (with an infinite Reynolds number (Re→∞; equivalently, a vanishing viscosity, ν→0) for the incompressible Navier-Stokes equations [e.g. *Frisch*, 1995]), or consider the case of non-zero eddy (rather than molecular) viscosity and eddy diffusivity, [e.g. *Schertzer and Lovejoy*, 1987]. In numerical models, where scales are split into resolved and (parametrized) sub-grid components this symmetry is unfortunately broken.

3.1.2. *Scaling and quantitative laws in turbulence.* Scale invariance (or scaling) is also a symmetry directly related to a striking and rather general feature of nonlinear systems: their high variability in space and time. Indeed, this extreme variability can easily be understood as the result of a scale invariant process that is repeated scale by scale, thus multiplicatively amplifying even small variability present at larger scales. This is related to the paradigm of cascades [Richardson, 1922] and is exemplified by the basic quantitative turbulence laws: the Richardson law [Richardson, 1926] and the Kolmogorov-Obukhov law [Kolmogorov, 1941; Obukhov, 1941]. The first relates the relative separation $r(t)$ of a pair of particles passively advected by turbulence:

$$\left\langle r(t)^2 \right\rangle \propto \bar{\varepsilon}\, t^3 \tag{16}$$

$\bar{\varepsilon}$ is the spatial average of the energy dissipation rate density, as well of the density of the energy flux ε to smaller scales. The second relates the shear of the velocity field $\delta u(\ell)$ to the scale ℓ:

$$\left\langle \delta u(\ell)^2 \right\rangle \propto \bar{\varepsilon}^{\,2/3} \ell^{\,2/3} \qquad E(k) \propto \bar{\varepsilon}^{\,2/3} k^{-5/3} \tag{17}$$

where $E(k)$ is the energy spectrum at the wave-number k. Their extensions to a passive scalar field [*Obukhov*, 1949; *Corrsin*, 1951] are:

$$\left\langle \delta\theta(\ell)^2 \right\rangle \propto \bar{\chi}\, \bar{\varepsilon}^{\,-1/3} \ell^{\,2/3} \qquad E_\theta(k) \propto \bar{\varepsilon}^{\,2/3} k^{-5/3} \tag{18}$$

where χ is the flux of scalar variance, $E_\theta(k)$ the spectrum of the scalar field. Surprisingly, these laws are still beyond the reach of analytical techniques, including the Quasi Normal Approximation [*Millionshchikov*, 1941], the Direct Interaction Approximation [*Kraichnan*, 1958; *Kraichnan*, 1959] (for review see [*Leslie*, 1973]), as well from the renormalization group [*Forster et al.*, 1977]. Indeed, rather ad-hoc modifications are required to obtain 'analytical closures' (Eddy-Damped QuasiNormal Model [EDQNM, *Orszag*, 1970] and the Test Field Model [TFM, *Kraichnan*, 1971]), which are compatible with these "mean field" laws. However, closures are unable to account for intermittency [*Frisch et al.*, 1980], see Sect. 3.4.1.

3.1.3. *Eddy turnover time and scaling space-time anisotropy.* In contrast to the exponential predictability decay law (Eq. (5)) for deterministic chaos (Sect. 2.2), Eqs. (16)–(18) are power laws. This is a consequence of the transformation group, which for any value of the exponents H, H' leaves the generating equations invariant (Eq. (15)). The particular values $H = H' = 1/3$, which correspond to the Kolmogorov-Obukhov (Eq. (17)) and to Corrsin-Obukhov law (Eq. (18)), can be found either by purely dimensional considerations or using the physical notion of "eddy turnover time" $\tau(\ell)$. The latter is the characteristic time for a structure of scale ℓ with a velocity shear across it $\delta u(\ell)$ to "turn over":

$$\tau(\ell) \propto \ell / \delta u(\ell) \tag{19}$$

Since the characteristic time of destruction of structures of this scale ℓ must be proportional to the eddy turn-over time [e.g. *Robinson*, 1971], one finds that the rate of transfer of energy to smaller scale is:

$$\varepsilon(\ell) \propto \delta u(\ell)^2 / \tau(\ell) \propto \delta u(\ell)^3 / \ell \tag{20}$$

therefore:

$$\delta u(\ell) \propto \varepsilon(\ell)^{1/3} \ell^{1/3} \qquad \tau(\ell) \propto \varepsilon(\ell)^{-1/3} \ell^{2/3} \tag{21}$$

By performing a spatial average and considering that $\bar{\varepsilon}(\ell)$ is scale independent and not too fluctuating (i.e. $\overline{\varepsilon^q} \approx \bar{\varepsilon}^q$), Eq. (21) yields the Kolmgorov-Obukhov scaling law Eq. (17) as well as an homogeneous eddy turn-over time $\bar{\tau}(\ell)$:

$$\bar{\tau}(\ell) \propto \bar{\varepsilon}^{\,-1/3} \ell^{2/3} \tag{22}$$

This confirms that there is a scaling anisotropy between time and space i.e. the (typical) lifetime of a structure varies as a power of the scale, in agreement with Eq. (14), with the "dynamical exponent": $1-H_t=2/3$, to be used in Sects. 3.2, 3.4.2, 3.4.3. It also yields the scaling laws for the passive scalar (Eqs. (18)).

3.2. Predictability in Homogeneous Turbulence

3.2.1. The phenomenology of error growth through scales and the MET. The general phenomenology of error growth through scales is rather straightforward: an error or uncertainty initially confined to small-scales will progressively 'contaminate' large-scale structures through these interactions. This is in sharp contrast with the MET that does not consider the problem of many nonlinearly interacting spatial scales. The problem of the evolution of spatially extended fields was first theoretically investigated by [Thompson, 1957]. Using initial time-derivatives and various meteorological models, Thompson studied the nonlinear uncertainty growth due to errors in the initial conditions resulting from the limited resolutions of the measurement network and of the models. He estimated the root mean square (RMS) doubling time for small errors to be about two days, whereas [Charney and al., 1966], using more elaborate meteorological models, estimated it as five days.

