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Discrete Angle Radiative Transfer
1. Scaling and Similarity, Universality and Diffusion

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In order to facilitate study of very inhomogeneous optical media such as clouds, the difficult angular part of radiative transfer calculations is simplified by considering systems in which scattering occurs only in certain directions. These directions are selected in such a way that the intensity field decouples into an infinite number of independent (e.g., orthogonal) families in direction space, each coupled only within its family. Further discretization, this time in space, lends itself readily to both analytical renormalization approaches (part 2) and to numerical calculations (part 2). We are particularly interested in scaling systems in which the optical density field has no characteristic size over a wide range of scales; these include internally homogeneous media of any shape but are more generally internally inhomogeneous and better described as fractals or multifractals. In this case, the albedo and transmission obey power laws in the thick cloud limit if scattering is conservative. By deriving powerful discrete angle (DA) similarity relations, we show that the scaling exponents that characterize these laws are "universal" in the sense that they are independent of the DA phase functions. We argue that these universality classes may be generally expected to extend beyond DA to include standard (continuous angle) phase functions and transfer equations. By comparing the DA equations with the diffusion equation, we show that in general the thick cloud limits of the two will be different: the thick cloud regime will only be "diffusive" in very homogeneous clouds, hence the term "universality class" is more appropriate. The DA similarity relations indicate that in scaling systems spatial variability is of primary importance, this suggests that far more research should be made to realistically model the spatial variability and to investigate its effect on radiative response, even if the angular aspect of the transfer process is made much less sophisticated than is possible in the classical plane-parallel type medium.

1. INTRODUCTION

1.1. *Context*

Geophysical and astrophysical systems ranging from terrestrial to interstellar clouds involve radiative transfer through extremely inhomogeneous optical media. Structures in both the scattering media and in the associated radiation field frequently occur over wide ranges in scale. The radiative transfer equation implies a linear radiative response with respect to the incident radiation or, more generally, the source function; however, the response relative to the optical properties of the scattering medium (such as optical density) is non-linear as soon as the medium is optically thick in any of its dimensions. Indeed, from this point of view, the multiple scattering process can be regarded as a kind of non-linear low-pass spatial filter yielding a smoothed image of the optical density field. As a result of this smoothing - and the difficulty in adequately accounting for the variability - the effects of inhomogeneity are often ignored.

Geophysical radiative transfer calculations have generally been carried out for plane-parallel (i.e., horizontally uniform) media, with vertical inhomogeneities confined to very narrow ranges of scale (see however Davis et al 1990 for multifractal plane parallel results). In clouds, our prime interest here, this homogeneity assumption has always been ad hoc, lacking both empirical and theoretical basis at least down to scales of a centimeter or so. With the advent of modern in situ or remote measurements, it is untenable even for the prototypical plane-parallel arctic stratus clouds [*Tsay and Jayaweera*, 1984]. Real clouds are known to be highly chaotic, turbulent structures with large variation of liquid water content down to the smallest observable scales.

The problem of determining the radiative properties of inhomogeneous clouds is notoriously difficult and remains an active field of research. The term "inhomogeneous clouds" is to be taken in a very broad sense: we include cloud fields as well as isolated internally homogeneous clouds of finite horizontal extent. A better description would be "non-plane-parallel" since the common feature (and main source of difficulty) in these radiative transfer problems is the presence of non-vanishing horizontal gradients in at least one horizontal direction. This field of research has become known as "multidimensional" radiative transfer and exactly complements the well developed theory of plane-parallel media where radiation field and/or optical properties vary in the vertical only, see *Lenoble* [1977] for an extensive review. However natural this nomenclature may seem, we only retain it for the purpose of cross referencing. We reserve usage of the word "dimension" for the quantitative description of the sparseness of various of the statistical properties with scale (i.e., its fractal dimensions) but also to indicate the dimension of space in which the scattering occurs since by reducing this number from 3 to 2 (even 1) we simplify the problem at hand without necessarily loosing physical insight. Moreover, we will argue that the description of the radiation field's statistical properties over a range of scales (its extreme, or nonlinear "variability") involves multiple fractal dimensions ("multifractals") hence a possible confusion that we wish to avoid.

In the following discussion we exclude from the outset work on "inhomogeneous" atmospheres where variability is confined to the vertical; for the purposes of this study, these stratified media exhibit plane-parallel (or one-dimensional) behaviour. Although the distinction is somewhat arbitrary, horizontally inhomogeneous systems can be divided into two categories: (1) those in which the clouds are internally homogeneous but in which the boundary conditions impose horizontal gradients in the radiation field and (2) those in which the internal optical density field varies in at least one horizontal direction. Arbitrariness

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comes from the fact that the former can be formally included in the latter by allowing discontinuous (step) functions of position and null density values. Category 1 has been the most extensively studied in the literature: simple geometrical shapes (e.g., cubes, cylinders, spheres) have been investigated using various methods [e.g., *Busygin et al.*, 1973; *McKee and Cox*, 1974; *Davies*, 1976, 1978; *Barkstrom and Arduini*, 1977; *Welch and Zdunkowski*, 1981*a,b*; *Preisendorfer and Stephens*, 1984; *Stephens and Preisendorfer*, 1984]. It has been suggested that statistical mixtures of these can model (noninteracting) cloud fields [e.g., *Mullaama et al.*, 1975; *Ronnholm et al.*, 1980; *Welch and Zdunkowski*, 1981*c*]. Genuine cloud fields (or extended clouds) modeled by one- and two-dimensional arrays of these homogeneous entities have also been studied [e.g., *van Blerkom*, 1971; *Busygin et al.*, 1977; *Avaste and Vaynikko*, 1974; *Aida*, 1977; *Wendling*, 1977; *Glazov and Titov*, 1979; *Titov*, 1979, 1980; *Davies*, 1984]. The much more physically relevant case 2 of internally inhomogeneous clouds have attracted somewhat less attention: for instance, *Giovanelli* [1959], *Weinman and Swartztrauber* [1968], *Romanova* [1975], *Romanova and Tarabukhina* [1981] and *Stephens* [1986] investigate systems with variability over a very narrow range of scales whereas *Cahalan* [1989] studies systems with broad (power law) spectra. All of these authors consider spatial variability in one horizontal dimension only, whereas *Stephens* [1988*a,b*] offers a general formalism and discusses arbitrary variability over many scales in connection with (two-dimensional) satellite imagery. Radiative transfer with variability in both vertical and horizontal direction(s) is more involved: see *Mosher* [1979] and *Welch* [1983] for deterministic (and narrow band) approaches and *Welch et al.* [1980] for white noise (uncorrelated) fields and *Gabriel et al.* [1986] for more physically justified fractal structures with spatial correlations and power law (scaling, rather than flat) spectra.

This very succinct review is only concerned with the problem of multiple scattering with (usually collimated) external illumination. Stellar astrophysicists have generally focussed on the spatial variability of internal (thermal) sources and frequency redistribution for the continuum and line spectra, respectively, as well as the effects of departure form local thermodynamical equilibrium [see *Jones and Skumanich*, 1980, and references therein]. Before leaving this topic, we might mention the other related problems notably that of inhomogeneous ground reflectance under a homogeneous scattering atmosphere, a problem with important remote sensing applications (e.g., *Malkevich* [1960], *Tanré et al.* [1981, 1987], *Diner and Martonchik* [1984*a,b*], for different geometries and approaches), and the thermal infrared problem for horizontally variable atmospheres (finite cuboidal clouds in particular; see, for instance, *Harshvardan et al.* [1981], *Stephens and Preisendorfer* [1984], and *Stephens* [1986]).

1.2. *Overview*

In this series, we present some recent work concerning a subclass of transfer systems in which the propagation occurs only in discrete directions, for example along mutually orthogonal directions; hence the generic name discrete angle (DA) radiative transfer (preliminary accounts have appeared in several places [*Gabriel et al.*, 1986; *Lovejoy et al.*, 1988, 1989; *Gabriel*, 1988; *Davis et al.*, 1989]). These systems can be viewed as a limit imposed on the phase functions in which the intensity field decouples into an infinite number of families in (absolute) direction space; within each family, interaction (coupling) only occurs between members. These self contained DA systems can then be treated separately, greatly simplifying the angular part of the transfer process which is a major source of difficulty in the conventional approach. DA formalism, presented in this paper, allows for arbitrary optical density field and boundary conditions and for systems with DA phase functions is exact, not approximate. The specific results obtained in part 2 [*Gabriel et al.*, this issue] and part 3 [*Davis et al.*, this issue] pertain to horizontally finite homogeneous or inhomogeneous clouds modelled by fractals. Part 2 focuses on an approximate but analytic renormalization approach to DA radiative transfer applicable to scaling systems; it is applied to homogeneous and deterministic fractal clouds. Part 3 follows up on this, using Monte Carlo simulation and extends the study to a simple class of random fractal clouds; it also includes a detailed discussion of some of the meteorological implications of this work. For some preliminary multifractal (rather than monofractal) results, see Lovejoy et al 1990 and Davis et al 1990.

After some preliminaries about scaling and similarity, the formal development proceeds as follows. Starting with the continuous angle radiative transfer equation in one two or three dimensions, we obtain its DA counterpart which is a finite system of coupled linear partial differential equations. By making the usual requirement that scattering probabilities only depend on relative (discrete) angles, we are able eliminate all but a countable infinite number of systems each characterized by highly symmetric coupling (matrices). These particularly interesting DA systems, in turn, are contrasted with the discrete ordinate solution of the standard transfer equation. Among these, only a small finite subset can be spatially discretized on a (regular) lattice, but DA equations on discrete spaces can also be obtained from first principles. The spatially continuous limits of the latter are readily compared with the previously obtained DA systems; this gives us some insight into potentially powerful analytical and numerical approaches. Next, some simple examples of DA systems are selected and described in more detail.

The "orthogonal" DA systems are then used to obtain very general similarity relations which exactly account for all phase functions (within that class) and seem to hold reasonably well for continuous angle radiative transfer as well (according to limited numerics). We show that diffusion equations can be obtained exactly as (non-physical) singular points of the similarity relations; hence diffusion and DA radiative transfer will generally be in different universality classes. We argue that, the main exception is for transfer through internally homogeneous media (or smoothly varying density fields) which are "trivially scaling". Finally, we examine the close relation between DA photons and the so-called "skating termite" Monte Carlo particles used to model conducting/superconducting mixtures in lattice statistical physics; this analogy proves useful in understanding the radiative behavior of media with embedded holes such as those investigated in parts 2 and 3.

2. PRELIMINARY CONSIDERATIONS

2.1. *Asymptotic Thick Cloud Scaling Laws*

Our overall objective is to simplify the radiative transfer problem sufficiently, so that it becomes analytically and numerically tractable while still remaining relevant to physically realizable clouds. In the thick cloud limit with conservative scattering, the physical size of the medium is the only relevant scale, conveniently measured in terms of optical thickness (τ). Hence we can further simplify our problem by studying cloud geometries that are invariant under simple scale changing operations or zooms, i.e., homogeneous or fractal structures (more empirical and theoretical motivation for the use of fractals as cloud models is postponed until part 3). In these scaling systems the transmittance (T) and albedo (R) will exhibit scaling (i.e., power law) behavior as $\tau \rightarrow \infty$:

$$T - T^* \approx h_T(\mathrm{P})\, \tau^{-\nu_T}$$
$$R^* - R \approx h_R(\mathrm{P})\, \tau^{-\nu_R} \qquad (1)$$

T^*, R^* are the "fixed points" of the scale-changing operation in the thick limit; for internally homogeneous clouds we can anticipate $T^* = 0$ and $R^* = 1$ as we have simply reconstructed the classical semi-infinite medium. We will prove that, when they exist, the scaling exponents ν_R and ν_T are "universal" in the sense that they are independent of the values taken by the DA phase functions, which are conveniently represented by matrix elements P_{ik} (for scattering from direction i into direction k). In contrast to this, the prefactors h_T, h_R are functions of (the matrix) **P**, as indicated explicitly in (1), where signs are chosen so that all the variables are positive (in this limit). The notion of "universality" is borrowed from nonlinear dynamical systems theory and its use is justified by the specific association of the thick cloud limit with an attracting (stable) fixed point of a scale changing operation as shall be seen in part 2. It is precisely this universal DA behavior that gives credence to the conjecture that in general, the thick cloud DA scaling exponents are identical with the corresponding continuous angle exponents; the results from part 3 generally support this idea.

