Fractal aircraft trajectories and nonclassical turbulent exponents

S. Lovejoy
Physics, McGill University, Montréal, Quebec, Canada H3A 2T8

D. Schertzer
CEREVE, École Nationale des Ponts et Chausées, 77455 Cedex 2, Marne-la-Vallée, France and Météo-France 75007, Paris, France

A. F. Tuck
National Oceanographic and Atmospheric Administration, Aeronomy Laboratory, Boulder, Colorado 80305, USA

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The dimension (D) of aircraft trajectories is fundamental in interpreting airborne data. To estimate D, we studied data from 18 trajectories of stratospheric aircraft flights 1600 km long taken during a "Mach cruise" (near constant Mach number) autopilot flight mode of the ER-2 research aircraft. Mach cruise implies correlated temperature and wind fluctuations so that $\langle \Delta Z \rangle \approx \Delta x^{H_z}$ where Z is the (fluctuating) vertical and x the horizontal coordinate of the aircraft. Over the range $\approx 3-300$ km, we found $H_z \approx 0.58 \pm 0.02$ close to the theoretical $5/9=0.56$ and implying $D=1+H_z=14/9$, i.e., the trajectories are fractal. For distances <3 km aircraft inertia smooths the trajectories, for distances >300 km, D=1 again because of a rise of 1 m/km due to fuel consumption. In the fractal regime, the horizontal velocity and temperature exponents are close to the nonclassical value 1/2 (rather than 1/3). We discuss implications for aircraft measurements as well as for the structure of the atmosphere.

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I. INTRODUCTION

A fundamental goal of atmospheric science is a statistical understanding of the extreme variability of the atmosphere over the entire range from planetary down to small viscous scales; some nine orders of magnitude. Since there are several competing theories, a major difficulty is in obtaining appropriate data. This is particularly true in the mesoscale range where aircraft wind and temperature measurements are essential. Clearly the interpretation of aircraft data depends on the dimensions (D) of the trajectories.

Standard data, whether remotely sensed or from meteorological networks (even if sparse [1], D<2), are independent of the state of the atmosphere. In comparison—except in an ideal 2D atmosphere—aircraft measurements will be sensitively *dependent* on the observed medium. This is because aircraft, no matter how well controlled, cannot fly in perfectly flat horizontal lines; wind fluctuations and thermal plumes cause systematic deviations. Under manual control the operator intervenes over periods of minutes to hours so that the behavior is not so easy to analyze; in contrast autopilots act over time scales of seconds, so that the structure of the trajectories at kilometer scales or larger will be the result of the combined action of the control algorithm and the long-range scaling properties of the atmosphere. Indeed, for the ER-2 aircraft discussed here, analysis of manually flown segments confirmed that they had quite different statistics (no clear scaling behavior) compared to the scaling followed in the "Mach cruise mode." In this mode, the autopilot acts to maintain the Mach number (Ma) fixed to ±2%–3% so that the control algorithm introduces correlations in the velocity, temperature, and vertical position via the iso-Mach condition as follows:

$$v_g = v_{long} + (\gamma RT)^{1/2}\ Ma \qquad (1)$$

with Ma=0.7, v_g is ground speed, v_{long} is the longitudinal wind speed (in the direction of the aircraft), γ the specific heat ratio, R is the gas constant, and T the absolute temperature (see Sec. III).