The scale dependency of the predictability times was underlined by [*Robinson*, 1967; *Robinson*, 1971]. Indeed, if the notion of characteristic error time τ is still relevant, it should depend on the spatial scale ℓ in a hierarchical manner. For $t > \tau(\ell)$ two fields initially similar at scale ℓ become quite different (e.g. rather decorrelated) at this scale, but may remain similar at larger scales. This is in agreement with the following estimates of the Lyapunov exponent $\mu(\ell) \propto 1/\overline{\tau}(\ell)$ and the characteristic space scale ℓ_e reached by the error at time *t* (see Eq. (22)):

$$\mu(\ell) \propto 1/\overline{\tau}(\ell) \propto \overline{\varepsilon}^{\,1/3}\ell^{-2/3} \qquad \ell_e(t) \propto \overline{\varepsilon}^{1/2} t^{3/2} \tag{23}$$

This shows—contrary to the usual assumption (Eq. (6))—that unless a break in the scaling occurs leading to smooth small scale behavior, the Lyapunov exponent μ will diverge at small scales.

3.2.2. Energetics and spectral analysis of the error growth. Let $\underline{u}^1(\underline{x},t)$ and $\underline{u}^2(\underline{x},t)$ be two solutions of a nonlinear system (e.g. velocities for Navier-Stokes equations) initially identical, but with a perturbation (error) $\delta\underline{u}(\underline{x},0) = \underline{u}^2(\underline{x},0) - \underline{u}^1(\underline{x},0)$ at t=0, confined to infinitesimally small spatial scales. The time-evolution of $\delta\underline{u}(\underline{x},t)$ corresponds to the effect of butterflies homogeneously distributed in space, rather than the effect of a single butterfly. When the nonlinear interactions preserve the kinetic energy (e.g. Navier-Stokes equations), it is convenient but not sufficient to consider both the correlated (kinetic) energy (per unit of mass):

$$e^c(\underline{x},t) = \tfrac{1}{2}\underline{u}^1(\underline{x},t)\cdot\underline{u}^2(\underline{x},t) \tag{24}$$

and the decorrelated energy:

$$e^{\Delta}(\underline{x},t) = \tfrac{1}{2}\left(\delta\underline{u}(\underline{x},t)\right)^2 = \tfrac{1}{2}\left(\underline{u}^2(\underline{x},t) - \underline{u}^1(\underline{x},t)\right)^2 \tag{25}$$

as well as the total energy and the energy of each solution:

$$e^T(\underline{x},t) = e^1(\underline{x},t) + e^2(\underline{x},t);\ \ e^n(\underline{x},t) = \tfrac{1}{2}\left(\underline{u}^n(\underline{x},t)\right)^2 \tag{26}$$

This implies the relation:

$$e^T(\underline{x},t) = e^c(\underline{x},t) + e^{\Delta}(\underline{x},t) \tag{27}$$

hence, if the total energy is statistically stationary (conserved on average), there will be a flux of correlated energy $e^c(\underline{x},t)$ to decorrelated energy $e^{\Delta}(\underline{x},t)$. This also holds for the corresponding energy spectra $E^T(k,t) = E^c(k,t) + E^{\Delta}(k,t)$, since the latter corresponds to a linear decomposition of the former with respect to wave number, *k*. Therefore, the decorrelated energy spectrum $E^{\Delta}(k,t)$ steadily increases in magnitude from large to small wavenumbers, converging to the total energy spectrum $E^T(k,t) \approx k^{-5/3}$ (Figure 3). The critical wave number $k_e(t)$ of the transition from dominant correlation to dominant decorrelation can be defined by $E^c(k_e(t),t) = E^{\Delta}(k_e(t),t)$, scales as $1/\ell_e(t)$ (Eq. (23)) and decreases as: $k_e(t) \approx t^{-3/2}$.

3.2.3. *Consequences and limitations.* If the constant of proportionality in the definition of the eddy-turn over time (Eq. (22)), as well as that relating the latter to the error time, is of order unity, then taking a "typical values" $\overline{\varepsilon} = 10^{-3}\,\mathrm{m^2 s^{-3}}$ and $\eta \approx 10^{-3}\,\mathrm{m}$ (energy flux and the viscous scale), one obtains $\tau_e(\eta) \approx \overline{\varepsilon}^{-1/3}\eta^{2/3} \approx 10^{-1}\,\mathrm{s}$, as well as $\tau_e(\ell) \approx \tau_e(\eta)(\ell/\eta)^{2/3}$ and therefore $\tau_e(\ell) \approx 10\,\mathrm{s}$; 1/2 hr; 28 hr; 5.4 days respectively for $\ell = 1\mathrm{m}$, 1 km, 10^3 km, 10^4 km. These estimates are close to those obtained by [Lorenz, 1969] (Figure 3), but slightly lower than the numerical (closure, nonintermittent) results obtained by [Leith, 1971; Métais and Lesieur, 1986] and [Kraichnan, 1970; Leith and Kraichnan, 1972].

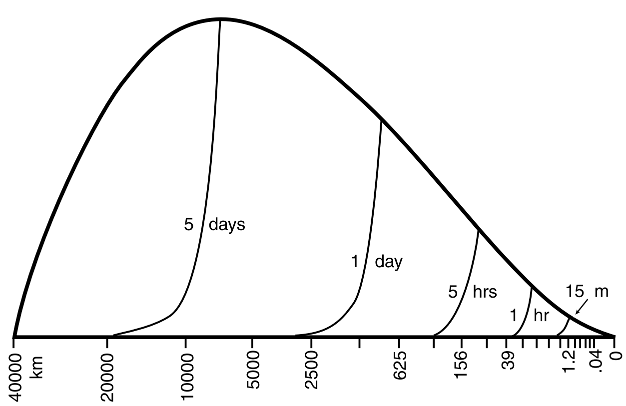

Figure 3: Atmospheric error growth according a quasi-normal closure simulation [*Lorenz*, 1969]. The decorrelated energy spectrum $E^{\Delta}(k, t)$ initially confined to a few meters (on the right) "pollute" the larger scales (up to 20,000 km (on the left) after various time intervals (from few minutes to 5 days).

3.3. Questioning the Lorenz-Leith-Kraichnan Approach

3.3.1. Isotropic models of atmospheric motions. The use by [Lorenz, 1969] of a $k^{-5/3}$ spectrum up to synoptic scales was considered as paradoxical and speculative, e.g. [Robert and Rosier, 2001], in any case as in contradiction with the standard model of atmospheric dynamics that considers large-scale atmospheric motions as quasi-two dimensional (quasi-2D) and small-scale motions as quasi-three dimensional (quasi-3D). This standard model emerges from quasi-linear approximations, in particular the quasi-geostrophic approximation [Charney, 1948] and the related notion of quasi-geostrophic turbulence [Charney, 1971]. Furthermore, it seems eminently sensible since at large scale the atmosphere appears to be a thin film of thickness $h \approx 10$ km with large horizontal scale $L \approx 20{,}000$ km.