It is not hard to anticipate how important external boundary conditions will be: for example, if these are reflecting (or periodic) in the horizontal rather than absorbing (or "open"), energy (flux) conservation ($T+R=1$) implies that $\nu_R = \nu_T$, $h_R = h_T$, $T^*+R^* = 1$ whether the medium is internally homogeneous or not; on the other hand, for open boundary conditions, light can "leak" out through the sides and we will see that $\nu_R < \nu_T$ which can be interpreted physically since our problem is highly up/down asymmetric (in terms of illumination at the various boundaries). The degree of internal (in)homogeneity is equally important: diffusion-controlled (quasi-homogeneous) systems will have ($\nu_R \leq$) $\nu_T = 1$ which is the plane-parallel value; but in highly inhomogeneous fractal structures where diffusion is likely to fail as a model for radiative transfer, we find ($\nu_R \leq$) $\nu_T < 1$.

There exists a large body of literature on asymptotics but, as far as we are aware of, it is entirely focussed on the subtle variations of (continuous) angular distribution of (specific) intensity with phase function, viewing and illumination geometry, all restricted to plane-parallel systems [see *van de Hulst*, 1980, and references therein]. With DA radiative transfer, we are able to look in the opposite direction: the phase function dependence of DA responses being generally confined to the prefactors is quite secondary compared with the effect of cloud geometry on the scaling exponents.

2.2. *Similarity Relations and the Nonlinear Aspect of Radiative Transfer*

We use the word "scaling" in the very broad sense accepted in the physics literature: invariance of various exponents with respect to scale changes. As we shall see further on, the concept it covers is not unrelated to the scaling analysis of the radiative transfer equation initiated by *van de Hulst and Grossman* [1968] which yields their widely used "similarity relations" originally devised to obtain approximate results for anisotropic scattering from known solutions for isotropic scattering by making the following substitutions:

$$\tau \rightarrow \tau\,(1-\varpi_0 g)$$
$$\varpi_0 \rightarrow \varpi_0\left(\frac{1-g}{1-\varpi_0 g}\right) \qquad (2)$$

where ϖ_0 designates the single-scattering albedo and g is the asymmetry factor (which are related, respectively, to the zeroth and first coefficients in the Legendre expansion of the phase function). It is notable that their analysis does not depend on the optical density ($\kappa\rho$) field; the similarity relations hold only when $\kappa\rho$ is rescaled everywhere. Applications are therefore not limited to plane-parallel geometry, see *Davis et al.* [1989] and part 3 for results on horizontally finite media, in two and three dimensions that obey (2) very well with $\varpi_0 = 1$. These relations in turn inspired the δ-Eddington [*Joseph et al.*, 1976] and δ-M [*Wiscombe*, 1977] approximations; only the former has been used outside of plane-parallel geometry by *Davies* [1978].

This distinction is important as the relevant "rescaling" of τ for conservative scattering (ϖ_0=1) is $(1-g)\tau$, where the $(1-g)$ can be folded into the prefactors of (1). In DA systems, we will show that phase function characteristics such as g do not affect the scaling exponents of optical thickness τ. As previously mentioned, in DA radiative transfer a more general version of similarity is obeyed exactly; in particular, this guarantees the phase function independence (universality) of any scaling exponents. Preliminary analyses show that if applied to continuous angle systems that it is an improvement to the approximate relations (2) above.

The opposite of the limit considered previously (i.e., optically thin clouds) is also interesting. In part 2, we are able to associate it with a repelling (unstable) fixed point (namely, R^*=0, T^*=1) and is therefore very sensitive to the choice of DA phase function. We also retrieve the usual criterion for the crossover from thin to thick regimes, viz. $(1-g)\tau \approx 1$. This regime can also be described by expressions (1). For example for thin homogeneous systems, ν_R and ν_T are trivially universal being both equal to -1. This reflects the well-known fact that thin systems respond linearly to a global change in optical density (since they are dominated by low order scattering). Thus we can view the rescaled optical thickness $(1-g)\tau$, i.e., the "effective" optical thickness (for isotropic conservative scattering), as the basic measure of nonlinearity in the transfer system (with respect to optical density).

2.3. *Radiative Transfer in Any Number of Spatial Dimensions*

The radiative transfer equation is customarily stated (often implicitly) for three spatial dimensions with a two-dimensional direction space which is uniquely parameterized by (polar) coordinates on the unit sphere. We will however be equally interested in systems embedded in only two spatial dimensions simply because they are easier to analyze yet sophisticated enough to gain insight into the radiative effects of inhomogeneity in all (available) directions. Let $I_s(x)$ be the (specific) intensity, i.e., the flux of radiant energy in direction s at position x per unit of "solid angle" and unit of "area" perpendicular to s; these last quantities and units must of course be taken in their d-dimensional sense. In d spatial dimensions, the basic radiative transfer equation (in absence of internal sources and neglecting polarization) can be written:

$$(s\cdot\nabla)\, I_s(x) = -\,\kappa\rho(x)\left\{I_s(x) - \int_{\Xi_d} p(s',s) I_{s'}(x)\, d\Omega_{s'}\right\} \qquad (3)$$

where we have introduced the following notation: $p(s',s)$ is the phase function for scattering from direction s' into s with its usual probabilistic interpretation, Ξ_d is the d-dimensional unit sphere with $d\Omega_s$ representing an element of its surface around s, optical density (or total cross-section per unit of d-volume) is designated by $\kappa\rho$ (to which we confine all spatial variability), where ρ is particle density (by mass or number) and κ the extinction coefficient (opacity or total cross-section per particle respectively). When appropriate, the definitions and units of these quantities must take into account the dimensionality of space. We adopt the following normalization conventions:

$$n_d = \int_{\Xi_d} d\Omega_s = 2\pi \text{ (for } d{=}2),\ 4\pi \text{ (for } d{=}3) \quad (4)$$

$$\int_{\Xi_d} p(s',s)\,d\Omega_{s'} = \varpi_0 \text{ (for all } s \in \Xi_d) \quad (5)$$

In essence, (3) is a monokinetic transport (Boltzmann) equation with its right-hand side describing streaming in phase space and its left-hand side a collision integral with a sink term (extinction) and a source term (multiple scattering). The absence of the external force contribution to the right-hand side makes (3) appropriate for the description photon transport [*Mihalas*, 1979]. In the important case of radiative transfer with conservative scattering, (3) can be viewed as a detailed balance between spatial gradients (left-hand side) and angular anisotropy (right-hand side). To see this suppose I_s is independent of s (the radiation field is locally isotropic), using (5) the right-hand side becomes $-(1-\varpi_0)\kappa\rho I_s = 0$ (here); in other words, anisotropy drives directional gradients. The converse is easily proven in the case of isotropic but not necessarily conservative scattering, i.e., take $p(s',s) = \varpi_0/n_d$: using (4), we see that vanishing left-hand side implies either that $\kappa\rho = 0$ (medium is locally void) or, more interestingly, that I_s is equal to its average over Ξ_d (I_s is locally isotropic). This interpretation of (3) takes on all its importance in extremely variable optical density fields $\kappa\rho(x)$, which is bound to influence the spatial variability of $I_s(x)$ and in view of the highly asymmetric/anisotropic boundary (illumination) conditions for the multiple scattering problem.

We acknowledge the fact that, insofar as $\kappa\rho$ is independent of I (or any other measure of radiant energy density), (3) is linear in I; this fact is used implicitly as soon as we talk about albedo or transmittance. This considerably simplifies the scaling (similarity) analysis of (3): an overall change in $\kappa\rho(x)$ is equivalent to a zoom on x (hence $s\cdot\nabla$). The basic idea in similarity theory is to relate intensity fields in systems identical except for phase functions and optical density, hence thicknesses (in all directions); this is done using (3). More precisely, the two systems will have the same intensity fields if their rescaled optical thicknesses and phase functions are the same. The similarity will only be approximate if the rescaling is performed only up to a given order in some expansion of the scattering/extinction kernel $K(s',s) = p(s',s) - \delta(s'-s)$ which can be used to regroup the two terms on the right-hand side of (3) [*McKellar and Box*, 1981].

The first exploitation of angular discretization in radiative transfer, apart from the original "two-flux" theory by *Schuster* [1905], was *Chandrasekhar's* [1950] systematic generalization of it, known as the discrete ordinate solution of (3) for axially symmetric phase functions in plane-parallel geometry where the streaming operator $(s\cdot\nabla)$ becomes $\mu d/dz$ where μ is the vertical direction cosine of s. In its original d=3 setting it proceeds as follows: by using N-point Gaussian quadrature (after Fourier expansion in azimuth) for the polar part of the solid-angular integral, one obtains a solvable "$2N$-stream" approximation to $I_s(z)$, the accuracy of which increases with N along with computational effort. Our approach is very different, since we are interested in systems which obey (3) exactly with DA phase functions.

3. DA Radiative Transfer: Fundamentals

3.1. *DA Radiative Transfer Systems With Phase Functions Dependent on Scattering Angle Only*

The basic idea of DA radiative transfer is to choose phase functions $p(s',s)$ which are finite sums of (Dirac) δ-functions, i.e., that describe scattering within a finite number of directions. In this case, the integral in (3) reduces to a matrix multiplication by a finite scattering matrix P_{ik} (for scattering from direction i to k), and $I_s(x)$ to a finite (formal) vector $I_i(x)$. We obtain:

$$(k\cdot\nabla)I_k = -\kappa\rho(x) \sum_k (1-P)_{ki}\, I_i \quad (6)$$

It is worth noting right away that there is no intrinsic difference between the physical definitions of DA intensity I_k which is governed by (6) and its continuous angle counterpart which is governed by (3): both are conserved quantities along the beam (in absence of extinction). On the other hand, we will be tempted to associate (and indeed we will compare quantitatively, in part 3) - this DA "intensity" (or radiance) with a continuous angle "flux" (or irradiance), which is essentially diluted by space along a bundle of rays, i.e., it obeys the "$1/r^{d-1}$" law, which is a basic tenet of standard radiative transfer. A corollary of this is that in DA radiative transfer one is no longer interested in distinguishing between collimated and diffuse illumination conditions at boundaries.

The elements of P_{ik} in (6) can still be interpreted as the relative probability of scattering from direction i into k and the finite set of selected directions $\{k\}$ is effectively "decoupled" from the continuum of other available directions Ξ_d-$\{k\}$. As usual, when dealing with δ-functions, it can helpful to view the DA phase function (the matrix P) as the limit of a sequence of continuous angle phase functions such that the intensity field decouples more-and-more into (infinitely many) finite families of beams. However, at this level of generality, the scattering probabilities depend in general on the absolute directions i and k, not just on the relative scattering directions, implying a strong anisotropy in the system possessed by relatively few physically interesting systems. A further disadvantage of such general DA systems is that the matrix elements P_{ik} give no information about the behavior of the system for intensities at directions other than over the finite set $\{k\}$.

In the following, we therefore restrict ourselves DA systems in which scattering probabilities depend only on the relative (scattering) angle between i and k; this is the DA version of the usual assumption in continuous angle radiative transfer that $p(s',s) = p(\theta_{s,s'})$ where $\theta_{s,s'} = \cos^{-1}(s\cdot s')$(necessary in particular for the discrete ordinate method). In the continuous angle case, this implies axial symmetry for the phase function; in the DA case, it implies an even higher degree of symmetry, e.g., along three mutually orthogonal axes in $d = 3$. In this case, the absolute orientation of any coordinate system introduced to describe the transfer process can be arbitrary (even if it is used to break the axial symmetry as just mentioned); there are no absolute directions, hence the matrix P_{ik} specifies the coupling within an infinite number of independent families in direction space.

The requirement that the (finite) matrix P_{ik} depends only on $i\cdot k$ greatly restricts the number of possible DA systems. To determine those systems which satisfy this requirement, we first note that it is equivalent to saying that the set of transformations needed to map unit vectors i unto k form a (nondegenerate and nontrivial) finite subgroup of the corresponding rotation/reflection group O(d). By "nondegenerate", we mean a subgroup that cannot be projected unto a finite subgroup of O(d-1) and by "nontrivial", we mean a subgroup that does not reduce to the identity element ($x \to x$) of O(d-1). We shall use the notation DA(d,n) for n beams in d dimensions.

Enumeration in d=1. On a line, only two directions are possible; hence only one DA system that we shall denote DA(1,2) (the "two-flux" model). O(1) is itself finite as it contains only the identity and parity ($x \to -x$) transformations, the condition for nondegeneracy is therefore irrelevant.