In order to understand how control algorithms affect trajectories, we must consider the turbulent statistics. The standard model of the atmosphere (e.g., [2]) involves a quasi-isotropic 3D turbulence at scales smaller than the "mesoscale gap" ($\approx$10 km), and a quasi-isotropic much smoother 2D turbulence at large scales where aircraft trajectories would remain approximately smooth and flat. *A priori* this is no longer possible in the framework of the "unified scaling model" [3], which involves a unique but strongly anisotropic turbulent regime and where vertical (Δz) and horizontal scales (Δx) of atmospheric structures are related through

$$\Delta z \approx \Delta x^{H_z}. \qquad (2)$$

The effective "elliptical" dimension of this model is $D_{el}=2+H_z$; $H_z=0$, 1 for isotropic 2D and 3D atmospheres. At scales smaller than a critical Δx_i, the aircraft inertia smooths out the fractality. However for larger scales, one may expect that the scaling relation [Eq. (2)] will also hold for the (random) altitude fluctuations of the aircraft (denoted ΔZ). In this case, the aircraft trajectories will be fractal ($H_z \neq 0$).

The unified scaling model originated as an alternative to the increasingly implausible "mesoscale gap" near 10 km. Indeed the horizontal wind spectrum is continuous from kilometers to thousands of kilometers [4–10]. In addition, radiosonde studies have consistently found much steeper power spectra in the vertical than in the horizontal the atmosphere is anisotropic over much of its range [11–15]. Radar

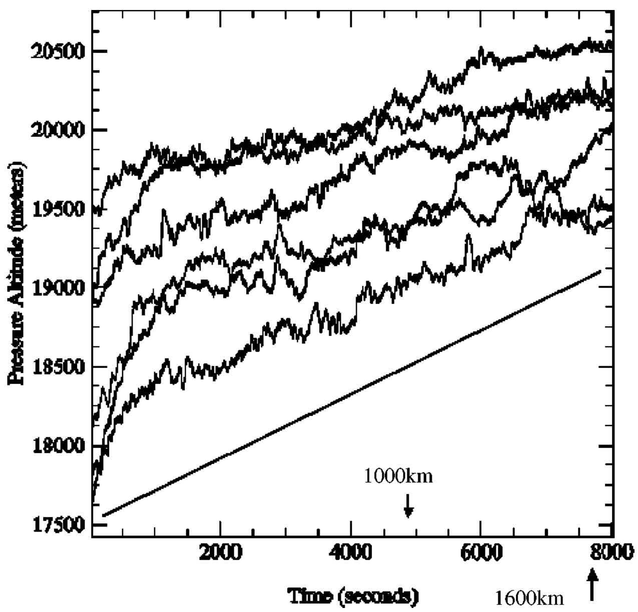

FIG. 1. This shows six representative flight segments, each using "Mach cruise," and each about 8000 s long (corresponding to about 1600 km). A reference trend of 1 m/km is shown.

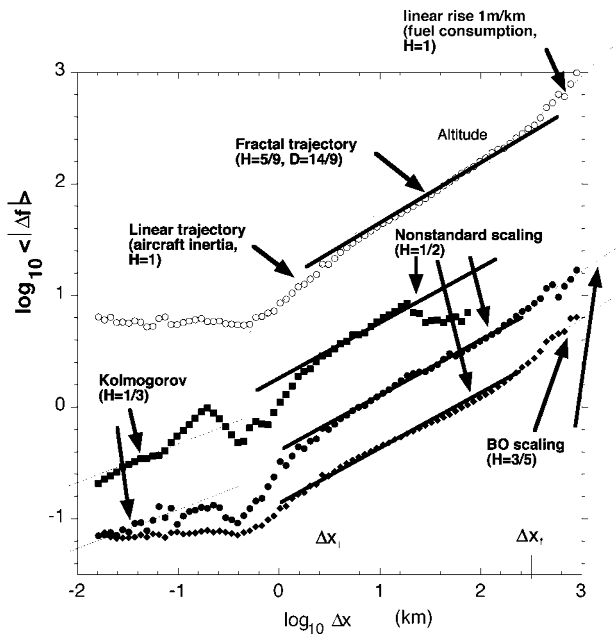

FIG. 2. First-order structure functions for the mean of the 18 flights. The top reference line has slope $H=H_z=5/9$; it corresponds to $l_s=4$ cm [Eq. (3)]. f represents the field (z, v, or T). The units of z (top, open circles) are m, for v (middle, longitudinal squares, transverse circles), are m/s, for T (bottom, diamonds), K. Thick reference lines have slopes $H=1/2$. From the amplitude of the small v, T fluctuations, we can estimate the noise as ±7 m, ±0.7 m/s, ±0.15 m/s, ±0.08 K for altitude, longitudinal v, transverse v, T, respectively.