However, 2D and 3D turbulence have quite distinct dynamics and transport properties. 2D turbulent flows are fairly smooth, since the main dynamical mechanism of 3D turbulence—vortex stretching—is impossible. Whereas 3D dynamics generate vorticity explosively, 2D dynamics only advect vorticity conservatively. Among various consequences of this vorticity conservation, there is no longer a cascade of energy towards smaller scales, but a cascade of enstrophy (the vorticity squared) with a different spectral slope [*Fjortoft*, 1953; *Kraichnan*, 1967] and nearly scale invariant eddy-turn over times.

In order to hold together, the standard model requires a "meso-scale gap", otherwise the 3D turbulence destabilizes the 2D turbulence. The existence of this gap was initially given some empirical support (Figure 4) by estimates of wind spectra [*Panofsky and Van der Hoven*, 1955; *Van der Hoven*, 1957]. In spite of strong criticism [*Pinus*, 1968], it was eventually consecrated by [*Monin*, 1972; *Pedlosky*, 1979]. As a consequence, [*Leith and Kraichnan*, 1972] studied 2D and 3D turbulent flow predictability.

More fundamentally, the atmosphere is buoyancy driven so that we must consider the fact that buoyancy forces generate another conservative flux [Bolgiano, 1959; Obukhov, 1959], related to both the potential energy defined by gravity, and to the large scale stable stratification of the atmosphere. Buoyancy was not considered in the standard model, whereas it is expected to dominate the kinetic energy flux at large scales. Unfortunately, this "buoyancy subrange" was originally hypothesized as an isotropic regime i.e. with the same $k^{-11/5}$ spectrum in both horizontal and vertical directions. However, as it was never observed along the horizontal, the idea languished. As discussed below an anisotropic generalization is required to yield a coherent model of atmosphere.

[*Lilly*, 1985] argued that empirical findings in the 1980's seriously undermined the standard model. The GASP experiment (Figure 5), the first large-scale campaign to measure the horizontal velocity spectrum [*Lilly and Paterson*, 1983; *Nastrom and Gage*, 1983b], found no evidence of a mesoscale spectral gap. This result was confirmed by the more recent MOZAIC experiment [*Lindborg*, 1999]. Instead, they found a Kolmogorov $k^{-5/3}$ scaling extending to at least hundreds of kilometers. This was confirmed in a variety of climatological and meteorological regimes, including tropical cylonic conditions [*Chigirinskaya et al.*, 1994]. However, atmospheric dynamics that aircrafts are assumed to measure can induce fractal aircraft trajectories [*Lovejoy et al.*, 2004] and hence possible biases due to mixed (nontrivially correlated) measurements of vertical and horizontal fluctuations.

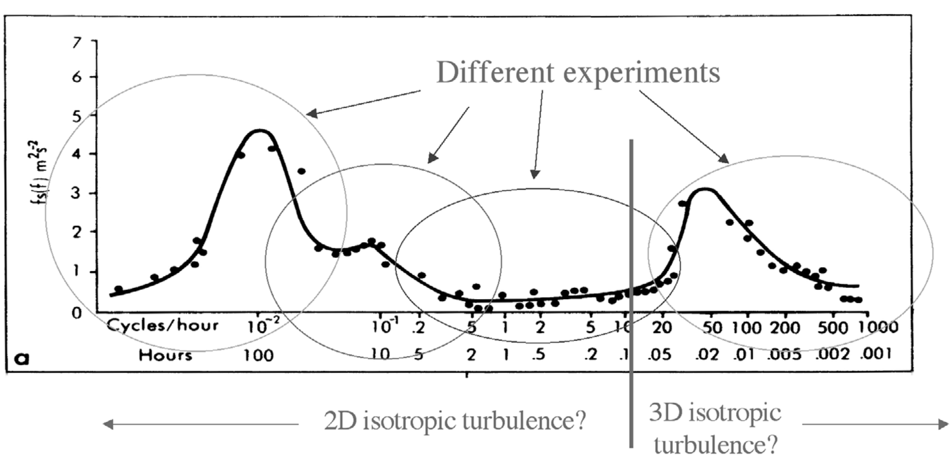

Figure 4: Atmospheric wind spectrum according to [*Van der Hoven*, 1957]. There are four different experiments at different frequencies. According to [*Vinnichenko*, 1969], the high frequency data was taken at "hurricane -like conditions", hence the spurious gap.

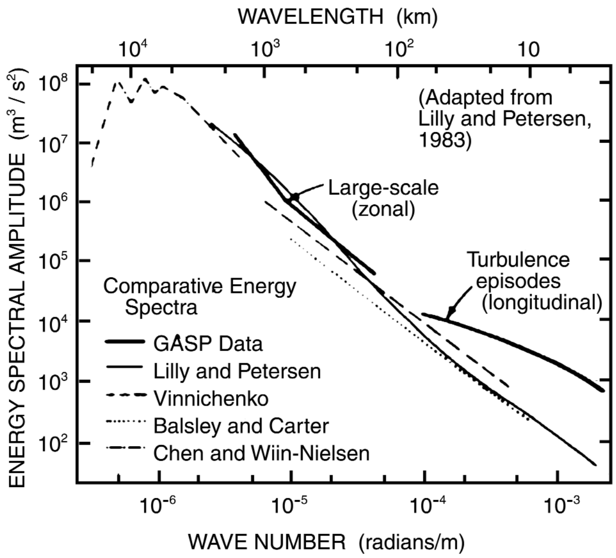

Figure 5: GASP atmospheric wind spectra [*Nastrom and Gage*, 1983a] from instrumented commercial aircraft. The basic spectrum is close to Kolmorogov out to at least several hundred kilometers and there is no evidence for a "meso-scale gap".

As an alternative, [*Lovejoy et al.*, 2001] used the fact that the infra red and visible radiances of cloud fields are strongly coupled to their structures and to the dynamics and therefore should be scaling over the same range: the analysis of nearly 1000 satellite images show that the (multi) scaling of the radiances was respected from planetary scales down to at least kilometer scales.