Enumeration in d=2. In the plane, we have a countable infinity of acceptable DA systems, each one corresponding to a nondegenerate finite subgroup of O(2) generated by a rotation through $2\pi/n$ for n=3,4,5,$\cdots$ which we shall designate by DA(2,n). Notice that the case n=1 is trivial and the case n=2 is excluded because rotation through π is equivalent to parity and is therefore degenerate.

Enumeration in d=3. In space, we have but five possibilities each corresponding to one of the five Platonic solids (or fully regular polyhedra): DA(3,4) for the tetrahedron, DA(3,6) for the cube, DA(3,8) for the octahedron, DA(3,12) for the dodecahedron, and DA(3,20) for the icosahedron. This indeed is the only way to divide the 4π steradians of Ξ_3 equally while maintaining the same (discrete) isotropy around every beam, this excludes the 13 semirregular (or Archimedian) solids, their duals (or Catalan) solids obtained by truncation or stellation of the above; see *Smith* [1982] for details.

Notice that the "trivial" (single-beam) DA(d,1) system is completely solved by the Bouger-de Beer law of (exponential) extinction. In many applications (spatial discretization in particular), it is desirable that the DA system allow backscattering; this implies that the associated subgroup of O(d) contains parity. Eligible DA systems would then be DA(1,2), DA(2,n) (with n=4,6,8,$\cdots$) and DA(3,6), DA(3,8), DA(3,12), DA(3,20), since the tetrahedron does not have "opposite" faces. The simplest of these are DA(d,2d) will be used extensively in the following, they correspond to mutually orthogonal beams (when d>1). The dodecahedron approach to radiative transfer (DART) [*Whitney*, 1974] is closely related to the DA(3,12) system and has been used primarily to optimize radiative transfer codes; along with *Chu and Churchill* [1955], *Siddal and Selçuk* [1979], *Mosher* [1979], and *Cogley* [1981], we favor the DA(3,6) model for its conceptual simplicity.

3.2. *DA Equations on a Lattice and Their Spatially Continuous Limit*

Spatially discretized DA radiative transfer equations can be obtained from first principles by considering a lattice regularly covering the d-dimensional space of interest. These spatially discrete equations are interesting for several reasons. First, they allow us to establish a relationship between radiative transfer in inhomogeneous clouds and certain diffusion problems in lattice statistical physics, see section 6. Second, they allow us to apply approximate real space renormalization (i.e. "doubling") techniques; see part 2. Third, in media with arbitrary optical density fields they can be used in straightforward numerical calculations (i.e., as finite difference equations), if all intensity fields are desired they can be numerically more efficient than the alternative Monte Carlo methods; see part 3. For the moment, we are interested in obtaining their spatially continuous limit and comparing it with the corresponding DA radiative transfer equation (6).

Consider a space-filling collection of identical cells in d dimensions. Denote the fundamental lattice constant by l and the vectors joining the neighboring cells by $k_n l$. The optical properties of each cell are such that scattering can occur only along the lattice directions defined by the k_n. The "interaction principle", which is a statement of linearity of radiative response with respect to sources [*Preisendorfer*, 1965], then yields in absence of internal sources:

$$I_i(ml) = \sum_{k} \sigma_{ik}(ml)\, I_k[(m-k)l] \qquad (7)$$

where we sum over all the DA scattering directions **k** (dropping subscripts); see Figure 1 for an illustration. The $\sigma(ml)$ are

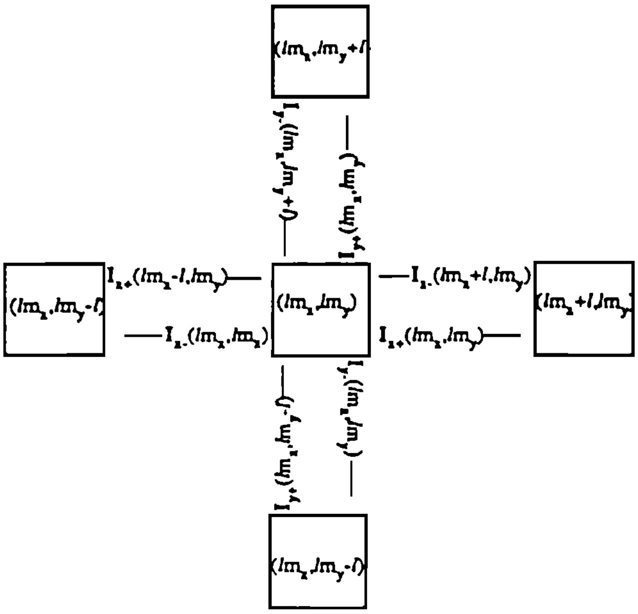

Fig. 1. Radiative interaction between square cells of size l is shown. If not absorbed, light scatters only along lattice directions towards nearest neighbors scattering elements, with probabilities denoted R,T, and S: For instance, $I_{+y}(m_x l,m_y l) = RI_{-y}(m_x l,(m_y+1) l) + TI_{+y}(m_x l,(m_y-1)l) + SI_{+x}((m_x-1)l,m_y l) + SI_{-x}((m_x+1)l,m_y l)$.

matrices of transfer coefficients whose existence is assured by the interaction principle itself. In this formulation, $I_i(ml)$ is a single number (neglecting polarization) that characterizes a whole distribution of intensity along the interface (corresponding to direction i) of the cell (positioned at ml). Since (7) expresses the fact that output of one cell is input to another, it is desirable to think of all these intensities as uniform along cell interfaces; this will only be the case in the limit where all the cells are optically thin. In other words, in this limit only does σ depends solely on the optical properties of the scattering medium filling the cell, i.e., it becomes independent of the (normalized) field I as gradients become negligible along cell interfaces.

In (7) we have implicitly chosen the DA beam directions $\{i\}$ as described in the previous subsection. Notice that the cells are dual to these direction sets and their associated solids and recall that they must now fill their embedding spaces.

Enumeration in d=1. On a line, spatial discretization poses no special problem (the"two-beam" model can even be solved without recourse to calculus, see Appendices A and C in part 2).

Enumeration in d=2. In the plane, we can exploit the three well-known regular tesselations of the plane: by squares, by equilateral triangles (both are used in part 2) or by regular hexagons; these lattices are associated respectively with a subclass of DA(2,6) (indeed "up" and "down" triangles must be alternated, see below), DA(2,4) and DA(2,6) models.

Enumeration in d=2. In space, we are interested in those Platonic polyhedra that are also (or can be combined into) parallelohedra or Fedorov solids, i.e., they fill space: "up" and "down" tetrahedra and a subclass of DA(3,8), cubes and DA(3,6), octahedra and DA(3,8); only DA(3,6) is exloited in parts 2 and 3.

Consider the case of a triangular lattice: the DA(2,6) subclass of interest corresponds to the inhibition of "transmittance" (σ_{ii}=0) since there is no opposite face as well as "scattering" through ±120°. Thus "transfer" of a given beam through a single cell feeds radiant energy into its opposite (at 180°) and the two at ±60°; of course, the same source of energy will feed the three other beams (including itself) upon crossing a second cell (or more).

As usual in radiative transfer problems, (7) can be given a probabilistic interpretation. We simply require particles to move along the lattice directions from one cell to the next, changing from direction i to k with probability σ_{ik}. This general case (where σ is arbitrary) is called a "correlated random walk" [*Renshaw and Henderson*, 1981]; it is also a first order Markov process. When σ is not far from the identity matrix, there is a small probability per step of scattering, the single cell equivalent optical thickness (in direction i), $-\log_e\sigma_{ii}$, is small and the particles will perform ballistic trajectories over exponentially distributed distances. In this case, we recover the standard Monte Carlo method for radiative transfer calculations: the particles model the behavior of photons (other kinds of Monte Carlo particles are discussed in section 6).

The only direct applications of (7) to inhomogeneous (including simply finite) clouds of which we are aware are by *Mosher* [1979], who called a cubic lattice system a "building block model", and by *Cogley* [1981], who primarily examined quite thin clouds.

In order to relate (7) to the DA radiative transfer equation (6), we now take the small l limit by first expanding I_k into a Taylor series around $x = ml$:

$$I_k[(m-k)l] = [1 - l\,(k\cdot\nabla) + \frac{l^2}{2}(k\cdot\nabla)^2) - \ldots]\, I_k(x) \tag{8}$$

Assuming that σ^{-1} exists and letting $\mathbf{1}$ denote the identity matrix (i.e., $1_{ik}=\delta_{ik}$), we obtain by using (7) to eliminate $I_k[(m-k)l]$ from (8):

$$(k\cdot\nabla)I_k = \frac{1}{l}\sum_i (1-\sigma^{-1})_{ki}\, I_i + \frac{l}{2}[(k\cdot\nabla)^2 - \ldots]\, I_i \tag{9}$$

Furthermore, assuming that the quantities Q_{ki} which are defined by

$$\mathbf{Q} \equiv \lim_{l\to 0} \frac{1-\sigma^{-1}}{l} \tag{10}$$

exist everywhere, then for vanishing l (and increasing m so that $x=ml$: remains constant), we obtain

$$(k\cdot\nabla)I_i = \sum_i Q_{ki}\,(x) I_i(x) \tag{11}$$

Comparing eqs. (11) and (6), we see that the two are identical if $\mathbf{Q}(x)=-\kappa\rho(x)(1-\mathbf{P})$. Recall that the former is valid in the conditions specified in Appendix A, where the higher order terms in (9) are negligible. Hence, in terms of the transfer matrix σ, we obtain from (10) taken at finite l:

$$\sigma = [\mathbf{1} + (1-\mathbf{P})\tau_0]^{-1} \tag{12}$$

where we have written $l\kappa\rho(x)$ as τ_0, which is the optical thickness of the single cell (at $x=ml$:) in the spatially discrete system (7). As expected, (12) reduces to the single-scattering result in the limit of small τ_0:

$$\sigma = \mathbf{1} - (1-\mathbf{P})\tau_0 + O(\tau_0^2) \tag{13}$$

In the above $(1-\tau_0)\mathbf{1}$ corresponds to zeroth order scattering (direct transmission) and $\tau_0\mathbf{P}$ to first order scattering. Identifying the diagonal elements of (13) with (1a), we find $T^*=1$, $h_T=t-1<0$ and $v_T=-1$; comparing similarly with (1b), we find $R^*=0$, $h_R=-r<0$ and $v_R=-1$.

Alternatively, (7) can be regarded as a finite difference approximation to (6), as long as l is taken small enough (and the boundary conditions such) that the high order terms in $(k\cdot\nabla)^2 I_k$, etc., are small compared with the first order term $(k\cdot\nabla)I_k$ in (9); see Appendix A for necessary and sufficient conditions for this to occur. A sufficient condition is that $\tau_0<<1$, i.e. when the diagonal elements of σ corresponding to forward scattering or direct transmission are dominant. In terms of the particle interpretation of (7), this means that the particles will behave as photons as long as they have only a low probability of changing direction in each cell. However, numerical results to be found in part 2, as well as theoretical arguments developed in Appendix A, indicate that this condition is unnecessarily restrictive; the solutions of (7) will be smooth enough to represent good approximations to (6) as long as all the eigenvalues of σ^2 are ≈ 1 and vary smoothly. However the latter case involves unphysical phase functions, hence meaningful results will always require σ nearly diagonal. Note that the numerical solution of (7) is easily obtained by over-relaxation, iteration, or other straightforward methods.

3.3. *Some Examples of DA Radiative Transfer Systems*

The simplest examples of DA radiative transfer are the "orthogonal" DA(d,2d) systems with d=1,2,3. The discrete space approximations corresponding to (7) involve $(2d)\times(2d)$ transfer matrices σ, $2d$ is the number of (mutually perpendicular) beams (when $d>1$). Since we are considering only the cases where the scattering coefficients depend only on the relative angle through which scattering occurs (i.e., 0, $\pi/2$, π), we obtain the following highly symmetric matrices:

$$\sigma = \begin{pmatrix} T & R \\ R & T \end{pmatrix} \quad \begin{pmatrix} T & R & S & S \\ R & T & S & S \\ S & S & T & R \\ S & S & R & T \end{pmatrix} \quad \begin{pmatrix} T & R & S & S & S & S \\ R & T & S & S & S & S \\ S & S & T & R & S & S \\ S & S & R & T & S & S \\ S & S & S & S & T & R \\ S & S & S & S & R & T \end{pmatrix} \tag{14}$$

in d=1, 2, 3, respectively. The k-sets are {1,-1}, in d=1; in d=2,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{15}$$

and, finally, in d=3,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{16}$$

T is the transmission coefficient of the cell, R is its albedo, and S represents transfer through a side; these need not correspond to single-scattering only. Note that $R+T+2(d-1)S+A = 1$, where $1 \geq A \geq 0$ is the absorption coefficient, which vanishes for conservative scattering. A slightly more complex situation arises when the lattice cells do not all share the same orientation, such as in the case of the plane covered by equilateral triangles, this model is described and used in part 2.

