

MULTIFRACTALS AND EXTREME RAINFALL EVENTS

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Abstract. Based on a multifractal structure hypothesis for temporal rainfall processes, a general formula relating maximum possible point rainfall accumulations is derived as a function of the duration and sample size. This formula appears to be in agreement with empirical observations. Such a result may reconcile some opposite points of view regarding extreme rainfall events, and suggests new ways of exploiting the scaling properties of rain processes.

From fractal sets to multifractal fields:

Fractals were first seen from a purely geometrical point of view [Mandelbrot, 1975, 1982]. Applied to rainfall time series studies this approach only enabled us to cope with occurrence or non-occurrence [Hubert and Carboneil, 1989] of rainfall or monofractal functions whose fractal dimension was independent of the intensity threshold [Lovejoy, 1981; Lovejoy and Mandelbrot, 1985; Lovejoy and Schertzer, 1985]. The occurrences must be carefully defined according to some kind of rainfall intensity threshold. The observed dependence of the fractal dimension upon the value of this threshold then yields an infinite hierarchy of dimensions [Schertzer and Lovejoy, 1985, 1991; Lovejoy et al, 1987; Lovejoy and Schertzer, 1990]. At first sight, we have lost the

simplicity of a unique fractal descriptor, but this problem can be overcome by using (universal) multifractal fields rather than fractal sets. The basic equation of multifractal fields, derived from the theory of multiplicative cascades, is the following [Schertzer and Lovejoy, 1987a,b, 1989]:

$$\Pr(R_\lambda > \lambda^\gamma) \approx \lambda^{-c(\gamma)} \quad (1)$$

Where λ is the scale ratio and is T/τ in time series of length T divided in elementary time-periods τ , R_λ is the intensity of the field at scale ratio λ , γ is an order of singularity. This equation states how the probability distribution of singularities of order higher than a value γ is related to the fraction of the space they occupy as determined by the codimension $c(\gamma)$. The " \approx " sign in the right side of equation (1) indicates equality to within slowly varying (e.g. log) functions of λ . For some results on multifractals and rain (including a review) see Schertzer and Lovejoy (1987a, 1989), Lovejoy and Schertzer, (1992).

Scaling laws for extreme rainfalls

We may now consider extreme rainfall events occurring within a duration τ . Whenever there is a maximum order of singularity γ_{\max} , then the maximum accumulation A_λ will be:

$$A_\lambda = \lambda^{-1} R_\lambda T \propto \lambda^{\gamma_{\max}-1} \propto \tau^{1-\gamma_{\max}} \quad (2)$$

As predicted by eq. 2, the maximum recorded point rainfall depths for different durations (from minutes to several years), as presented in most hydrological or meteorological textbooks

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and encyclopedia [Gilman, 1964; Réménieras, 1965; Raudviki, 1979] exhibit more or less the same algebraic behavior (figure 1). On a log-log diagram, the slope is found to be about 0.5 ($\gamma_{max} \cong 0.5$).

There are many mechanisms which can give rise to finite γ_{max} . In multifractal processes, the highest possible order of singularities can be artificially restricted by construction (e.g. "geometric" or "microcanonical" processes [Schertzer and Lovejoy, 1992]). However, such a maximum is naturally produced by a large class of universal processes [Schertzer Lovejoy, 1987a,b, 1991; Schertzer et al, 1991; Brax et Pechanski, 1991; Schmitt et al, 1992]. In this paper, we don't discuss the various issues related to universality in multifractals, i.e. existence of attractive and stable processes which are scaling nonlinear analogues of the gaussian noises that often arise in linear processes. We rather concentrate on the exploitation of their codimension function below which depends only on two fundamental parameters C_1 and α (for non conservative process, a third parameter (H) is necessary):

$$c(\gamma) = C_1 \left(\frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha} \right)^{\alpha'} \quad \alpha \neq 1$$

$$c(\gamma) = C_1 \exp\left(\frac{\gamma}{C_1} - 1\right) \quad \alpha = 1 \quad (3)$$

where $\alpha^{-1} + \alpha'^{-1} = 1$. C_1 measures the departure from homogeneity ($C_1=0$) and α is the Lévy index bounded between 0 and 2, which measures the departure from monofractality ($\alpha=0$).

