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UNIVERSAL MULTIFRACTALS IN TURBULENCE.

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INTRODUCTION

It has been only rather recently [1-5] recognized that scaling fields involve a whole hierarchy of (fractal) dimensions ($D(\gamma)$) characterizing the sparser and sparser regions of space occupied by higher and higher orders ($\gamma>0$) of singularities of densities ε of a turbulent flux Π (e.g. the energy, scalar variance flux or "density" of points on a strange attractor):

$$\Pi_l(A) = \int_A \varepsilon_l \, d^dx : \; \Pr(\varepsilon_l \geq l^{-\gamma}) \approx l^{c(\gamma)} \qquad D(\gamma)=d-c(\gamma); \qquad (1)$$

where l is the (smaller and smaller) homogeneity scale, in a partial construction of the process down to scale l, $D(\gamma)$ is the corresponding dimension function to $c(\gamma)$ (itself a co-dimension) when $c(\gamma)\leq d$, d being the dimension (fractal or otherwise) of the set A on which the process is observed. In the case of strange attractors where the singularity exponent of the measure itself corresponding to the flux (not its density) is considered a slightly different notation is often used; the order of singularity of the measure is denoted α and D(g) is considered as the "spectrum of singularities" $f(\alpha)$. The two notations are related by the following:

$$\alpha = d - \gamma \qquad f(\alpha) = d - c(\gamma) \qquad (2)$$

which clarifies the dependence of α and $f(\alpha)$ on the dimension d of the observation set. In the following, the symbol α will be reserved for the (quite different) Levy index (for the stable Levy distributions) and the related divergence of statistical moments. Indeed let us point out, for $h\geq1$ this type of divergence resulting from eq.1 (ensemble average denoted <.>):

$$\langle\Pi_l(A)^h\rangle \geq \langle\int_A \varepsilon^h_l \, d^{hd}x\rangle \geq N(\gamma)\, l^{-h\gamma}\, l^{hd} \qquad N(\gamma) \approx l^{c(\gamma)-d} \qquad (3)$$

for any singularity γ ($N(\gamma)$ being the nuber of occurences of singularities stonger than γ), thus:

$$\int_A \varepsilon^h_l \, d^{hd}x \approx l^{-K(h)+(h-1)d} \qquad K(h) = \sup_\gamma\{h\gamma - c(\gamma)\} \qquad (4)$$

which diverges as soon as $K(h)>(h-1)d$. Note that singularities contributing to divergences, correspond to $\gamma>d$ and $c(\gamma)>d$ (apparent negative $D(\gamma)$!) and are extremely rare, in fact almost surely not present in a realisation. These wild singularities are missed by the formalism of the spectrum of singularities (α, $f(\alpha)$ are considered as non-negative).

THEORETICAL DEVELOPMENTS

A-prioris any increasing function $c(\gamma)$ could be suitable for eq.1. However, it can be demonstrated [6] that:

i) multiplicative processes are generic processes leading to precisely such multifractal behavior. For example, in the simplest example, the "α-model"; each step of the process determines the fraction of the flux transmitted to smaller scales and respects a canonical conservation of the flux [6], i.e its ensemble average is scale invariant:

$$<\Pi_l(A)> = <\Pi_{l_0}(A)> \quad (5)$$

this property assures a "weak measurable" convergence [7] of the process.

ii) due to its (semi-) group property, each multiplicative process is fully determined by its generator Γ, which (random) values correspond to order of singularities:

$$\varepsilon_l = e^{\Gamma_l} \quad (6)$$

Γ_l being the filtered generator at the scale l (i.e. is homogeneous at the scale l). Multiple scaling introduces the exponent function K(h) of the h^{th} moment of the density respecting:

$$< \varepsilon_l^h > = l^{-K(h)} = e^{-K(h)\,Log(l)} \quad (7)$$

(K(h)Log(l)/h, h being analogues -in respect to classical thermodynamics- of free energy and inverse of temperature). As the statistical moments are the Laplace transforms of the probability distributions, the exponents K(h) and $c(\gamma)$ are related by a Legendre transform:

$$c(\gamma) = \sup_h\{h\gamma - K(h)\} \qquad K(h) = \sup_\gamma\{h\gamma - c(\gamma)\} \quad (8)$$

and are completely monotone functions (hence convex).

iii) the only generators being stable and attractive under addition are Gaussian (Lévy index $\alpha = 2$) or Lévy-stables ($0<\alpha<2$) "1/f noises" -their (generalized) spectra should be proportional to the inverse of the frequency or wave-number. The case $\alpha=0$ corresponds to the (rather trivial) β-model. α determines the universality classes. Note that we have to consider only "extreme stable Lévy" generators, i.e. having only negative extremes values (positive values must be distributed exponentially - non extremally- otherwise we obtain divergences of all statistical moments of the density, and the process is not normalisable).