[16] and lidar [17] have also been used to draw similar conclusions; see also [18,19]. On the empirical side, the current debate is whether the spectrum in the vertical is close to k^{-3} (Lumley-Shur, gravity waves) or to $k^{-11/5}$ (Bolgiano-Obukhov, buoyancy driven). If in addition, the horizontal spectrum is $k^{-5/3}$, then these competing models have "elliptical dimensions" =7/3, 23/9 respectively, either of which will likely yield fractal trajectories (note that elliptical dimensions are primarily of interest when the structures are localized). Lilley [20] concludes that all the available data is compatible with $D_{el}=23/9$, and Lilley *et al.* [17] provides a direct empirical estimate using passive scalar lidar backscatter, obtaining 2.55 ± 0.02.

II. THE DATA AND STATISTICAL ANALYSIS OF THE TRAJECTORIES

We use data from the NASA ER2 high-altitude aircraft taken from the AASE (Airborne Arctic Stratospheric Expedition) during a campaign based in Stavanger, Norway, from January to February 1989. AASE was designed to study stratospheric polar ozone concentrations and their relations with other meteorological variables [10,21]; data were taken at 10 Hz($\approx$20 m). The aircraft position was measured particularly accurately in the vertical ($\approx\pm6$ m). Due to the particular aerodynamic characteristics of the ER2 aircraft, it must fly within 2%–3% of Mach 0.7 ($\approx$200 m/s) over much of the flight path [21].

We chose segments from 18 flights in which the aircraft was flying (nearly) along a great circle route using Mach cruise, each of which was over 1600 km long. Fig. 1 shows a vertical cross-section of six representative flights. With the exception of the vertical drift of roughly 1 m/km, caused by the slow lightening of the aircraft due to its fuel consumption, the segments appear to be random walks.

In order to test Eq. (2), we calculated structure functions (i.e., statistical moments of the increments) as follows:

$$\langle|\Delta Z|^q\rangle = A_q \Delta x^{\zeta_z(q)}, \tag{3}$$

where $\langle\rangle$ indicates ensemble averaging, q is the order of the structure function, $\zeta_z(q)$ its scaling exponent, and A_q is a prefactor. Note that we ignore the (relatively) small random horizontal fluctuations and treat Δx as a sure variable. Before looking at the data, let us consider the theoretical predictions of the unified scaling model for the logarithmic derivative $\zeta_z(1)$ of $\langle|\Delta Z|\rangle$. At scales $<\Delta x_i$, aircraft inertia will overcome the turbulence, the trajectory will be smooth and, if not perfectly flat will have $\zeta_z(1)=1$. For the wind and temperature we expect Kolmogorov statistics; $\zeta_v(1)=\zeta_T(1)=1/3$ [note: whenever noise dominates the signal, $\zeta_z(1)=0$]. Similarly, at scales $>\Delta x_f$ the trend imposed by the linear 1 m/km rise again implies $\zeta_z(1)=1$, we expect to recover the vertical exponents $\zeta_v(1)=\zeta_T(1)=3/5$ (see below). For scales $\Delta x_i<\Delta x<\Delta x_f$ the inertia is not enough to smooth out the fluctuations, and the effect of the fuel consumption rise is negligible; this is the fractal region.