[*Gage*, 1979] proposed that the meso-scale $k^{-5/3}$ spectrum could be explained by a 2D inverse energy cascade. This modification of the standard model is nevertheless inconsistent with the empirical evidence that atmospheric variability is also scaling along the vertical. Using balloon measurements over heights of 10–20 km the vertical spectrum of the horizontal wind was consistently found (Figure 6) to follow Bolgiano-Obhukhov (BO) $k^{-11/5}$ scaling throughout the troposphere [*Adelfang*, 1971; *Endlich et al.*, 1969; *Lazarev et al.*, 1994; *Schertzer and Lovejoy*, 1985].

3.3.2. *Anisotropic models of atmospheric motions.* [*Schertzer and Lovejoy*, 1985a] proposed that in the horizontal the statistics are dominated by the energy flux (leading to $k^{-\beta_h}, \beta_h \approx 5/3$ in the horizontal), while in the vertical the conservative flux generated by the buoyancy force is dominant leading to $k^{-\beta_v}, \beta_v \approx 11/5$. Contrary to the original BO framework, neither the Boussinesq approximation, nor other stable stratified reference states are required. Fluid particles respond to local gradients, not gradients with respect to theoretical reference values.

In this new model, the atmosphere is anisotropic at all scales and effectively becomes progressively flatter and flatter along the horizontal at larger and larger scales. The differential flattening can be characterized by an intermediate "elliptical dimension" $2 \le D_{el} \le 3$ [*Schertzer and Lovejoy*, 1984]: $D_{el} = 2 + (\beta_h - 1)/(\beta_v - 1)$ (where β_h, β_v are the spectral slopes along the horizontal and vertical directions). When the horizontal extent of a structure is increased by λ, its volume increases by $\lambda^{D_{el}}$. The atmosphere is therefore neither 3D isotropic at small scales nor 2D isotropic at large scales (D_{el} = 3 and 2 are the 3D and 2D isotropic cases respectively). In terms of D_{el} the current atmospheric debate is between buoyancy driven flows with D_{el}=23/9=2.555 and flows resulting from a gravity wave mechanism leading to D_{el}=7/3=2.333 ($\beta_v \approx 3$), [*Lumley*, 1963; *Shur*, 1962; *Van Zandt*, 1982; *Weinstock*, 1978]. Recent lidar based aircraft data from pollution (considered as a passive scalar surrogate), using simultaneous vertical and horizontal backscatter measurements, yield D_{el}=2.55±0.02 over the range 3 m to 120 km, very close to the 23/9 model and incompatible with the 2D, 3D and 7/3D models (Figure 7 and [*Lilley et al.*, 2004]). Note that empirical evidence of anisotropic scaling have been also found for the magnetic susceptibility of the earth crust [*Lovejoy*, 2001] and

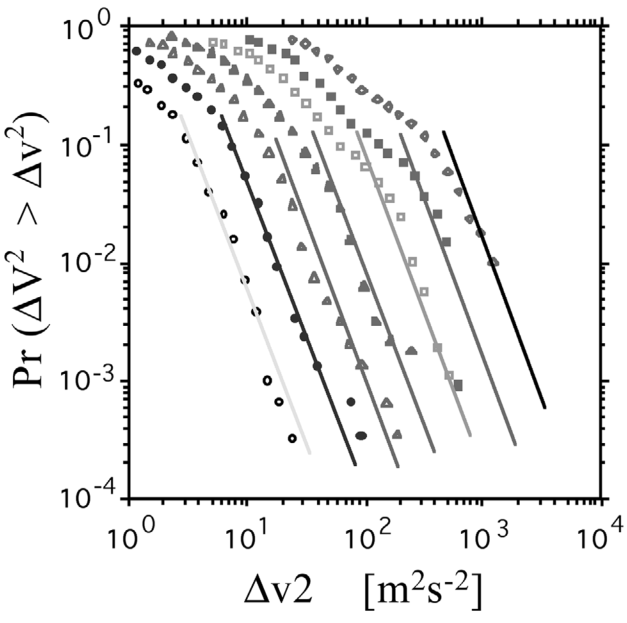

Figure 6: Scaling of the probability distribution of the vertical shear of the horizontal wind [*Schertzer and Lovejoy*, 1985b]. The left curve for 50m thick layers, with thickness increasing by factors of 2 to right (80 radiosonde ascents). The straight lines indicate a reference slope –5 (corresponding to a power-law probability falloff, divergence of moments of order q>5), and the line spacing indicates Bolgiano-Obukhov (BO) scaling in the vertical.

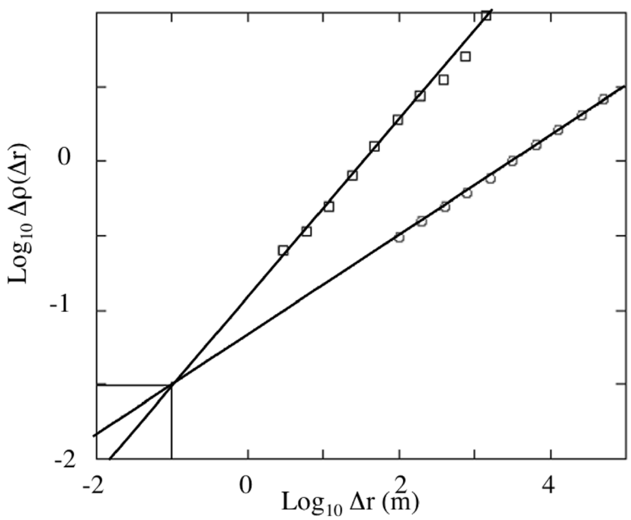

Figure 7. Log-log plot of the first order structure function of the aerosol backscatter ratio [*Lilley et al.*, 2004], a surrogate for the concentration of a passive scalar, obtained from 9 vertical cross-sections (4.5 km thick, 120km long) with vertical resolution 3m, horizontal resolution 100m. The straight lines indicate the theoretical exponents (horizontal), (vertical), their intersection determines point the "sphero-scale" l_s=10cm. Structures larger than l_s are flattened in the horizontal, whereas smaller structures are vertically aligned.

for the soil hydraulic conductivity [*Tchiguirinskaia*, 2002], so that geophysical stratification may be generally scaling.