The universality classes and corresponding $c(\gamma)$ are fully determined by the codimension C_1 and the Lévy's index α:

$$\frac{1}{\alpha}+\frac{1}{\alpha'}=1 \qquad c(\gamma)=C_1\left[\frac{\gamma}{\alpha' C_1}+\frac{1}{\alpha}\right]^{\alpha'} \qquad K(h)=C_1\frac{h^\alpha - h}{\alpha - 1} \quad (9)$$

(note that $\gamma = C_1$ is the order of singularity contributing to the (ensemble) average of the flux and satisfies: $c(C_1)=C_1$).

PRACTICAL IMPLICATIONS

We discuss the relevance of the preceding developments to geophysical fields either for stochastic simulations or data analysis. Some theoretical arguments, going back to the derivations of the Kolmogorov spectrum, indicate that the velocity field (or passive scalar field) can be obtained by fractional integrations (order H) on (non-integer) powers (P) of the fluxes. For instance, the passive scalar field [6] can be considered as resulting from a fractional integration of order H=1/3 on a flux raised to the power P=1/3. Such transformations are linear with respect to the generator, and hence preserve the general form of eqs.7 (they modify only the coefficients), hence they lead to a rather general normalized co-dimension function:

$$c_N(\gamma) = c(\gamma)/c(0) = (1+\gamma/\gamma_1)^{\alpha'} \quad (10)$$

For instance, it is possible to check that this index for the rain radar reflectivity is $\alpha \approx 2$ (cf.Fig.1) in estimating $c(\gamma)$ with the help of the functional box counting algorithm on 10 rain cases, γ_1 via a least squares regression. The standard error of the fit (of $c(\gamma)$) in all 10 cases, over the entire range of $c(\gamma)$, was ±0.062 which is comparable to the errors in determining $c(\gamma)$ from the functional box-counting algorithm [8]. We then plot the curves $<c_N(\gamma)>$ v.s. $<(1+\gamma/\gamma_1)2>$ in Fig. 1 (here averaging on all available cases). As predicted, the curves all closely follow the line x=y (shown for reference). Similar results were obtained in [9] for satellite and infra red radiances.

In conclusion we stress the fact that the existence of universality classes allows us to avoid the complex problem of dealing with high order moments (and/or low observational

dimension d) which are difficult to empirically estimate since one encounters statistical divergences (for K(h)>(h-1)d) introducing spurious scaling (due to the break down of the large number law). On the contrary, the determination of α requires only low order moments (the convexity of K(h) for $h \approx 1$). It has many practical implications in geophysics [10,11].

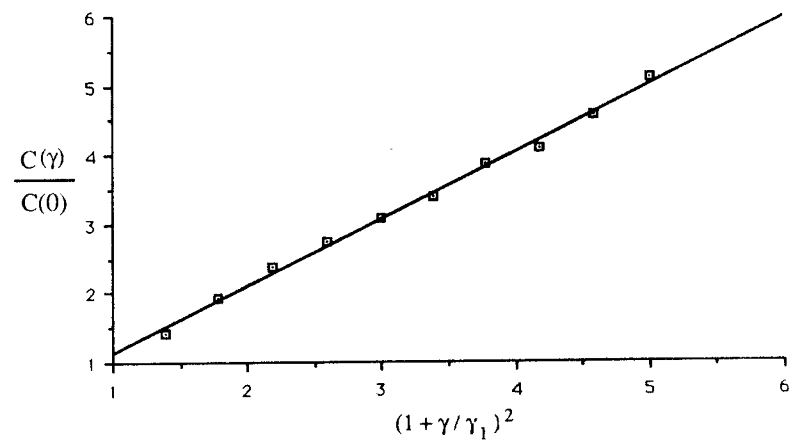

Fig. 1 The mean normalized co-dimension, $<c_N(\gamma)>$ for the radar data, plotted against the mean $<(1+\gamma/\gamma_1)2>$ showing that $c(\gamma)$ belong to the universality class defined by $\alpha=2$. A perfect fit (the line x=y) is shown for reference.

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