For $\alpha \geq 1$, the orders of singularity are unbounded, on the contrary when $0 \leq \alpha < 1$, a finite maximum order of singularity γ_0 does exist:

$$\gamma_0 = \frac{C_1}{1 - \alpha} \quad (4)$$

Sampling dimension to quantify limited information

We must now consider a complication which arises because of the finite size of the data sets. A finite sample will miss extreme events which are sufficiently rare: the empirical singularities will be bounded by an effective maximum γ_s smaller than the theoretical γ_0 . Indeed, if we have N_s independent series, each with range of scales λ , then we may introduce the "sampling dimension" [Schertzer and Lovejoy, 1989, 1991; Lavallée, 1991; Lavallée et al., 1991] (D_s) to quantify the fraction of the (infinite) dimensional probability space actually explored:

$$D_s = \frac{\log N_s}{\log \lambda} \quad (5)$$

The actual total dimension is $D+D_s$, where D is the dimension of each sample, $D=1$ for time series. We now use the geometric interpretation of $c(\gamma)$ as a codimension: whenever the observing space $D+D_s > c(\gamma)$, the set of singularities of order γ will be a fractal set of dimension $D+D_s-c(\gamma)$. The maximum attainable γ yielding a nonnegative dimension (see figure 2), is $\gamma_s = c^{-1}(D+D_s) = \gamma_0 [1 - \alpha(C_1/(D+D_s))^{-1/\alpha}]$ which satisfies (for $\alpha < 1$, $\alpha' < 0$):

$$\gamma_0 \left(1 - \alpha \left(\frac{C_1}{D} \right)^{-1/\alpha} \right) \leq \lambda_s \leq \gamma_0 \quad (6)$$

with upper bound corresponding to an infinite sample size ($D_s \rightarrow \infty$), the lower bound to a single sample ($D_s=0$).

Confrontation with empirical observations

Glancing at the empirical values gathered in table 1, it is noteworthy that α is quite consistently in the vicinity of 0.5 and certainly bounded by 0 and 1. As a consequence, in the framework of universal multifractals, rainfall time series should have a maximum order of singularities. Averaging

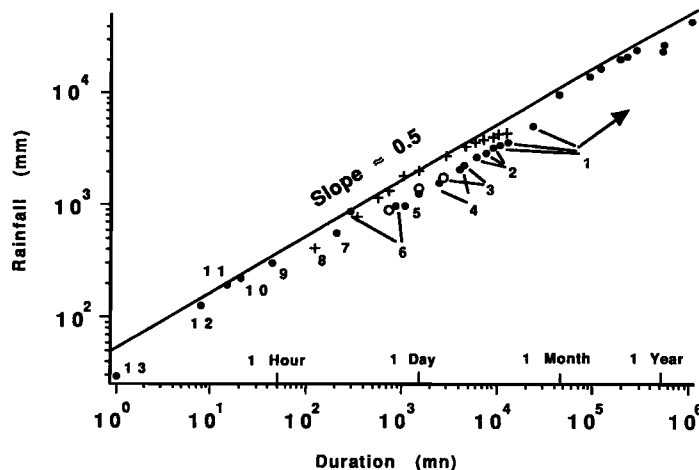


Fig. 1: The world's record point rainfall values, reproduced from Raudkivi (1979). 1 - Cherrapunji, India; 2 - Silver Hill Plantation, Jamaica; 3 - Funkiko, Taiwan; 4 - Baguio, Philippine Is.; 5 - Thrall, Texas; 6 - Smethport, Pa; 7 - D'Hani, Texas; 8 - Rockport, W.Va; 9 - Holt, Mo.; 10 - Cutea

de Arges, Romania; 11 - Plumb Point, Jamaica; 12 - Fussen, Bavaria; 13 - Unionville, Md.; values from Jennings Jenning, (1950). (+) La Reunion, France; (o) Paishih, Taiwan; values from Paulhus (1965).

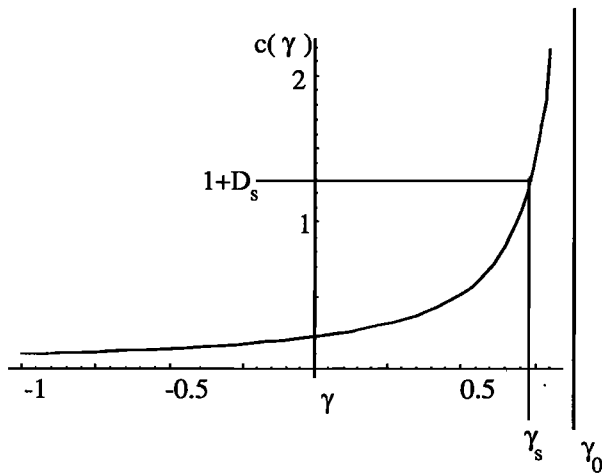


Fig. 2: The $c(\gamma)$ curve corresponding to the estimated parameters $\alpha = 0.51 \pm 0.05$, $C_1 = 0.44 \pm 0.16$, with γ_s for $N_s \approx \lambda^{D_s}$ samples ($N_s \approx \lambda^{D_s}$, λ being the scale ratio and D_s the sampling dimension), and γ_0 ($=\gamma_s$ for an infinite number of samples; $D_s = N_s = \infty$).