Using eq. (12) and the symmetry of the σ matrices, we see that the corresponding $\mathbf{P}$ matrices are of the same form:

$$\mathbf{P} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \quad \begin{pmatrix} t & r & s & s \\ r & t & s & s \\ s & s & t & r \\ s & s & r & t \end{pmatrix} \quad \begin{pmatrix} t & r & s & s & s & s \\ r & t & s & s & s & s \\ s & s & t & r & s & s \\ s & s & r & t & s & s \\ s & s & s & s & t & r \\ s & s & s & s & r & t \end{pmatrix} \tag{17}$$

for DA(1,2), DA(2,4), and DA(3,6), respectively. Again $a = 1\text{-}t\text{-}r\text{-}2(d\text{-}1)s$ is a measure of (local) absorption; in the terms of Appendix D, we have $a = 1\text{-}\varpi_0$. Fission-type scattering can, of course, be modeled by allowing negative (true) absorption $a<0$ ($\varpi_0>1$).

Writing out the DA radiative transfer equation (6) explicitly in the DA(3,6) case, we obtain

$$[\mathbf{A}_x\frac{\partial}{\partial x} + \mathbf{A}_y\frac{\partial}{\partial y} + \mathbf{A}_z\frac{\partial}{\partial z}]\,\mathbf{I} = -\,\kappa\rho(x)\,(1\text{-}\mathbf{P})\,\mathbf{I} \tag{18}$$

where

$$\mathbf{A}_x = \begin{pmatrix} 1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad \mathbf{A}_y = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad \mathbf{A}_z = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&-1 \end{pmatrix} \tag{19}$$

$$\mathbf{P}\text{-}1 = \begin{pmatrix} t\text{-}1 & r & s & s & s & s\\ r & t\text{-}1 & s & s & s & s\\ s & s & t\text{-}1 & r & s & s\\ s & s & r & t\text{-}1 & s & s\\ s & s & s & s & t\text{-}1 & r\\ s & s & s & s & r & t\text{-}1 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} I_{+x}\\ I_{-x}\\ I_{+y}\\ I_{-y}\\ I_{+z}\\ I_{-z} \end{pmatrix}$$

The DA(2,4) model can be retrieved formally by putting $I_{+x} = I_{-x} = 0$, $\partial/\partial x = 0$ in (18) and, similarily, the DA(1,2) model with $I_{+x} = I_{-x} = I_{+y} = I_{-y} = 0$, $\partial/\partial y = \partial/\partial x = 0$. It is noteworthy that the latter is equivalent to the popular "two-flux" approximation to standard radiative transfer in plane-parallel geometry, which is widely used when fluxes rather than intensities are desired; it is briefly reviewed in Appendix B. As in the equations on a lattice (7), the complete problem is not defined until the boundary conditions on the "vector" $\mathbf{I}$ are specified. Note that although this is a system of linear partial differential equations, many standard methods of solution, such as characteristics, do not work since $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$, are singular.

Except for notation, the full DA radiative transfer system described by (18) and (19) is identical to the "six-beam model" of *Chu and Churchill* [1955] or *Siddal and Selçuk* [1979], who seem to have worked independently. The former authors used it as an approximation to continuous angle scattering in plane-parallel geometry (obtained by taking $\partial/\partial x = \partial/\partial y = 0$ hence $\partial/\partial z = d/dz$), the latter (who incorporate internal sources) compare its performance with other solutions of the radiative transfer problem for enclosures (which is of importance in furnace design). Our exploitation of this idea differs substantially from theirs: we do not consider the DA case as an approximation scheme for continuous angle radiative transfer but rather we study it as a theoretically realizable model interesting in its own right; we return to this question in the discussion at the end of section 5.

4. From Similarity Relations to Universality Classes in DA Radiative Transfer

4.1. *Scattering and/or Absorbing Media*

The simplest DA system of some interest is DA(1,2); its symmetry is such that it exactly obeys a one-dimensional steady state diffusion equation with internal sinks and yields accordingly exponential-type behavior with a characteristic optical length scale known as the "diffusion length"; see Appendix B. In this section, we study some general properties of the much more interesting DA(2,4) and DA(3,6) systems. We start by introducing the following notation:

$$I_{i\pm} = I_{+i} \pm I_{-i} \tag{20a}$$

$$\delta_i = \frac{1}{\kappa\rho(x)}\frac{\partial}{\partial x_i} \tag{20b}$$

where $i = 1, 2, 3$ for the x, y, z directions, respectively. The I_{i-} terms are the d components of the flux d-vector (at position $\mathbf{x}$) in DA radiative transfer; the I_{i+} can be viewed as the contribution of radiation flowing along the i-axis to total intensity, e.g., $J = I_{x+}+I_{y+}+I_{z+}$ (in d=3). Rather than expressing the phase function in terms of r,t,s, we introduce the following equivalent parameterization when and where convenient:

$$a = 1\text{-}t\text{-}r\text{-}2(d\text{-}1)s \tag{21a}$$

$$q = 1\text{-}t\text{+}r \tag{21b}$$

$$p = 1\text{-}t\text{-}r \tag{21c}$$

where $d = 1, 2, 3$ is the dimensionality of the space in which the scattering occurs. Notice that the relative weights of the P_{ik} in (21a)–(21c) are $(\mathbf{i}\cdot\mathbf{k})^n$ with $n = 0,1,2$, respectively; the above are therefore simply related to the zeroth through second Legendre coefficients (ϖ_n) of the phase function; see Appendix D. For instance,

$$a = 1\text{-}\varpi_0 \tag{22a}$$

$$q = 1\text{-}\varpi_0 g \tag{22b}$$

Recall that $\varpi_1/\varpi_0 = 3g$ (in d = 3). We have from (21a) and (21c),

$$p = a\text{+}2(d\text{-}1)s \tag{22c}$$

this parameter can therefore be viewed as a measure of the combined effect of absorption and side scattering; it could be used to model Rayleigh-type scattering which has only zeroth and second Legendre coefficients. For simplicity, we shall assume in the following that the phase functions are constant, i.e., only the optical density varies (typically via $\rho(x)$). Adding and subtracting consecutive pairs of rows in (18), we obtain, respectively:

$$pI_{i+} + 2s\,(J - I_{i+}) = -\delta_i I_{i-} \tag{23a}$$

$$qI_{i-} = -\delta_i I_{i+} \tag{23b}$$

differentiating (23b) and substituting into (23a), we obtain

$$\left(1 - \frac{1}{pq}\delta_i^2\right)I_{i+} = \frac{1\text{-}a/p}{d\text{-}1}\,(J - I_{i+}) \tag{24a}$$

$$I_{i-} = -\,\frac{1}{q}\,\delta_i I_{i+} \tag{24b}$$

Essentially the same equations are obtained and solved numerically by *Siddal and Selçuk* [1979].

We observe that these equations naturally separate into two groups: the first of which (24a) can be solved independently of the second for the I_{i+}, from whence the remaining I_{i-} can be obtained by differentiation using (24b); finally, the various beam intensities can be obtained by the linear combinations dictated by (24a). Note that the basic character of equation (24a) and its solutions is determined by the values of its parameters pq and, say, $1\text{-}a/p = 2(d\text{-}1)s/p$. In spite of this separation of (dependant) variables, this system is still difficult to handle directly, since the d equations in (24a) are still fully coupled (via side scattering). This implies that they cannot be combined into a scalar equation for J. Similarly (24b) is not the usual kind of Fickian law that converts a scalar (measure of the radiation field) into a vector (measure of the flow of radiation). The only exception is when $p=a$ (implying $s=0$) or $d=1$ (making $I_{i+}=J$) the right-hand side of (24a) then vanishes identically, and we recover one-dimensional diffusion equations for each of the I_{i+} separately; in this case we obtain d noninteracting ($s=0$) one-dimensional diffusion fields.

An additional complication comes from the boundary conditions which are more naturally expressed in terms of the $I_{\pm i}$ than the $I_{i\pm}$. Using 20a, 24b, we obtain: we obtain$[I_{i+} \pm \frac{1}{q}\delta_i I_{i+}]_{x \in S}=I_0$ where I0 is an (appropriate) external sources over the boundary S, and the sign ± is the same as that of the normal vector to S.

4.2. *Conservatively Scattering Media*

In the rest of this section, we consider only the important but special case of conservative scattering where $a = 0$. The DA(1,2) system analyzed in Appendix B now obeys a second order ordinary differential equation and yields accordingly $\nu_T = 1$ with $\nu_R = \nu_T$ by conservation of radiant energy (this combination of scaling exponents characterizes plane-parallel geometry throughout this study).

In this important case, the basic character of the equations for higher dimensional systems (24a) depends only on the sign of the product pq; four possibilities exist: $-\infty<pq < 0$, $pq = 0$, $\infty>pq > 0$, $|pq|=\infty$ each associated with a different universality class, as we shall see. For positive finite (physical) phase functions, $1 \geq p \geq 0$, $2 \geq q \geq 0$ and $1 \geq pq \geq 0$ since $pq = (1-t)^2-r^2$ and $(1-t) \geq r$ here. Again, the case $p = 2(d-1)s = 0$ is singular and (24a) reduces to d one-dimensional Laplace equations for each of the I_{i+}. In terms of the discretization (7), the physical regime $pq > 0$ is obtained with $T \approx 1$, $R \approx 0$ for each cell, and the unphysical regime $pq < 0$ can be numerically simulated using (7) with $T \approx 0$, $R \approx 1$ for each cell; see Appendix A. A much more interesting case occurs in the limit $|pq| \to\infty$ because (24a) reduces to a singular matrix equation which implies that all the I_{i+} components are equal, each satisfying exact (two or three dimensional) diffusion equations. This shows that diffusion is not in the same universality class as DA radiative transfer.

Concentrating our attention on the set of equations (24) for the $I_{i\pm}$, and taking $pq > 0$, we introduce the notation

$$\delta_i' = \frac{1}{\sqrt{pq}\,\kappa\rho(x)}\frac{\partial}{\partial x_i} \tag{25}$$

we obtain, using (24)

$$(1 - \delta_i'^2)I_{i+} = \frac{1}{d-1}(J - I_{i+}) \tag{26a}$$

$$I_{i-} = -\sqrt{\frac{p}{q}}\,\delta_i' I_{i+} \tag{26b}$$

We now remark that the first set (26a) no longer contains any explicit reference to the phase functions. In other words, for the I_{i+}, changing the phase functions is exactly equivalent to uniformly rescaling the optical density (hence all optical thicknesses) by $\sqrt{pq}$. (Actually, we may obtain (26) under slightly more general conditions about the constancy of the phase functions: all that is required in order to eliminate explicit reference to p,q in conservative scattering is that the ratio p/q is constant everywhere. This is equivalent to the requirement of a constant side-to-backscattering ratio: s/r = const, which can be seen by expressing $1-t$ in terms of r and s in (21b) and (21c) using (21a) with $a = 0$.) We will exploit this fact to obtain powerful similarity relations which for continuous angle systems can be considered as second order corrections to the similarity relations (2) which involve only $q = 1-g$ when $\varpi_0 = 1$.