3.3.3. Which statistics? [*Lilly*, 1985] questioned the quasi-normal closure framework , which implies that the analyses of [*Lorenz*, 1969] [*Leith*, 1971] and [*Kraichnan*, 1970; *Leith and Kraichnan*, 1972] are local as well as global. This is not consistent with the observation that various atmospheric structures (e. g. rotating thunderstorms [*Lilly*, 1983]) maintain a stable identity much longer than their turnover time. [*Schertzer and Lovejoy*, 1984] reported that the probability distributions of vertical wind shear amplitude $|\delta u|$, energy flux density ε and potential temperature θ have power law tails. This means that for example (Figure 6) the probability of the wind shear $|\delta u|$ (as well as it squared δu^2) to exceeding a (large) fixed threshold is a power-law :

$$\Pr(|\delta u| \geq s) \approx s^{-q_D} \tag{28}$$

the power-law exponent q_D is a critical exponent. [*Lilly*, 1985] therefore argued that the error statistics should be similarly divergent and much more variable and extreme than those estimated in the quasi-normal framework of closures. Indeed, power law probability tails**,** are often considered a hallmark of Self-Organized Criticality [*Bak et al.*, 1987]; the exponent q_D being a critical order of divergence of statistical moments This means that the (theoretical) statistical moments of order $q \geq q_D$ are infinite. On finite (e.g. empirical) samples the moment estimates are finite but grow (i.e. diverge) as the number of samples increases. This divergence results from the fact that the weights or frequencies of extremes is much higher than usual. This is consistent with the fact that the probability of finding a 10 times larger fluctuation, decreases only by a factor 10^{q_D} (e.g. according to Figure 6 the probability to have a 10 times larger wind shear decreases only by 10^5, not by an exponential factor).

3.3.4. The fundamental role of intermittency. As early as 1942 Landau [*Landau and Lifshitz*, 1987; *Yaglom*, 1994] questioned the assumption of the homogeneity of fluxes used by [*Kolmogorov*, 1941; *Obukhov*, 1941] to derive their scaling law of velocity shears (Eq. (17)). [*Batchelor*, 1953; *Batchelor and Townsend*, 1949] observed that not only does the "activity" of turbulence induce inhomogeneity, but the activity itself is very inhomogeneously distributed: there are "puffs" of active turbulence inside of puffs of (active) turbulence. This inhomogeneity has been termed "intermittency", which may be even more fundamental for active scalars such as rain and cloud fields than for the dynamics. These considerations lead to the general idea [e.g. *Leslie*, 1973] that a turbulent flow is only turbulent in tiny fractions of space and time. A precise meaning to the term "fraction" was achieved with the help of cascade models.

3.4. Strongly Non Gaussian Statistics and Multifractal Modeling

In contrast to the limitations of closure models, multifractal models yield strongly non-gaussian statistics and therefore structures of very different intensities. A key ingredient is a multiplicative property of the models that describe a cascade of instabilities, i.e. incrementally the heterogeneity of fluxes flowing through smaller and smaller structures increases.

3.4.1. *Multifractals and the phenomenology of cascades.* Stochastic multifractal processes originated from the phenomenological assumption [e.g. *Yaglom*, 1966] that in turbulence successive cascade steps define independent fractions of the flux, F, transmitted to smaller scales and that a cascade is scaling (Figure 8). To be more precise, let $\lambda = L/\ell$ be an intermediate scale ratio ("resolution"), where L is the outer scale and ℓ the scale corresponding to scale ratio λ, and let $\Lambda = L/\ell' = \lambda\lambda'$ be the total scale ratio of the cascade. Scaling means that the cascade from λ to Λ corresponds to a cascade from ratio 1 to λ' contracted by T_λ of scale ratio λ; $T_\lambda(f(x)) = f(T_\lambda(\underline{x}))$; in the self-similar (isotropic) case $T_\lambda(\underline{x}) =$

$\underline{x}/\lambda$. When combined these two properties imply that the flux is a multiplicative group, with the equality symbol indicating that the random variables on each side have identical probability distributions:

$$F_{\Lambda=\lambda\lambda'} \overset{d}{=} F_\lambda \cdot T_\lambda(F_{\lambda'}) \qquad (\lambda, \lambda' \geq 1) \qquad (29)$$

Hence we obtain the following scaling law for the statistical moments:

$$< F_{\lambda\lambda'}{}^q >= \lambda'^{K(q)} < F_\lambda{}^q > \qquad (\lambda, \lambda' \geq 1) \qquad (30)$$

where the exponent $K(q)$ is the moment scaling function. The probability distribution for the event $\{F_\lambda \geq \lambda^\gamma\}$ is also scaling (mathematically, this may be obtained using the Mellin transform [*Schertzer and Lovejoy*, 1993; *Schertzer et al.*, 2002a]):

$$\Pr\{F_\lambda \geq \lambda^\gamma\} \propto \lambda^{-c(\gamma)} \qquad (31)$$

the arbitrary exponent γ, which defines a given level of activity or intensity at all resolutions λ, is a "singularity": the larger it is, the faster F_λ grows with resolution/scale λ. The scaling exponent $c(\gamma)$ of the probability is a statistical codimension [*Schertzer and Lovejoy*, 1987] also called the "Cramer" function [*Mandelbrot*, 1991; *Oono*, 1989]. When the embedding dimension $d > c(\gamma)$, it corresponds to a geometric notion of codimension, so that on a given sample of this process, the event $\{F_\lambda \geq \lambda^\gamma\}$ is almost surely a fractal set of dimension: $D(\gamma) = d - c(\gamma)$. In other words a multifractal field can be understood as an infinite hierarchy of embedded fractal sets of dimension $D(\gamma)$ and supporting a given singularity γ, i.e. F_λ grows faster than λ^γ with increasing resolution λ. The highest singularities are the rarest, hence $c(\gamma)$ increases with γ, whereas $D(\gamma)$ decreases. For instance, the schematic Figure 8 displays a unique extreme singularity, three more intermediate ones, and extremely low ones for the rest of the (2D) space. In any case, multifractality cannot be understood as a dimension depending on scale. Whereas scale invariant geometric sets of points are fractals, scale invariant fields (i.e. with a value at each point) are multifractals.