over all the analyzed series we obtain $\alpha = 0.51 \pm 0.05$, $C_1 = 0.44 \pm 0.16$. In the following we will assume that the embedding space dimension D is equal to 1 and that the sampling dimension D_s is equal to zero (i.e. considering a unique sample). Due to the nonlinearity of eqs. 4 and 6, there are two direct ways of obtaining the statistically best estimate of γ_0 and γ_s , yielding different results. On the one hand, as displayed in table 1, we can compute estimates of γ_0 and γ_s from each estimate of α and C_1 , then average. On the other hand, we can compute estimates of γ_0 and γ_s from averaged values of α and C_1 . However, it turns out that the differences between the two results are not so important: $\gamma_0 \approx 0.88$ and $\gamma_s \approx 0.66$ with the first method; $\gamma_0 \approx 0.90$ and $\gamma_s \approx 0.69$ with the second method. We actually prefer the first method

because it readily yields a confidence interval: $\gamma_0 = 0.88 \pm 0.31$; $\gamma_s = 0.66 \pm 0.20$.

The observed slope of figure 1 (≈ 0.5), is compatible within one standard deviation with that estimated from γ_s values ($1 - \gamma_s = 0.34 \pm 0.20$) and quite compatible within this deviation from that estimated from γ_0 values. More refined theories could take into account a possible degree of non conservation of the rain cascade, as well as the observed breaks in the scaling. Other improvements could include new estimation algorithms as well as the processing of larger data sets and may lead to even closer theoretical and empirical agreement.

Conclusions and perspectives

There exist presently two rather opposite views on extreme precipitation. One school of thought relies deeply on the notion of the "possible maximum precipitation" (PMP) considered as a physically based notion. In order to estimate the possible maximum precipitation at a given location (and implicitly at a given scale) a sophisticated analysis of the rainfall process in an attempt to address all its relevant and physical aspects (meteorology, orography, etc.) is required. However, such an approach is often considered as remaining too speculative or qualitative, especially with respect to engineering needs.

On the other hand, supporters of statistical analysis consider rainfall rate as a random variable and time series as a stochastic process. Statistical approaches lead to rainfall rate probabilities useful in engineering designs. However, without any reference to any physical processes, the role of hydrologists could easily be reduced to fitting empirical data to ad hoc statistical laws.

These early results may help to reconcile the two points of view since they are based on both physics and statistics. Indeed, in our approach the multiplicative cascade accounts for turbulent processes resulting from nonlinear interactions between different scales and fields and leads to the statistical

Table 1: A comparison of various gage estimates of α , C_1 , γ_0 , γ_s over various time scales. Parameters were mostly estimated from PDMS (probability Distribution/Multiple Scaling) the DTM (Double Trace Moment) techniques [Lavallée, 1991; Lavallée et al, 1991; Lovejoy and Schertzer, 1991; Schertzer and Lovejoy, 1991]. (*Private communications; +Ladoy et al., submitted to C. R. Acad. Sci. Paris, 1992).

Data Type	Gage, daily accumulation	Gage, 6 minutes resolution	Gage, daily accumulation	Gage, daily accumulation	Gage, 15 minutes resolution	Mean and standart deviation
Location	Global Network	Réunion Islands (France)	Nîmes (France)	Dédougou (Burkina Faso)	Alps (France)	
Stations	4000	1	1	1	28	
Duration	1 year (scaling regime up to 16 days)	1 year (scaling regime up to 30 days)	40 years (scaling regime up to 16 days)	45 years (scaling regime up to 16 days)	4 years	
α	0.5	0.5	0.45	0.59	0.50	0.51 ± 0.05
C_1	0.6	0.20	0.6	0.32	0.47	0.44 ± 0.16
γ_0	1.20	0.40	1.09	0.78	0.94	0.88 ± 0.30
γ_s	0.84	0.36	0.83	0.57	0.72	0.66 ± 0.18
reference	Tessier et al, 1992	Hubert et al*	Ladoy et al ⁺	Hubert et al*	Desurosne et al*	

description of rainfall (eqs.1, 3). We are thus able to give a precise (statistical) definition of the possible maximum precipitation at a given scale: we not only clarify the role of scales for the definition of the PMP, but also the role of the limited size of samples used for its estimation. We furthermore showed that the two basic multifractal exponents (C_1 , α) determine the maximum attainable singularities (γ_0 and γ_s) and hence the possible maximum precipitation at a given scale and on a given sample.