To see how this works, consider the solutions $I_i^{(1)}(x;\tau)$ for phase functions defined by p_1 and q_1, where, for notational convenience, we have replaced the given $\kappa\rho(x)$ field by the single parameter τ. This is possible, since as long as we keep the same cloud geometry (only increasing optical densities by constant factors), the density can be parameterized by the optical thickness across an arbitrary part of the system, call it optical "size". Introducing the idea of an "effective" optical size $\tau_{\rm eff} = \tau/\sqrt{pq}$, as long as the boundary conditions on I_{i+} (not on I_{+i} or I_{-i} individually; see below) are the same, we obtain

$$I_{i+}^{(2)}(x;\tau_{\rm eff}) = I_{i+}^{(1)}(x;\tau_{\rm eff}) \tag{27}$$

(Note that the same kind of analysis can be made for cases involving $a>0$; we must then introduce an "effective absorption $=a/p$). Dropping explicit reference to x, and using (26b), we obtain

$$I_{i-}^{(2)}\left(\frac{\tau}{\beta}\right) = \alpha\,\delta_i' I_{i+}^{(1)}(\tau) \tag{28a}$$

$$\beta = \sqrt{\frac{p_2 q_2}{p_1 q_1}} \tag{28b}$$

$$\alpha = \sqrt{\frac{p_2 q_1}{p_1 q_2}} = \frac{q_1}{q_2}\beta \tag{28c}$$

where β is the ratio of optical thicknesses required to give equivalent effective optical thicknesses. Using (27), i.e. the fact that $I_{i+}^{(2)}(\tau/\beta) = I_{i+}^{(1)}(\tau)$, hence $\delta_i I_{i+}^{(2)}(\tau/\beta) = \delta_i I_{i+}^{(1)}(\tau)$, and (26$b$), we obtain

$$I_{i-}^{(2)}\left(\frac{\tau}{\beta}\right) = \alpha\, I_{i-}^{(1)}(\tau) \tag{29}$$

Combining this with (28a) and the definitions of the $I_{i\pm}$, yields

$$I_{+i}^{(2)}\left(\frac{\tau}{\beta}\right) = \frac{1}{2}(1+\alpha)\, I_{i+}^{(1)}(\tau) + \frac{1}{2}(1-\alpha)\, I_{i-}^{(1)}(\tau) \tag{30a}$$

$$I_{-i}^{(2)}\left(\frac{\tau}{\beta}\right) = \frac{1}{2}(1-\alpha)\, I_{i+}^{(1)}(\tau) + \frac{1}{2}(1+\alpha)\, I_{i-}^{(1)}(\tau) \tag{30b}$$

The above generalized similarity relations are valid for all positions x for all conservative scattering DA(d,2d) phase functions (for $d > 1$). In particular, these relations show that an understanding of the behavior of the system for one phase function for increasing optical thicknesses is sufficient. An interesting point which could be useful numerically, is that the isotropic DA phase function (which yields $pq = 1-1/d$) is not the system that will converge fastest to the thick cloud limit, since taking $p = q = 1$ (the maximum possible: $r = t = 0$, all side scattering), yields $\beta = \sqrt{d/(d-1)} > 1$.

In practical applications, relations (30) are not immediately useful, since natural boundary conditions involve specifying I_{+i}, I_{-i} on various boundaries rather than I_{i+} directly. We now show how the appropriate boundary conditions can be found in the latter cases. This result will directly establish universality (DA phase function independence) for quite general thick cloud scaling exponents. For simplicity, in order to demonstrate the method, we will also require symmetry of the scattering medium: either twofold rotational symmetry or reflectional symmetry about a central horizontal plane, although more complex relations can be derived in less symmetric media. For the moment, we also assume either reflective or cyclic horizontal boundary conditions. Using the boundary conditions

$$\int_{top} I_{-z}(\text{top})\, dS_{+z} = 1$$
$$\int_{bottom} I_{+z}(\text{bottom})\, dS_{-z} = 0 \qquad (31)$$

where the elements $dS_{\pm z}$ are projections of the surface elements of the (upper/lower) boundaries on the y- axis (d=2) or x-y planes (d=3). We define the reflection and transmission coefficients as

$$R(\tau) = \int_{top} I_{+z}(\text{top})dS_{+z}$$
$$T(\tau) = \int_{bottom} I_{-z}(\text{bottom})\, dS_{-z} \qquad (32)$$

The shape of the top and bottom boundaries can be quite arbitrary, we can even use an arbitrary incident (top) intensity distribution (the normalization to 1 is just for convenience).

We now exploit the linearity of the radiative transfer equation which ensures that new solutions can be generated by superposition as well as multiplication of old ones by arbitrary constants. We must do this in order to ensure that the resulting (superposed) top and bottom boundary conditions satisfy $I^{(2)}_{i+}(\tau/\beta) = I^{(1)}_{i+}(\tau)$. The appropriate boundary conditions are obtained by illuminating the top as above, but by also illuminating the bottom with an identical radiation pattern except in the +z direction (this is where the symmetry of the the scattering medium is required), and with intensities equal to the negative of the previous ones. We therefore obtain

$$\int_{top} I^{(1)}_{+z}(\text{top})dS_{+z} = R_1\text{-}T_1 \qquad (33a)$$

$$\int_{top} I^{(1)}_{-z}(\text{top})dS_{+z} = 1 \qquad (33b)$$

hence

$$\int_{top} I^{(1)}_{z+}(\text{top})dS_{+z} = 1\text{+}R_1\text{-}T_1 \qquad (33c)$$

On the bottom, we have

$$I^{(1)}_{+z}(\text{bot}) = -I^{(1)}_{-z}(\text{top}) \qquad (34a)$$

$$I^{(1)}_{-z}(\text{bot}) = -I^{(1)}_{+z}(\text{top}) \qquad (34b)$$

$$I^{(1)}_{z+}(\text{bot}) = -I^{(1)}_{z+}(\text{top}) \qquad (34c)$$

In the corresponding medium with the second phase function for which we wish to develop similarity relations, we impose identical radiation patterns, only rescaled by a factor γ determined so that $I^{(2)}_{i+} = I^{(1)}_{i+}$ on the boundaries:

$$\int I^{(2)}_{z+}(\text{top})dS = \gamma(1\text{-}T_2\text{+}R_2) = \int I^{(1)}_{z+}(\text{top})dS = 1\text{-}T_1\text{+}R_1 \qquad (35)$$

With this choice of γ, the bottom boundary conditions will automatically be satisfied, since only the signs will change. Applying definitions (32) and boundary conditions (31) to similarity relations (30), we obtain

$$\int I^{(2)}_{+z}dS = \gamma(R_2\text{-}T_2) = \tfrac{1}{2}(1\text{+}\alpha)(R_1\text{-}T_1)\text{+}\tfrac{1}{2}(1\text{-}\alpha) \qquad (36a)$$

$$\int I^{(2)}_{-z}dS = \gamma = \tfrac{1}{2}(1\text{-}\alpha)(R_1\text{-}T_1)\text{+}\tfrac{1}{2}(1\text{+}\alpha) \qquad (36b)$$

Taking ratios

$$(R_2\text{-}T_2) = \frac{(1\text{+}\alpha)(R_1\text{-}T_1)\text{+}(1\text{-}\alpha)}{(1\text{-}\alpha)(R_1\text{-}T_1)\text{+}(1\text{+}\alpha)} \qquad (37)$$

Since we have conservative scattering, and the horizontal boundary conditions are such that the sides can act neither as net sources nor as sinks, we must have $R_2\text{+}T_2 = R_1\text{+}T_1 = 1$. We therefore obtain

$$T_2\left(\frac{\tau}{\beta}\right) = \frac{T_1(\tau)\alpha}{1\text{+}T_1(\tau)(\alpha\text{-}1)} \qquad (38a)$$

Equivalently:

$$\frac{1}{T_2(\tau/\beta)} - 1 = \frac{1}{\alpha}\left(\frac{1}{T_1(\tau)} - 1\right) \qquad (38b)$$

Note that the factor γ calculated above is all that is required to completely rescale the internal radiation fields in the second medium. In this case, we obtain

$$I^{(2)}_{+i}\left(x,\frac{\tau}{\beta}\right) = \frac{(1\text{+}\alpha)\, I^{(1)}_{+i}(x,\tau) \text{+}(1\text{-}\alpha)\, I^{(1)}_{-i}(x,\tau)}{(1\text{-}\alpha)(R_1\text{-}T_1)\text{+}(1\text{+}\alpha)} \qquad (39a)$$

$$I^{(2)}_{-i}\left(x,\frac{\tau}{\beta}\right) = \frac{(1\text{-}\alpha)\, I^{(1)}_{+i}(x,\tau) \text{+}(1\text{+}\alpha)\, I^{(1)}_{-i}(x,\tau)}{(1\text{-}\alpha)(R_1\text{-}T_1)\text{+}(1\text{+}\alpha)} \qquad (39b)$$

Deriving similarity relations for less symmetric media with more general boundary conditions is possible, although the (matrix) manipulations required can be quite tedious; we shall give two more examples without detailed derivation. First, if we drop the condition of twofold symmetry (but maintain cyclic or reflective horizontal boundary conditions), we must have more information about the response of the system in order to obtain a similarity relation. Specifically, we require the response of the system when illuminated with an arbitrary radiation pattern from below, as well as from above. Denoting the corresponding transmission coefficients T_b, T_t for the bottom and top, respectively, we obtain

$$T_{t2}\left(\frac{\tau}{\beta}\right) = \frac{T_{t1}(\tau)\alpha}{1\text{+}\,(T_{t1}(\tau)\text{+}T_{b1}(\tau))(\alpha\text{-}1)/2} \qquad (40)$$

and similarly for T_{b2}. As expected, this reduces to (38) when $T_t = T_b$.

As another example, we can obtain the corresponding equations for open horizontal boundary conditions (we still require media with $2d$-fold symmetry) which yield

$$Q_2\left(\frac{\tau}{\beta}\right) = \frac{Q_1(\tau)\alpha}{1\text{+}\,\tfrac{1}{2}Q_1(\tau)(\alpha\text{-}1)} \qquad (41a)$$

$$P_2\left(\frac{\tau}{\beta}\right)=\frac{P_1(\tau)\alpha}{1+\frac{3}{4}P_1(\tau)(\alpha-1)} \quad (41b)$$

where we have used the following definitions

$$Q = 1-R+T \quad (42a)$$

$$P = 1-R-T \quad (42b)$$

The equations (38) with cyclic or reflective boundary conditions are readily retrieved as the special case of (41) since $T+R=1 \Rightarrow P=0$, $Q=2T$, from (42). It is interesting to note that although this result is exact only for the DA(2,4) and DA(3,6) systems, preliminary numerical evidence indicates that it provides a remarkably good approximation in standard (continuous angle) radiative transfer at least in plane-parallel geometry. A detailed account of these generalized similarity relations with their derivations and their implications will be given elsewhere.

Except for the symmetry requirements, the above results are still quite general for conservative DA(d,$2d$) radiative transfer (d>1). We now consider the thick and thin cloud limits ($\tau\to\infty$, $\tau\to 0$, respectively) where we expect the functional forms (1). Combining these with the similarity relations (38a) or (38b), we obtain

$$T_2^* = \frac{\alpha T_1^*}{1+(\alpha-1)T_1^*} \quad (43a)$$

$$\nu_{T2} = \nu_{T1} \ (=\nu_T) \quad (43b)$$

$$h_{T2} = h_{T1}\left(\frac{\alpha\beta^{-\nu_T}}{(1+(\alpha-1)T_1^*)^2}\right) \quad (43c)$$

From (43a) we see that if $T_1^* = 0$ ($\tau\to\infty$) then $T_2^* = 0$ too and if $T_1^* = 1$ ($\tau\to 0$) then $T_2^* = 1$ also (i.e., 0 and 1 are the fixed points of this similarity transformation for T). Of course, we have here $R_2^* = 1-T_2^*$ and $h_{R2} = h_{T2}$ as expected in this case of closed horizontal boundary conditions. The exponents $\nu_R = \nu_T$ are left unchanged (similar arguments in the more general case with open sides where $\nu_R \neq \nu_T$ show that ν_R and ν_T are separately conserved by the similarity transformation). This establishes that if scaling limits exist, that the exponents are universal, i.e., phase function independent.

Aside from their utility in deducing the complete radiation fields for any DA phase function given that of any other, these generalized similarity relations are also useful in deducing corrections to the standard (approximate) similarity theory. In the thick cloud limit (1), optical thicknesses must no longer be rescaled by $q_2 = 1-g_2$ as specified in (2) but by

$$\left(\frac{\beta}{\alpha}\right)^{1/\nu_T}\beta^{1-1/\nu_T} = q_2^{(\nu_T+1)/2\nu_T}\left(\frac{p_2}{p_1}\right)^{(\nu_T-1)/2\nu_T} \quad (44)$$

where we have put $q_1 = 1$ ($g_1 = 0$). This reduces to q_2 only when ν_T=1, i.e., the (periodic) medium is homogeneous (hence plane-parallel) or at the very least dominated by diffusion, see next section. Since (even anomalous) diffusion behavior is expected to be different from radiative transfer in the same fractal (see section 6), we do not expect the standard similarity relations (2) to perform well in this kind of medium.