The two scaling functions are related by the Legendre transform [*Parisi and Frisch*, 1985]:

$$K(q) = \max_\gamma \{q\gamma - c(\gamma)\} \quad c(\gamma) = \max_q \{q\gamma - K(q)\} \qquad (32)$$

Thus the main multifractal properties common to all the various formalisms are an infinite hierarchy of statistical exponents [e.g. *Badii and Politi*, 1984; *Grassberger*, 1983; *Grassberger and Procaccia*, 1983; *Hentschel and Procaccia*, 1983; *Schertzer and Lovejoy*, 1984; Stanley and Meakin, 1988] and an infinite hierarchy of singularities [e.g. *Benzi et al.*, 1984;

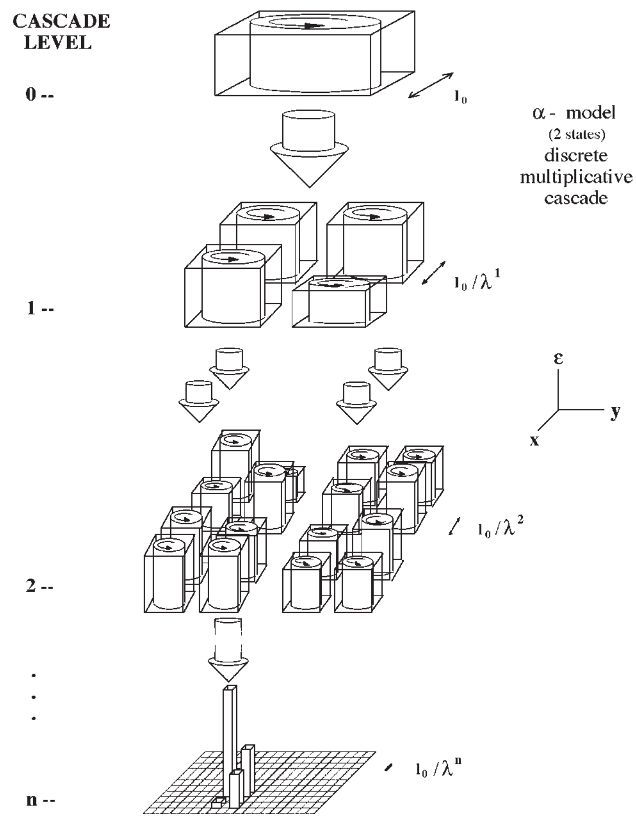

Figure 8. Illustration of a discrete (in scale) cascade process. The displayed first few steps show how the energy flux at large scales multiplicatively modulates the flux at successively smaller scales. Since the energy is conserved on average, the flux is concentrated in a hierarchy of fractal sets whose fractal dimension decreases with intensity threshold.

Halsey et al., 1986; *Parisi and Frisch*, 1985]. However, there are substantial differences. For example, there is an upper bound for singularities of "geometrical" multifractals [*Halsey et al.*, 1986; *Parisi and Frisch*, 1985], where each singularity is assumed to be supported by a well–defined geometrical (fractal) set, and for singularities of "microcanonical" multifractal processes that conserve fluxes on each realization as well as scale by scale [e.g. *Benzi et al.*, 1984; *Meneveau and Sreenivasan*, 1987; *Pietronero and Siebesma*, 1986]. In contrast, more general canonical multifractals, which conserve only ensemble flux averages, do not generally have any upper bound [*Schertzer and Lovejoy*, 1992]. The resulting extreme singularities yield power probability distributions having power-law tails (Eq. (28)). which are discussed in Sect. 3.3.3.

3.4.2. Multifractal modeling. Static multifractal models (pure spatial cuts, without time) have become useful tools for simulations of clouds [e.g. *Arneodo et al.*, 1999; *Naud et al.*, 1996; *Wilson et al.*, 1991] and of other geophysical fields [e.g. *Deidda*, 2000; *Pecknold et al.*, 1996; *Pecknold et al.*, 1997; *Pecknold et al.*, 1993]. Their dynamic versions, i.e. space-time processes, have been developed for studying turbulence, rain and the predictability [*Marsan et al.*, 1996; *Over and Gupta*, 1996; *Schertzer et al.*, 1997; *Marsan*, 1998]. Multiplicative processes, in particular when continuous in scale, can be generated from white noises. Indeed, they can be obtained by the exponential of an additive process $\Gamma_\lambda(\underline{x})$ called the "generator "of the flux ($\underline{x} = (x, y, z)$ for a 3D spatial process, $\underline{x} = (x, y, z, t)$ for a time-space process):

$$F_\lambda = e^{\Gamma_\lambda} \tag{33}$$

In order to respect the scaling property of the statistical moments (Eq. (30)) the generator must have a logarithmic divergence with resolution λ; $\Gamma_\lambda \propto Log(\lambda)$; $\lambda \to \infty$. This is achieved with an appropriate "fractional integration", i.e. a power-law filtering in the Fourier space [for details e.g. *Schertzer et al.*, 1997]) of a white noise, which can be chosen as a Levy stable noise [*Schertzer and Lovejoy*, 1987; *Schertzer and Lovejoy*, 1997]. One may note that the anisotropy between space and time (Sect. 3.1.3, in particular Eq. (22)), as well as the necessary causality condition (asymmetry between past and future) can easily be taken into account [*Marsan et al.*, 1996].

3.4.3. Multifractal predictability limits. In order to generalize the approach followed in the spectral analysis of predictability (Sect. 3.2.2) to multifractals, we consider the time evolution of a pair of fields of common resolution Λ. They are identical up to the time t_0 when one lets the fluxes become independent at small scales [*Schertzer and Lovejoy*, 2004]. For simplicity, consider the scalar rain rate R(x,t) illustrated by Plate 1 (time t along the horizontal, location x along the vertical). Plates 1a–b display a pair of rain rate fields $R^1_\Lambda(x, t)$ and $R^2_\Lambda(x, t)$ and Plate 1d their absolute difference $|\delta R_\Lambda(x,t)|$. One may qualitatively note the role of intermittency: most of the difference $|\delta R_\Lambda(x,t)|$ is due to a small number of extremely large values.