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References

- Brax, P. and R. Pechanski, Levy stable law description on intermittent behaviour and quark-gluon phase transitions. *Phys. Lett. B*, **253**, 225-230, 1991.
- Gilman, C. S., Rainfall, in *Handbook of applied hydrology*, edited by V. T. Chow, pp. 9.1-9.68, Mc Graw Hill, 1964.
- Hubert, P. and J. P. Carbonnel, *Dimensions fractales de l'occurrence de pluie en climat Soudano-Sahélien*, Hydrol. continent., **4**, 3-10, 1989.
- Jennings, A.H., World's greatest observed point rainfall, *Monthly Weather Rev.*, **78**, 4-5, 1950.
- Lavallée, D. Multifractal techniques: Analysis and simulation of turbulent fields, Ph.D. thesis, 144 p., McGill University, Montréal (Québec), 1991.
- Lavallée, D., S. Lovejoy, and D. Schertzer, On the determination of the codimension function. In *Nonlinear variability in geophysics: Scaling and Fractals*. Edited by D. Schertzer and S. Lovejoy, pp. 99-110, Kluwer, Dordrecht, 1991.
- Lovejoy, S., Analysis of rain areas in terms of fractals. *Proc. of 20th conference on radar meteorology*, AMS Boston, 476-484, 1981.
- Lovejoy, S. and B. Mandelbrot, Fractal properties of rain and a fractal model. *Tellus* **37A**, 209, 1985.
- Lovejoy, S. and D. Schertzer, Generalized scale invariance and fractal models of rain. *Wat. Resour. Res.* **21**, 8, 1233-1250, 1985.
- Lovejoy, S., D. Schertzer, and A.A. Tsonis, Functional Box-Counting and Multiple Dimensions in rain. *Science*, **235**, 1036-1038, 1987.
- Lovejoy, S. and D. Schertzer, Multifractals, Universality classes and satellite and radar measurements of cloud and rain fields. *J. Geophys. Res.*, **95**, 2021, 1990.
- Lovejoy, S. and D. Schertzer Multifractal analysis techniques and the rain and clouds fields from 10^{-3} to 10^6 m In *Non-linear variability in geophysics: Scaling and Fractals*. Edited by D. Schertzer and S. Lovejoy, pp. 111-144, Kluwer, Dordrecht, 1991.
- Lovejoy, S. and D. Schertzer, Multifractals and Rain. In *New Uncertainty Concepts in Hydrology and Water Resources*, Cambridge, Edited by Kunzewicz, Z.W., University Press, 1992.
- Mandelbrot, B.B., *Les objets fractals, forme, hasard et dimension*, Flammarion, Paris, 1975.
- Mandelbrot, B.B., *The fractal geometry of Nature*, Freeman, San Francisco, 1982.
- Paulhus, J.L.H., Indian ocean and Taiwan rainfall set new records, *Monthly Weather Rev.*, **93**, 331-335, 1965.
- Réménieras, G., *L'hydrologie de l'ingénieur*, 456 pp., Eyrolles, Paris, 1965.
- Raudkivi, A.J., *Hydrology*, 479 pp., Pergamon Press, Oxford New-York, 1979.
- Schertzer, D. and S. Lovejoy, Generalised scale invariance in turbulent phenomena. *Physico-Chemical Hydrodynamics Journal*, **6**, 623-635, 1985.
- Schertzer, D. and S. Lovejoy, Singularités anisotrope, divergence des moments en turbulence. *Ann Sc. math. du Québec*, **11**, 139-181, 1987a.
- Schertzer, D. and S. Lovejoy, Physical modeling and Analysis of Rain and Clouds by Anisotropic Scaling of Multiplicative Processes, *J. Geophys. Res. D*, **92**, 9693-9714, 1987b.
- Schertzer, D. and S. Lovejoy, Nonlinear variability in geophysics: multifractal analysis and simulation. In *Fractals: Physical Origin and Consequences*, Edited by L. Pietronero, pp. 49-79, Plenum, New-York, 1989.
- Schertzer, D. and S. Lovejoy, Nonlinear geodynamical variability: multiple singularities, universality and observables. In *Nonlinear variability in geophysics: Scaling and Fractals*, edited by D. Schertzer and S. Lovejoy, pp. 41-82, Kluwer, Dordrecht, 1991.
- Schertzer, D., S. Lovejoy, D. Lavallée, F. Schmitt, Universal hard multifractal turbulence: theory and observation, in *Nonlinear Dynamics of Structures*, Edited by R. Z. Sagdeev, U. Frisch, F. Hussain, S.S. Moiseev and N.S. Erokhin, World Scientific, pp. 213-235, 1991.
- Schertzer, D., S. Lovejoy, Hard and soft multifractal processes, *Physica A*, **185**, 187-194, 1992.
- Schmitt, F., D. Lavallée, D. Schertzer, and S. Lovejoy, Empirical Determination of Universal Multifractal Exponents in Turbulent Velocity Fields. *Phys. Rev. Lett.*, **68**, 305-308, 1992.
- Tessier, Y., S. Lovejoy, and D. Schertzer, Universal Multifractals: theory and observations for rain and clouds. *J. Appl. Meteor.*, in press, 1992.

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