Turning to the data in Fig. 2, we have indicated the various theoretical slopes discussed above by thin reference lines. We see that the above simple theory reasonably accounts for the extremes. In addition, over the intermediate range $\Delta x_i<\Delta x<\Delta x_f$ with $\Delta x_i\approx3$ km, $\Delta x_f\approx300$ km, we see scaling with $\zeta_z(1)=H_{zt}=0.58\pm0.02$, implying fractal trajectories with $D=1+H_{zt}$. Over the same range, we also find $\zeta_v(1)=0.50\pm0.02$, $\zeta_{vlong}(1)=0.52\pm0.03$, $\zeta_T(1)=0.45\pm0.02$ for v transverse (orthogonal to the aircraft direction), v longitudinal (parallel) and T, respectively [$\zeta_{vlong}(1)$ is only for $3<\Delta x<10$ km; see below]. The corresponding energy spectra are $E(k)=k^{-\beta}$ with $\beta=1+\zeta(2)$; $\zeta(2)=2H-K(2)$ where $K(2)$ is a multifractal (intermittency) correction estimated here as ≈ 0.1 so that over the fractal range $\beta_T\approx\beta_v\approx 1.9$. To within errors and possibly small intermittency corrections, the $\zeta(1)$ are sufficiently close to each other and to $1/2$ that a corresponding reference line was added. However, detailed analysis showed that these values are definitely smaller than $\zeta_z(1)$, which is close to the theoretical $5/9$ value discussed below. The exponent $\zeta(1)=1/2$ is a classical exponent, which is ubiquitous in physics, e.g., the Lorentzian, normal diffusion etc. However here, this apparently "normal" value is, in fact, anomalous with respect to the classical Kolmogorov $1/3$. In any case, as indicated, T, v are multifractal and so cannot be modeled by "normal" diffusion.

The (aircraft) inertial scale $\Delta x_i=3$ km corresponds to ≈ 15 s; it limits the maximum vertical accelerations to $\approx 1\%-2\%$ of g which are not too noticeable. For scales below 400 m, although the trajectory is smooth, the z, T signals are dominated by measurement errors (± 6 m, ± 0.07 K, respectively), while the velocity shows evidence of Kolmogorov $1/3$ scaling.

The usual assumption is either that aircraft altitude is constant to within a fixed amplitude noise (implying $H_{zt}=0$) or that it is deterministic (with negligible vertical fluctuations), but with a nonzero slope ($H_{zt}=1$). Figure 2 shows that neither is valid over the fractal range. By comparing the curves for the transverse, longitudinal velocities and temperature, we see that it is plausible that the nonscaling variations in these structure functions can be explained by measurement errors and by variations in aircraft altitude. Recalling that the standard isotropic 3D and 2D values for $\zeta_v(1)$ are $H=1/3$ and $H=1$, respectively, we see that the observed value ($\approx 1/2$) is nonclassical. Finally, the v_{long} fluctuations seem to "saturate" at about 7–8 m/s, which is about $2\%-3\%$ of the mean 200 m/s. This saturation is presumably the result of the autopilot feedback, which keeps the Mach number within exactly this tolerance (otherwise the aircraft stalls or experiences Mach buffet). Since the transverse velocity is little affected by this mechanism, it shows wider-range scaling and will be used below.

Additional analyses show that over the range of reliable statistics ($0<q<3$), $\zeta(q)$ is nearly linear. Equation (3) with $\zeta_z(q)\approx qH_{zt}$ can be written as:

$$T_\lambda(\Delta R(\Delta x))\overset{d}{=}\Delta R(\Delta x/\lambda);$$

$$T_\lambda=\lambda^{-G_t};$$

$$G_t=\begin{pmatrix}1 & 0\\ 0 & H_{zt}\end{pmatrix}, \qquad \text{(d4)}$$

where $\Delta R(\Delta x)=\mathbf{(}\Delta x,\Delta Z\ (\Delta x)\mathbf{)}$ is a random vector displacement, λ is an arbitrary scale ratio, and $\overset{d}{=}$ means equality in probability distributions. T_λ is a generalized (anisotropic) "zoom" (contraction) by factor λ, and G_t is the generator of this scaling anisotropy [3]. The interpretation of this is that the fluctuations in the z direction over a small horizontal flight segment $\lambda^{-1}\Delta x$ have the same probability distribution as those over a long segment Δx, as long as they are rescaled by $\lambda^{-H_{zt}}$. This anisotropic scaling property of the trajectory will be useful below.