5. DA Radiative Transfer and Diffusion Processes

5.1. *Diffusion as an Approximation to DA Radiative Transfer*

We now take a different approach in order to explore the thick cloud limits of DA radiative transfer, in particular to determine under which conditions such limits can be approximated by solutions of a diffusion equation. We have clearly seen that two diffusion regimes can be obtained exactly: one at pq=0 involving d independent (uncoupled) one-dimensional diffusion equations and the other at $|pq|=\infty$ with a single d-dimensional diffusion equation. Since both correspond to singular points of the similarity transformations for conservatively scattering DA systems, we will expect both diffusion regimes to be generally in different universality classes than DA transfer. In this section we discuss some examples.

We therefore want to search for circumstances under which we might expect to obtain approximate d-dimensional diffusion equations for physically relevant DA phase functions. From our analysis of (24) we expect to obtain a diffusion equation as first order approximation (in $(pq)^{-1}$) if $\delta_z^2 I_{i+} \ll pqI_{i+}$.

For simplicity, we shall only consider the DA(2,4) system; Appendix C deals with the slightly more complicated DA(3,6) case. Considering (24a) for x_i=y,z (d=2), we can readily obtain fourth order equations for I_{y+},I_{z+}:

$$[\delta_z^2\delta_y^2 - pq(\delta_y^2+\delta_z^2)+q^2a(2p-a)]\, I_{y+} = 0 \quad (45a)$$

$$[\delta_y^2\delta_z^2 - pq(\delta_y^2+\delta_z^2)+q^2a(2p-a)]\, I_{z+} = 0 \quad (45b)$$

Recall from definition (45b) that δ_z^2 and δ_y^2 do not commute in general. When $a \neq 0$, the ratio of the zeroth to second order term enables us to define a characteristic length L such that

$$\kappa\rho(x)L \approx \frac{1}{\sqrt{2aq(1-a/2p)}} \quad (46)$$

since, according to Appendix D, $aq = (1-\varpi_0)(1-\varpi_0 g)$, we retrieve the usual "diffusion length" to within a factor $(1-a/2p)^{-1/2}$ dependent on the second Legendre coefficient; 2 standing for d. We expect in this case exponential type behavior; however, as $a\to 0$ (conservative scattering), this length scale diverges; we then anticipate algebraic solutions.

We see immediately from the above, by adding (45a) and (45b) and using $J(x)=I_{y+}+I_{z+}$, that when the highest order terms ($\delta_y^2\delta_z^2$ and $\delta_z^2\delta_y^2$) are (both) negligible, we obtain a bona fide two-dimensional diffusion equation for the scalar quantity J. Using definition (20b), it reads

$$\nabla\cdot\{D(x)\,\nabla J(x)\} = a\kappa\rho(x)\,J(x) \quad (47)$$

where the right-hand side is the rate of destruction of radiant energy (at the given wavelength) by true absorption and the (local) radiative diffusion constant is given by

$$D(x) = \frac{1}{2q(1-a/2p)\kappa\rho(x)} \quad (48)$$

The condition that the high-order derivatives are negligible implies a high degree of smoothness in both the density and radiation fields; it holds best far from sources (e.g., cloud top) and sinks (e.g., cloud sides, especially near the top).

5.2. *Comparison of DA and Diffusion as Approximations to Radiative Transfer*

In essence, the Eddington approximation to continuous angle radiative transfer in d dimensions is an expansion of the local specific intensity field $I_s(x)$ into a scalar field (J) that represents its isotropic part and a d-vector field (flux) that models direction and intensity of the "flow" of radiation. Substitution of this ansatz into the radiative transfer equation (3) yields a second-order (diffusion) equation in J and a Fickian relation for obtaining the flux, given J [*Giovanelli*, 1959]. A δ-Eddington approach is also possible [*Davies*, 1978]; apart from this rescaling, the phase function must be limited to a two-term Legendre expansion within this approximation. In short we have $d+1$ functions of x to be determined, but one obeys a second-order partial differential equation and constrains all the others, two parameters are available to describe the scattering/absorption process (typically ϖ_0 and g). It is well-known that this approximation fails near boundaries: it is intrinsically incapable of adjusting to the prevailing highly anisotropic (boundary) conditions. For the multiple scattering problem, the boundary conditions must indeed be modified yielding the "mixed" or "radiative" boundary conditions for the diffusion equation in J. As mentioned in subsection 2.3 in the case of conservative scattering, anisotropy of $I_s(x)$ with respect to s means strong gradients in $I_s(x)$ with respect to x, implying that higher order terms are at work as in (45*a*).

In summary, we see that diffusion is a poor approximation to radiative transfer whenever spatial gradients are important or (equivalently, in the conservative case) highly anisotropic intensity fields prevail. We therefore expect that in general, (thick cloud) transfer exponents for both radiative transfer and diffusion will be different. It may in some circumstances still be possible to use diffusion approximations: as argued above (for DA) the best case for this is the internally homogeneous medium with or without sides. Unsurprisingly, *Davies* [1978 and 1976] succeeds in reproducing very well his Monte Carlo results for horizontally finite clouds of various aspect ratios by using a three-dimensional version of the diffusion approximation of (continuous angle) radiative transfer. Also, in part 3, we find the diffusion (d=1) value of ν_T (namely, 1) for d>1 for both DA and continuous angle calculations on media horizontally finite or not but only when they are internally homogeneous. Finally we note that (even for plane-parallel media) diffusion poorly models the angular distribution of the diffusely reflected intensity largely because of low order scattering. Diffusion theory can be combined with single-scattering to improve its performance at this task as in the *Sobolev* [1956] approximation, loosing its conceptual simplicity in the process.

In contrast to this DA(d,2d) radiative transfer models the intensity field with 2d functions of position that are fully coupled within a system of first order partial differential equations that can be combined into fourth order partial differential equations (d=2; see above) or integro-differential equations (d=3; see Appendix C). Moreover, they call for three parameters to describe the corresponding phase function (say, t,r,s) when d>1. Notice that 2d>d+1 except for d=1, where again only two phase function parameters need to be specified; another indication that we are retrieving a system that obeys diffusion exactly, namely the "two-flux" model. Boundary conditions for DA radiative transfer are the same as those of continuous angle theory. There are no a priori restrictions on the gradients of the density and/or intensity fields nor on the anisotropy of the latter which makes DA radiative transfer an ideal tool for investigating the most inhomogeneous media. Still DA systems are quite simple, greatly facilitating numerical calculation and, in some nontrivial cases, are sufficiently tractable to allow analytical approaches to be explored. Most of all, DA is a particular case of general radiative transfer which means that if (broad) continuous angle universality classes exist then DA systems are sufficient to study their characteristics.

In conclusion, we see that DA systems are, by construction, an exact description of the radiative transfer process in arbitrary optical density fields with DA phase functions, whereas diffusion will only be approximate (in d>1). DA systems avoid some of the more serious shortcomings of the diffusion approximation that prohibit its use in extremely variable optical density fields one expects to find in clouds. In order to allow spatial gradients of any degree in any (discrete) direction detailed angular information is sacrificed; in part 3, we will see that in many applications this information is not as precious as that gained by leaving the realm of plane-parallel geometry.

In terms of general (mass or radiative) transport theory, the Boltzmann equation with no external forces (3) yields the diffusion equation (47) and coefficient (48) in its continuum limit; the question is whether or not that is as good a model as a simplified version of the former, e.g., its DA counterpart (6)? The concept of universality allows us to reformulate this question in more precise terms, at least in the radiative case: if diffusion and (radiative) transfer do not share the same universality classes and if DA and (continuous angle) radiative transfer do, then DA can be viewed as a better approximation to radiative transfer than diffusion. In the following section we will argue (by analogy) that the first condition is expected to be generally true and in part 3 the second condition is shown to be well verified numerically in general. Since we have defined universality in terms of scaling laws and if the above conjecture proves to be true then, in the thick cloud limit, errors due to the diffusion approximation will diverge whereas those introduced by using DA phase functions will approach a constant factor.

6. The Relation Between (DA) Radiative Transfer and Lattice Statistical Mechanics: Photons as "Blinkered Termites"

A problem in statistical physics that has received considerable attention in the last few years, is the study of the electrical conductivity properties of random media. The prototypical case is of two materials (A,B) with different conductance properties, distributed on lattice sites with probabilities p and (1-p) respectively. The two extreme cases of interest are (1) A is an insulator, B a conductor (the random resistor network, or RRN limit) and (2) A is a superconductor, B a normal conductor (the random superconducting network, RSN limit). The interesting questions that arise in these limits concern the properties of the macroscopic conductance (Σ) of a large system, and how this conductance varies as $p \to p_c$ where p_c is the percolation threshold for the system. Recall that as $p \to p_c$, the size of the connected A regions grow until (at $p=p_c$) the largest is infinite in extent, the system said to "percolate", i.e., in RRNs (or RSNs) a path exists connecting opposite sides of the system with conducting (respectively superconducting) materials. At this point, the A material is distributed over a fractal; see *Stauffer* [1985] for an excellent review. In particular, as p approaches p_c from below, we obtain in the RRN and RSN limits respectively, $\Sigma \approx (p-p_c)^{\mu}$ and $\Sigma \approx (p-p_c)^{-s}$ where μ and s are the RRN and RSN exponents. Although the values of p_c depend on the lattice type, the exponents μ and s are "universal" (as is the fractal dimension of the percolating network); they are found to only depend on the dimension of space. In parts 2 and 3, we employ the same type of universality argument in DA and continuous angle radiative transfer, respectively.

The macroscopic conductance involves solving Kirchoff's (electrical circuit) laws on the lattice. When the lattice size $l \to 0$, we obtain a diffusion equation with local diffusion constant proportional to the local conductance. In a steady state, the voltage $V(x)$ obeys an equation identical to (47) with the diffusion coefficient $D(x)$ replaced by the local conductance. Since eq. (48) shows that the diffusion constant in the diffusion limit of DA radiative transfer is proportional to $(\kappa\rho)^{-1}$, we have a formal analogy between the diffusion regime of DA radiative transfer and the conduction problem, as long as we can ignore the high order terms in eqs. (45*a*) and (45*b*). This analogy is easily generalized from DA to the standard diffusion approximation in continuous angle radiative transfer. The RRN case corresponds to a cloud with $\rho_A = \text{const} < \infty$, $\rho_B = \infty$ and the RSN limit to $\rho_A = 0$, $\rho_B = \text{const} < \infty$. Although not relevant to the following discussion, we note that in the conduction problem the boundary conditions are either Dirichlet (given voltage) or von Neumann (given current) whereas in the problem of radiative diffusion they are always mixed, as previously mentioned.

De Gennes [1976] was the first to point out the diffusive nature of the conduction problem; he also suggested numerically solving the problem using random walk (Monte Carlo) methods. In the RRN limit, the diffusing particle called an "ant" diffuses in the conducting material (the "labyrinth") constrained by the insulator which act as walls. The ant is either "blind" or "myopic". In the former case, it selects a lattice direction at random and each time step, moves ahead one lattice unit in the corresponding direction. If there is a wall, it stops and waits for the next time step. In the myopic case, the ant selects directions only among those available which it chooses uniformly at random. As expected, the large scale properties (e.g., the exponents), are found to be the same in both cases: the myopic and blind ants belong to the same universality class.

The RSN limit is of more interest to us here, since it is the analogue of a cloud made up of uniform optical density with "holes". Unfortunately, it proves to be more complex to analyze. The primary problem is to develop rules that govern the behavior of the particle in the superconducting material. Where the conductance is zero, a particle should travel infinitely fast ($D(x) = \infty$) slowing down only to "burrow" through regions of finite $D(x)$. Hence *De Gennes* [1979] coined the term "termite" for such particles; see Figure 2. *Bunde et al.* [1985] describe a number of attempts to define appropriate rules so that the termite would model diffusion in the RSN limit. One early attempt that failed to reproduce singular behavior at $p = p_c$ (and hence was not a good model of diffusion) involves "skating termites" which perform (isotropic) random walks on the ordinary conductor and ballistic (photon-like) trajectories in the superconductor. It is clearly the ballistic trajectories that lead to its nondiffusive behavior.

It is not hard to see that the "skating" termite is identical to the Monte Carlo particle used in (7) with regions of isotropic transfer coefficients (all elements of σ are equal) mixed with regions with holes (i.e., $\sigma = 1$). Since we have shown that when σ is dominated by forward transfer, the particles behave as photons, we might call our photons "blinkered" termites which tend to deviate only with low probability per step from ballistic trajectories.