[*Marsan et al.*, 1996] checked that the spectral analyses of multifractal simulations of a velocity component are in agreement with homogeneous turbulence results (Sect. 3.2.2, and Figure 3). Bursts of violent fluctuations cannot be accounted for using second order statistical moments, in particular energy spectra; these are evident in Figure 9 which displays an 'elementary' decorrelated/error energy spectrum, i.e. not obtained by ensemble average, but only over a unique sample. It is no longer as smooth as an ensemble averaged decorrelated/error energy spectrum $E^\Delta(k, t)$ (e.g. Figure 3), but rather corresponds to a sequence of decorrelation (more

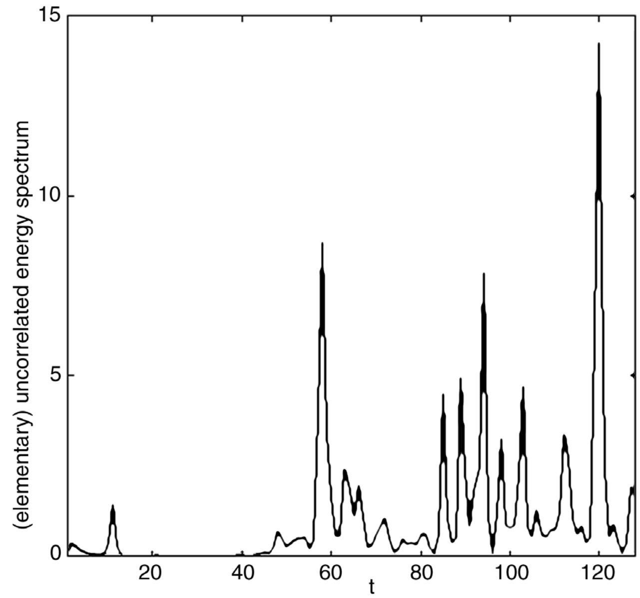

Figure 9. Elementary error energy spectrum displaying decorrelation bursts ([Schertzer et al., 1997]). This elementary error energy spectrum is obtained from a unique realization (is the resolution of the simulation) rather than from an ensemble average. It is no longer as smooth as of Figure 3, but rather corresponds to a sequence of decorrelation bursts at different scales.

generally of independence) bursts at different scales. These bursts result from the fact that although the energetics of the upscale cascade of errors remain basically the same, they do not constrain the largest fluctuations of the errors as much as in the homogeneous turbulence case.

We emphasized that statistics of second order moments, in particular their correlation that corresponds to the correlation energy for a velocity field, are unable to account for the co-evolution of a pair of multifractal fields. Therefore, we need to consider a covariance of order q for different values of q. This is rather simple for fluxes, e.g. the respective energy flux densities ε^i_Λ (i=1,2) of a pair of velocities $u^i(\underline{x}, t)$. Up to t_0 the fluxes are identical over the full range of the cascade process (i.e. over the possibly infinite cascade scale ratio Λ). After t_0, they remain rather similar only over a decreasing scale ratio $\lambda(t) \leq \Lambda$, which necessarily follows a power law. More precisely [*Schertzer and Lovejoy*, 2004], the latter is defined by the dynamical exponent $1-H_t$, which defines the scaling space-time-space anisotropy (Sect. 3.1.3, in particular Eq. (22)):

$$t \leq t_0 : \lambda(t) = \Lambda;\ \ t > t_0 : \lambda(t) \approx Min[\Lambda, \left(T/(t-t_0)\right)^{\frac{1}{1-H_t}}] \tag{34}$$

where T is the outer time scale. As a consequence, one obtains for the (normalized) covariance of order q:

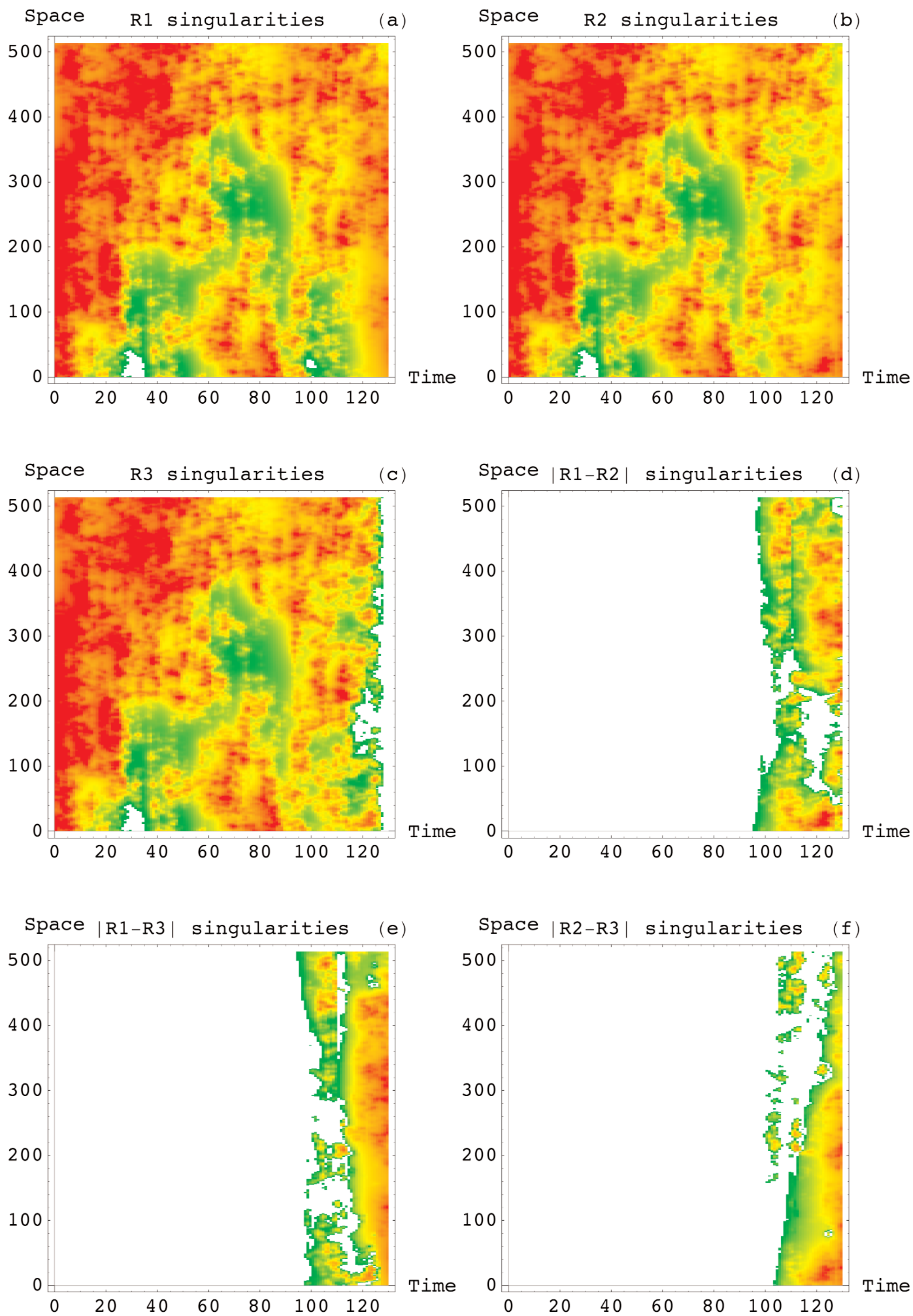

Plate 1.a–f Simulation of multifractal predictability decay for rain field: (a) and (b) are identical up to t_0=64, after which their fluxes become independent. (c) displays the forecast based on their common past and the deterministic conservation of the flux afterward. Singularities of the fields (i.e. their log divided by the log resolution), as well as of their absolute differences (d–f), are displayed according to the following palette: white for negative singularities; green to yellow for singularities contributing to statistics up to the mean; red for singularities contributing to second and higher order moments.