We conclude that in clouds with embedded holes, the photons (blinkered termites) are unlikely to approach diffusion limit ($v_T = 1$) for two reasons. First, like the skating termites, they follow ballistic trajectories in the holes hence do not follow standard (diffusive) random walks; in the case of variable $D(x)$, distributed over a fractal, one talks about "generalized" or "anomalous" diffusion processes or sometimes even "nondiffusive" random walks (see *Schlesinger et al.* [1986] or *Havlin and Ben-Avraham* [1987] for an extensive review). Second, the embedded holes imply that gradients in I are likely to be important everywhere; hence the diffusion equation (47) is not likely to be a good approximation to radiative transfer, even with a highly variable coefficient $D(x)$. Conversely skating termites, like their blinkered cousins, the photons, obey systems of partial differential equations such as (18) which are poor approximations to the diffusion equation at least in fractal media; this explains the failure of that model to reproduce RSN phase transitions.

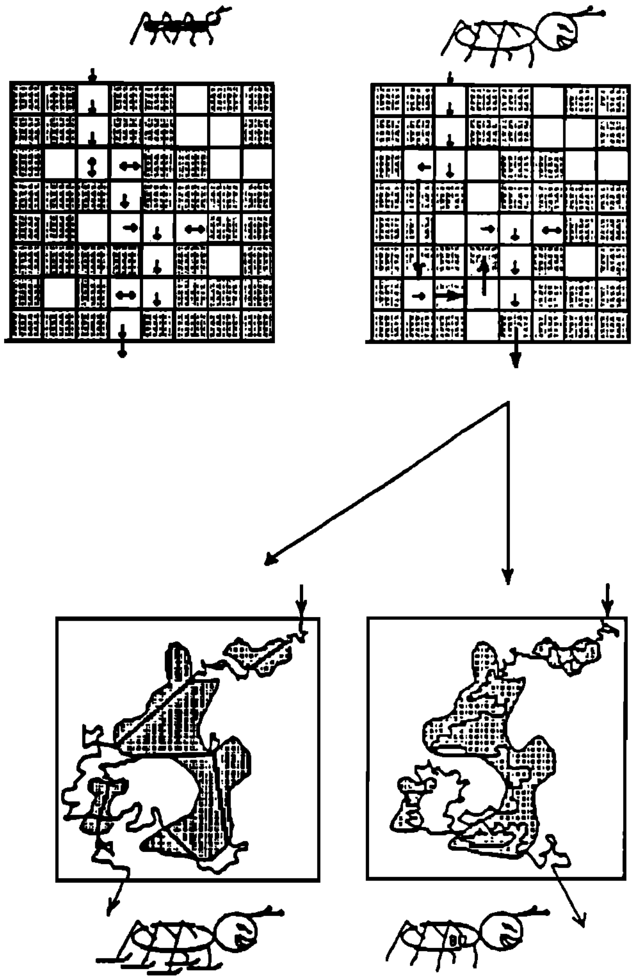

Fig. 2. Schematic illustration showing the "ants" and "termites" used to simulate diffusion in random conductor/insulator mixtures and random conductor/superconductor mixtures, respectively. The upper left shows the ant in the labyrinth; the mixture of normal conductors (white areas) and insulators (shaded areas), which act as walls, define the (RRN) labyrinth. Here the ant is "blind"; "myopic" ants have also been considered. Upper right shows the corresponding termite problem (RSN) which is used to simulate conduction in networks made of superconductor (white) and normal conductor (shaded) regions. The lower figures, right to left, show a much lower resolution view of the RSN problem. On the right, a possible skating termite path which obeys (7) with isotropic transfer coefficients in filled regions and ballistic trajectories in holes. The "blinkered" termite (which models photon trajectories) also obeys (7), with the same transfer coefficients in the hole regions, but with predominantly forward transfer in the cloud regions. On the left, we show a "Boston" termite, which does a random walk in both conducting and superconducting regions speeded up in the superconductor, and with special rules for handling the boundaries. Due to its ballistic trajectories, the "skating" termite is known to be a poor model for diffusion (it does not involve phase transitions in the percolation problem), and hence we suspect the blinkered termite will also have nondiffusive thick cloud behavior.

7. Conclusions

Motivated by a desire to understand radiative transfer in inhomogeneous systems, we have investigated a series of radiative transfer models involving scattering through discrete angles only. These discrete angle (DA) radiative transfer systems are special cases of continuous angle radiative transfer, involving DA phase functions which effectively decouple the intensity field into an infinite number of mutually independent families; within each family coupling only occurs among a small number of directions. We obtain both systems of first order partial differential equations for DA transfer on spatially continuous media and systems of linear algebraic equations for DA transfer on media spatially discretized on various lattices. Upon taking the continuous limit of the latter, conditions for the equivalence of the two formulations are given. This will prove useful as the discrete space equations are exploited analytically and numerically in parts 2 and 3.

The requirement that DA scattering probabilities depend only on the relative scattering angle considerably restricts the number of interesting DA systems; these are enumerated exhaustively. Although others are described, we mainly concentrate on systems with orthogonal axes in two and three dimensions (four and six beams, respectively). The basic mathematical character of these systems is determined by two parameters: one that measures the relative importance of absorption (equivalently, the zeroth Legendre coefficient of the DA phase function) and pq, a product of terms dependent on the first and second Legendre coefficients). In the case of conservative scattering, there are four regimes of interest: $-\infty<pq<0$ implies unphysical (negatively valued) phase functions, $pq = 0$ is singular (we obtain an uninteresting set of independent one-dimensional diffusion equations), $\infty>pq>0$ corresponds to the physically interesting regime and $|pq|=\infty$ corresponds to an exact two or three dimensional diffusion eqution.

In this case explicit phase function dependence can be entirely removed allowing us to derive powerful DA similarity relations. If the DA radiative transfer equation is solved with given but arbitrary boundary conditions and any spatial distribution of optical density for some (conservative scattering) phase function then the corresponding solutions for all other phase functions are obtained by rescaling optical thickness according to these relations; in this context "optical thickness" refers to an arbitrary cross-section of the medium. The only requirement is that the ratio of backward-to-side scattering is everywhere constant.

We are especially interested in media and regimes in which transmittance and albedo are described by power law functions of optical thickness, i.e., homogeneous clouds of any shape, fractals or multifractals either very thick or very thin. An important consequence of the similarity relations is that the scaling exponents are are invariant under the similarity transformation and are therefore "universal" (in the language of nonlinear dynamics); this means that (DA) phase functions can only influence prefactors and are therefore "irrelevant" variables (in the same usage).

We then investigate the relation between DA systems and processes satisfying diffusion equations in various dimensions. In general, the DA systems will obey diffusion equations as long as high-order derivatives can be neglected. This will only be possible in quasi-homogeneous systems; even there, near sources (i.e., cloud tops) they will not be negligible hence the transmission and albedo exponents are expected to differ (except in plane-parallel geometry where only one exponent arises). Finally, we compare DA radiative transfer through fractally inhomogeneous media with electrical conduction through conductor/superconductor mixtures at percolating threshold; this electron diffusion problem has been extensively studied in statistical physics. This comparison supports the idea that thick cloud (DA) limits will generally not be diffusive, even for transmittance.

In the following two parts, we will examine a variety of scaling media using approximate but analytical methods based on renormalization group ideas (part 2) as well as various numerical approaches (part 3); these examples will illustrate the formalism outlined here. In part 3, we examine the important question of extending DA universality classes to continuous angle radiative transfer as well as the meteorological implications of our findings.

Appendix A: On the Smoothness of the Intensity Field and the Spatial Discretization of DA Radiative Transfer Equations

In subsection 3.2 we argued that the spatially discrete DA equations (7) provided good approximations to the DA radiative transfer equation (6) provided that the intensity fields I were sufficiently smooth. A sufficient condition was shown to be that the dimensionless single cell optical thickness τ_0 was small everywhere. Here, we argue that this condition is somewhat more restrictive than necessary, in particular, we seek a condition relating the variations in the dimensionless transfer matrix σ to variations in the intensity fields. We find that when gradients in I imposed by boundary conditions are small, that the eigenvalues of σ^2 are nearly unity, and the relative variation of the eigenvalues of σ^2 are small, then the solution of (7) are also likely to be smooth enough. This has been numerically verified in certain cases discussed in part 3.

We start by introducing the following finite operators:

$$E_k f = f(x-kl)$$
$$\Delta_k = E_k - 1 \tag{A1}$$
$$\nabla^2 = \sum_k \Delta_k$$

where ∇^2 is a finite difference Laplacian and the sum is over all scattering directions $\boldsymbol{k}$. To shorten the notation, we write $E_k I_k = (EI)_k$. Equation (7) can then be written

$$I_k = \sigma_{ik}\,(EI)_k \tag{A2}$$

(In this appendix, summation is implied over repeated indices.)

We now use the modulus (squared) of the vector I_i (noted $|I|^2$) to characterize the amplitude of the intensity vector at each grid point, and use the finite Laplacian to characterize the smoothness of the latter. Bounds on the variation of $|I|^2$ are all the more restrictive, since the elements of σ are positive and with the boundary conditions of interest, I_i is positive everywhere.

$$|I|^2 = I_i^T I_i = (EI)_i^T (\sigma^2)_{ik}\,(EI)_k$$
$$\nabla^2 |I|^2 = 2(EI)_i^T(\sigma^2)_{ik}\,\nabla^2(EI)_k + (EI)_i^T(\nabla^2\sigma^2)_{ik}\,(EI)_k \tag{A3}$$

where we have dropped higher order difference terms and used the fact that due to symmetry $\sigma^T=\sigma$ (with the notation superscript "T" designating "transpose"). Diagonalizing the matrix σ (and introducing primes to indicate the diagonalized intensities), we obtain

$$\nabla^2 |I|^2 = 2\sum_k (EI')_k \Lambda_k^2\,\nabla^2(EI')_k \quad + \sum_k (EI')_k^2\,\nabla^2\Lambda_k^2$$
$$\nabla^2 |I|^2 = \sum_k (EI')_k[2\Lambda_k^2\,\nabla^2(EI')_k + (EI')_k\,\nabla^2\Lambda_k^2] \tag{A4}$$

where Λ_k designates an eigenvalue of σ. In the DA(2,4) case, these are T-R (twice), $T+R-2S$, and $1-A$; in the DA(3,6) case, we find T-R (three times), $T+R-4S$ (twice), and $1-A$.

Equation (A4) shows that the variation in smoothness of I arises from two sources, the first being essentially due to the gradients imposed by the boundary conditions, while the second being due to variations in σ. We first consider the case where the scattering medium is homogeneous (i.e., $\nabla^2\Lambda_k{}^2 = 0$), and we impose some intensity gradient across our system. We know that when τ_0 is small enough, σ is nearly diagonal and all the $\Lambda_k{}^2 \approx 1$, furthermore, in this case, $\nabla^2 |I|^2$ will be sufficiently small that high-order difference terms in the (9) can be neglected. Equation (A4) indicates that changing σ such that all the $\Lambda_k{}^2$ remain ≈ 1 will maintain smooth fields. This is important, since $\Lambda_k{}^2 \approx 1$ holds not only for $T \approx 1$, but also $R \approx 1$.

Now consider introducing spatial variations in σ. According to (A4), as long as

$$\frac{\nabla^2(EI')_k}{(EI')_k} \gg \frac{\nabla^2\Lambda_k{}^2}{\Lambda_k{}^2} \tag{A5}$$

for all k, then we do not expect spatial variations in σ to introduce large inhomogeneities in I. We expect the fields I to remain smooth, and hence to continue to yield good estimates of the solution of the radiative transfer equation.