$$C^{(q)}(\varepsilon_\Lambda^1, \varepsilon_\Lambda^2) = \left\langle \left(\varepsilon_\Lambda^1 \varepsilon_\Lambda^2\right)^q \right\rangle \Big/ \left(\left\langle \varepsilon_\Lambda^{1\,q} \right\rangle \left\langle \left(\varepsilon_\Lambda^2\right)^q \right\rangle \right) \propto \lambda(t)^{K(q,2)}$$

with $K(q, 2) \equiv K(2q) - 2K(q)$. The multifractality $K(q,2)$ of the joint field $\varepsilon_\Lambda^1 \varepsilon_\Lambda^2$ is purely defined by that of ε_Λ^i (K(q)). The distinctive feature is that instead of being fixed at Λ (as for $\varepsilon_\Lambda^i = \varepsilon_\Lambda^2$) the range of scale ratios $[1, \lambda(t)]$ also has a power-law decay (Eq. (34)). The same occurs for its probability distribution. It is important to appreciate that these power laws are valid for all time scales, not only the large scales. This is in a sharp contrast with the schematic sequence of predictability behaviors presented by Figure 2 and discussed in Sect. 2.4. The corresponding "Liouville+ MET' scenario is therefore not relevant for multifractal fields.

3.4.4. Forecasts and stochastic parameterizations. We may now explore the question of optimizing forecast procedures so that the decay law of the (normalized) covariance $C^{(q)}(\varepsilon_\Lambda^F, \varepsilon_\Lambda^0)$ of order q of the forecast field ε_Λ^F and of the observed field ε_Λ^0 is as close as possible to the theoretical $C^{(q)}(\varepsilon_\Lambda^1, \varepsilon_\Lambda^2)$ (Eq. (35)). For example, let us point out that the multifractal behavior of meteorological fields theoretically explains and confirms the recent empirical evidence that stochastic parametrizations do better than deterministic ones [*Buizza et al.*, 1999; *Houtekamer et al.*, 1996], in particular in the EPS framework. It suffices to use the fact that a multifractal field may be defined with the help of a white-noise. Indeed, past and future components of a white noise are independent and identically distributed. Therefore, any white-noise identically distributed to the past component is obviously a possible future component. In particular, the resulting process will in the future keep the same statistical properties, as well as the same scale ratio. On the contrary, a future component defined in a deterministic manner cannot have an identical statistical distribution. In particular, its scaling function $K_{\det}(q)$ is linear with respect to q, instead of being nonlinear as that of the observations $K(q)$. At best, one can only find a (deterministic) procedure to preserve the statistics of a given order q. This is illustrated by Plate 1 obtained by a numerical simulation, where the (deterministic) future component of the noise of the forecast field $R_\Lambda^3(x, t)$. (Plate 1c) was defined to preserve the mean ($q = 1$) of the flux. Plates 1e–f display the drastic loss of all extreme events ($q >> 1$) with respect to the samples $R_\Lambda^1(x, t)$ and $R_\Lambda^2(x, t)$. More quantitative statements can be readily obtained with the help of the covariance $C^{(q)}(\varepsilon_\Lambda^F, \varepsilon_\Lambda^0)$ of order q. This should encourage the radical EPS evolution to increasingly account for the randomness of meteorological fields at different scales.

4. CONCLUSION AND PROSPECTS

Prediction in geophysical systems is still in its infancy. Basic questions such as the nature of error growth and limits to predictability must be answered if only to allow the predictions to be seriously evaluated. Geophysics thus faces a situation somewhat analogous to that of celestial mechanics a few centuries ago when the ad-hoc epicycle framework became unmanageable: the definition of a suitable framework for predictability assessment of geophysical fields still requires much theoretical work.

We underlined that geophysical systems are not only complex in time, but also in space and therefore rather complex^{1+D}, where D is the dimension of the spatial extension. This requires a critical assessment of the relevance of concepts that emerged from the study of simple temporally complex systems.

In this direction, we have reviewed, compared and contrasted several frameworks that belong to two broad categories: dynamical systems (or deterministic chaos) and scaling in time and space. We discussed several critical issues related to the transition from (geometrical) finite dimensional phase space to (functional) infinite dimensional ones, in particular the importance of the anisotropic symmetry between time and space, as well as the question of singular behaviors at small scales. We argued that they imply strong limitations on the applicability of the Multiplicative Ergodic Theorem (MET) and of the Liouville equation. We reviewed the evidence brought by homogeneous turbulence phenomenology and statistical closure models that predictability decay laws are algebraic rather than exponential. Unfortunately, the quasi-normal framework of these models prevents them from dealing with intermittency: the strong heterogeneity of the activity of turbulence, i.e. the "bursts" of the energy fluxes through scales. We illustrated the potential of multifractals to quantify the fact that weak and strong events have different predictability limits, in particular with the help of a (normalized) correlation of increasing order. However, the most important single point is that these algebraic laws hold at all times, not only asymptotically. This contrasts to the standard predictability framework that involves a sequence of linear, nonlinear and chaotic regimes. It also shows and explains why stochastic sub-grid modeling in the context of Ensemble Prediction Systems may perform much better than deterministic modeling. In a more general manner, a better understanding of the intrinsic predictability limits of natural phenomena should help us to find alternative modeling strategies approaching the intrinsic predictability limits.

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D. Schertzer, CEREVE, Ecole Nationale des Ponts et Chaussées, 6-8 Avenue Blaise Pascal, Cité Descartes, 77455 Marne-la-Vallee Cedex 2, France. Daniel.Schertzer@cereve.enpc.fr

S. Lovejoy, Physics dept., McGill U., 3600 University st., H3A 2T8, Montreal, PQ, Canada. Lovejoy@physics.mcgill.ca