As an example of the relation between σ and **P**, we can perform the matrix inversion in (12) explicitly. In the DA(2,4) case this yields

$$\tau_0(t-1) = 1 - \frac{T(T+R)-2S^2}{(T-R)(T+R-2S)(1-A)} \tag{A6a}$$

$$\tau_0 r = \frac{R(T+R)-2S^2}{(T-R)(T+R-2S)(1-A)} \tag{A6b}$$

$$\tau_0 s = \frac{S}{(T+R-2S)(1-A)} \tag{A6c}$$

$$\tau_0 a = \frac{A}{1-A} \tag{A6d}$$

hence $a = 0 \Leftrightarrow A = 0$ as expected and as $A \to 1$, $\tau_0 a \to \infty$. In section 4 we showed that the basic character of the solution of a DA radiative transfer problem depends on the product of the fundamental parameters $q = 1-t+r$ and $p = 1-t-r$. Adding and subtracting (A6a) and (A6b), we obtain:

$$\tau_0 p = \frac{2S}{(T+R-2S)(1-A)} \tag{A7a}$$

$$\tau_0 q = \frac{1}{T-R} - 1 \tag{A7b}$$

Taking $A = 1$, the case $T \approx 1$ corresponds to $\tau_0 p > 0$, $\tau_0 q > 0$, whereas the case $R \approx 1$ implies $\tau_0 p > 0$ but $\tau_0 q \approx -2$ implying negative values for the DA phase function, since $q < p$ implies $r < 0$; physically realizable values of t,r,s,a (between 0 and 1) give $0 \le p \le 1$, $0 \le q \le 2$. Note that if we allow for fission-type scattering in which $a < 0$ (or $\varpi_0 > 1$), but maintaining $t,r,s > 0$, then p and q can be negative but we cannot have $p > 0$ and $q < 0$ as required here; hence $R \approx 1$, $T \approx 0$ is not a model of fission. Finally, the discretized diffusion equation which is obtained with R=T=S=1/4 corresponds to $|pq| = \infty$ as expected

Equations (45a) and (45b) show that, when both diffusion and higher order derivative terms are important everywhere (as in fractal clouds), the sign of the product pq is important in determining the character of the equations; indeed in conservative scattering, it completely determines the character. Therefore pq changes sign when we go from $T \approx 1$, $R \approx 0$ to $R \approx 1$, $T \approx 0$, and we expect to change universality classes (as defined by the scaling exponents) in the process. In part 3, we will see that this occurs in all cases except the DA(2,4) model applied to the homogeneous square medium.

Appendix B: The DA(1,2) System or the "Two-Stream" Approximation to Radiative Transfer Through Plane-Parallel Media

Although the results are well known, it is worth showing how the DA(1,2) system is strictly equivalent to "two-flux" approximation (without terms for the direct beam) for radiative transfer in plane-parallel media which was been extensively reviewed by *Meador and Weaver* [1980]. It can of course be solved exactly. Putting $I_{+x} = I_{-x} = I_{+y} = I_{-y} = 0$ in (18) and $\partial/\partial z = d/dz$ we obtain its $d = 1$ equivalent :

$$I_{+z} = -\frac{1}{\kappa\rho(z)}\frac{dI_{+z}}{dz} + tI_{+z} + rI_{-z} \tag{B1a}$$

$$I_{-z} = \frac{1}{\kappa\rho(z)}\frac{dI_{-z}}{dz} + rI_{+z} + tI_{-z} \tag{B1b}$$

Eliminating I_{-z} and using the usual change of variables $d\tau(z) = \kappa\rho(z)dz$, we obtain

$$\left[\left(1-t+\frac{d}{d\tau}\right)\left(1-t-\frac{d}{d\tau}\right) - r^2\right] I_{+z} = 0 \tag{B2}$$

writing $q = 1-t+r$ and $a = 1-t-r$ ($= p$ also, in this $d = 1$ case), which have been assumed constant here, we obtain an identical diffusion equation for either "flux", actually a DA intensity:

$$\left[aq - \frac{d^2}{d\tau^2}\right] I_{\pm z} = 0 \tag{B3}$$

This makes one-dimensional two beam DA radiative transfer a very special case, since we have seen in section 5 that, in the corresponding two- or three-dimensional systems (with $s = 0$), diffusion can only be obtained in the bulk of thick quasi-homogeneous clouds, far from all boundaries. The general solution of (B3), for $aq > 0$ is, say

$$I_{-z}(\tau) = I_0 e^{-\tau\sqrt{aq}} + I_1 e^{+\tau\sqrt{aq}} \tag{B4}$$

where I_0 and I_1 are constants determined by the boundary conditions $I_{-z}(0) = 1$, $I_{+z}(\tau_1) = 0$, where τ_1 is the total optical thickness; recall that $I_{-z}(\tau)$ determines $I_{+z}(\tau)$ via (B1b). $1/\sqrt{pq}$ is the well-known diffusion length scale measured (locally) in units of (photon mean free path) $1/\kappa\rho(z)$. When $a = 0$ (conservative scattering, infinite diffusion length), we obtain either by taking the limit as $a \to 0$ of (B4) or returning to (B3) with $aq = 0$:

$$I_{-z}(\tau) = I_0 + I_1\tau \tag{B5}$$

Using the above boundary conditions, we obtain $I_0 = 1$, $I_1 = -r/(1+r\tau_1)$; thus transmission (T) and albedo (R) are given by

$$T = \frac{I_{-z}(\tau_1)}{I_{-z}(0)} = \frac{1}{1+r\tau_1} \tag{B6a}$$

$$R = \frac{I_{+z}(0)}{I_{-z}(0)} = 1-T \tag{B6b}$$

Identifying with the asymptotic expansions (1), this yields for $\tau_1 \to \infty$, $\nu_T = \nu_R = 1$ with $h_T = h_R = 1/r$. Using the results of Appendix D, we see that $r = (1-g)/2$ for $\varpi_0 = 1$ and $aq = (1-\varpi_0)(1-\varpi_0 g)$ in general.

Appendix C: The DA(3,6) Radiative Transfer System Contrasted With Three-Dimensional Diffusion

As we shall see below, the DA(3,6) model is more complex to analyze than its two-dimensional counterpart, although the basic conclusions of the section 5 still hold. Introducing the notation

$$I_{x\pm}=I_{+x}\pm I_{-x}$$
$$\delta_x=\frac{1}{\kappa\rho(x)}\frac{\partial}{\partial x} \qquad \text{(C1)}$$
$$D_x=p-q^{-1}\delta_x^2$$

similarly for y and z, with the definitions (21) for p,q and a (with d=3). Starting with (18) and (19), some straightforward manipulation yields

$$\begin{bmatrix} D_x & -2s & -2s \\ -2s & D_y & -2s \\ -2s & -2s & D_z \end{bmatrix}\begin{bmatrix} I_{x+} \\ I_{y+} \\ I_{z+} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(C2)}$$

which, by substitution, leads to

$$[\frac{1}{2s}(D_zD_x-4s^2)-(D_z+2s)(D_y+2s)^{-1}(D_x+2s)]\,I_{x+}=0 \qquad \text{(C3)}$$

with similar equations for I_{y+} and I_{z+} obtained by cyclic permutation of the subscripts. In the general case where $\kappa\rho(x)$ is not uniform, the commutators $[D_x,D_y]$ and $[D_x,D_z]$ do not vanish and ordering in (C3) is important. Notice that (C3) is a integro-differential equation, since the $(D+2s)^{-1}$ are integral operators. Although complete analysis of the above is outside our present scope, a diffusion approximation to the thick cloud limit may be obtained, as well as an exact equation for the homogeneous case.

To obtain a diffusion equation from (C3), we take the limit of small gradients, i.e., $\delta\to 0$, and expand the integral operator in a Taylor series

$$(D+2s)^{-1}=\frac{1}{p+2s-\delta^2}=\frac{1}{p+2s}[1+\frac{\delta^2}{p+2s}-\frac{\delta^4}{[p+2s]^2}+...] \qquad \text{(C4)}$$

After some manipulation we obtain (to second order) the following diffusion equation for the total intensity $J=I_{x+}+I_{y+}+I_{z+}$:

$$(\delta_x^2+\delta_y^2+\delta_z^2)J=aq[3+\frac{p}{2s}]J \qquad \text{(C5)}$$

which is a diffusion equation as described in section 5, again holding when high order derivatives can be neglected (e.g., in quasi-homogeneous optical density fields and far from sources). In the very special homogeneous case where all the Ds commute, we obtain directly from (C2):

$$[16s^3+4s^2(D_x+D_y+D_z)-D_xD_yD_z]\,I_i=0 \qquad \text{(C6)}$$

for all i (because of commutation and linearity). Or, when written out in full (in terms of optical distances, where $\kappa\rho$=1):

$$[\frac{\partial^6}{\partial x^2\partial y^2\partial z^2}-pq(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial x^2\partial z^2}+\frac{\partial^4}{\partial y^2\partial z^2})+$$
$$q^2(p^2-4s^2)(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})-\frac{1}{4}aq^3(a^2+2a+8p-4pa)]\,I_i=0 \qquad \text{(C7)}$$

Note that, as in the DA(1,2) and DA(2,4) systems, the zeroth order term vanishes when $a=0$; this leads to scaling rather than exponential type behavior. Furthermore, in the thick cloud limit, we can again anticipate a diffusion-like transmission law and a substantially different albedo law due to the fact that the higher order terms in (C7) will be more prominent at the top boundary than the lower one.

Appendix D: The Single-Scattering Albedo and Asymmetry Factor of Various DA Phase Functions

When both scattering and (true) absorption can occur, the relative probability of scattering or single-scattering albedo is denoted "ϖ_0". In many applications, the relevant phase functions are highly forward scattering. In continuous angle radiative transfer, this has been customarily characterized by the asymmetry factor, which is the (cosine) weighted moment of the phase function denoted "g". In DA radiative transfer models, these definitions yield

$$\varpi_0=\sum_k P_{ik} \qquad \text{(D1}a\text{)}$$

$$g=\frac{1}{\varpi_0}\sum_k P_{ik}\cos\theta_{ik} \qquad \text{(D1}b\text{)}$$

where P_{ik} is the DA phase function scattering matrix for scattering from direction $\boldsymbol{i}$ into direction $\boldsymbol{k}$, and θ_{ik} is the angle between $\boldsymbol{k}$ and $\boldsymbol{i}$ so that $\cos\theta_{ik}=\boldsymbol{i}\cdot\boldsymbol{k}$. The results of the summations in (D1) are independent of $\boldsymbol{i}$ in the models studied throughout this series.

Applying this definition to the DA(d,2d) radiative transfer models used most often in the text (and using $\cos 0^\circ=1$, $\cos 90^\circ=0$, $\cos 180^\circ=-1$), we find that for those models whose beams are mutually perpendicular:

$$\varpi_0=t+r+2(d-1)s \qquad \text{(D2}a\text{)}$$
$$\varpi_0 g=t-r \qquad \text{(D2}b\text{)}$$

The most general DA(2,6) model with beams at $0^\circ,\pm60^\circ,\pm120^\circ$, and 180° (scattering probabilities t, s, s' and r respectively), we obtain

$$\varpi_0=t+r+2(s+s') \qquad \text{(D3}a\text{)}$$
$$\varpi_0 g=t-r+s-s' \qquad \text{(D3}b\text{)}$$

while for the subclass of DA(2,6) models with (primary) beams at $180^\circ,\pm60^\circ$ (and secondary beams at 0°, $\pm120^\circ$ as discussed in subsection 3.2), we need only two parameters (r,s). For ϖ_0 and g, only first scattering is considered, taking $t=s'=0$ in (D3a) and (D3b) we find

$$\varpi_0=r+2s \qquad \text{(D4}a\text{)}$$
$$\varpi_0 g=s-r \qquad \text{(D4}b\text{)}$$

This particular DA system which has no (direct) forward scattering can nevertheless be applied to a triangular lattice (with both "up" and "down" cells) and investigated using renormalization or relaxation methods; see sections 2 and 4 as well as Appendix D of part 2. Another two-parameter subclass of DA(2,6) is the DA(2,3) model with beams at $0^\circ,\pm120^\circ$, obtained by taking $r=s=0$. It is an acceptable DA system since it remains decoupled but it is odd (in more sense than one) since it has no (direct) backscattering, which means, in particular, that it has no spatially discrete counterpart even though it has the same symmetries as the (space-filling) equilateral triangle.

In part 2, we show that the regime where (conservative) scattering is linearly proportional to τ extends up to $\approx 1/(1-g)$; hence we expect the prefactors of our asymptotic expressions (1) to also be proportional to some power of $(1-g)$. This agrees with standard continuous angle results in plane-parallel clouds as well as the findings of *Davis et al.* [1989] and part 3 for finite homogeneous square and cubic clouds respectively.

The pair (ϖ_0, g) or, equivalently, the first two Legendre coefficients is sufficient in many popular approximate schemes in radiative transfer, e.g., "two-flux" theory (Appendix B), similarity relations (2), or diffusion (section 5). It is important to note that its specification is insufficient to describe completely the most interesting DA(d,n) models, i.e., with $n \geq 2d$ beams. As shown in section 4, the value of the second order Legendre coefficient is fundamental in the sense that it participates in the determination of the basic character of the mathematical problem associated with the (orthogonal) DA(d,$2d$) systems.

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