The value of the exponent H_{zt} ($\approx 0.58\pm0.02$) was found to be very close to the theoretical value $H_z=5/9$ in the $23/9$ D model of the atmosphere [3]. This model is motivated by the fact that, theoretically and empirically, the fluctuations of the horizontal wind (v) in the horizontal direction follow Kolmogorov ($k_h^{-5/3}$), while in the vertical Bolgiano-Obukhov statistics ($k_v^{-11/5}$, k_h, k_v are horizontal and vertical wave numbers). The model, therefore postulates: (a) in the horizontal $\Delta v\propto\varepsilon^{1/3}\Delta x^{1/3}$ with ε= energy fluxes and (b) in the vertical $\Delta v\propto\phi^{1/5}\Delta z^{3/5}$ with ϕ= buoyancy force variance fluxes. A single law valid for an arbitrary displacement vector $\Delta\underline{r}=(\Delta x,\Delta z)$ is obtained by introducing a physical "generalized" scale function $\|\Delta\underline{r}\|$ such that $\|T_\lambda\Delta\underline{r}\|=\|\Delta\underline{r}\|/\lambda$. A simple example is

$$\|\Delta\underline{r}\|=l_s\left(\left|\frac{\Delta x}{l_s}\right|+\left|\frac{\Delta z}{l_s}\right|^{1/H_z}\right);\quad l_s=\phi^{-3/4}\varepsilon^{5/4}, \qquad (5)$$

where l_s is the "sphero-scale" and $H_z=(1/3)/(3/5)=5/9$ is the stratification exponent (for passive scalars, Lilley *et al.* [17] obtain $H_z=0.55\pm0.02$). From Eq. (5) we see that for scales $>l_s$, structures are flattened in the horizontal, whereas at scales $<l_s$, they are vertically aligned; l_s is the scale at which horizontal and vertical shears in v are equal; the structures are "roundish" [3]. With the scale function, the full horizontal velocity statistics (any vector displacement Δr) are

$$\Delta v(\Delta\underline{r})\approx\varepsilon^{1/3}\|\Delta\underline{r}\|^{1/3}, \qquad (6)$$

which reduces to the Kolmogorov and Bolgiano-Obukhov scalings for $\Delta\underline{r}=(\Delta x,0)$ and $\Delta\underline{r}=(0,\Delta z)$, respectively.

The key physical idea underlying this model is that the dynamics determines the mean structures and that the mean structures in turn determine the "physical scale." In all statistical laws, the isotropic (Euclidean) distance function should be replaced by the (anisotropic) scale function. A consequence is that atmospheric dynamics are governed not by isotropic cascades, but rather by anisotropic cascades leading to anisotropic multifractal fields. Any isotropic results are translated in to the equivalent anisotropic ones by changing H_z from 1 to $5/9$.

III. THE HORIZONTAL AND VERTICAL SCALING ALONG A TRAJECTORY

The $23/9$ D unified scaling model predicts that the horizontal statistics as functions of Δx should be the same as vertical statistics as functions of $l_s(\Delta z/l_s)^{(1/H_z)}$ [estimated as $l_s(\langle|\Delta Z(\Delta x)|\rangle/l_s)^{(1/H_z)}$] with $l_s=4$ cm estimated from the equation $\langle|\Delta Z(l_s)|\rangle=l_s$ solved by graphical extrapolation in Fig. 2. In Fig. 3 we directly test this prediction for $q=1$. Over the fractal range the theoretically rescaled vertical statistics are remarkably close to the horizontal ones. Using Δz

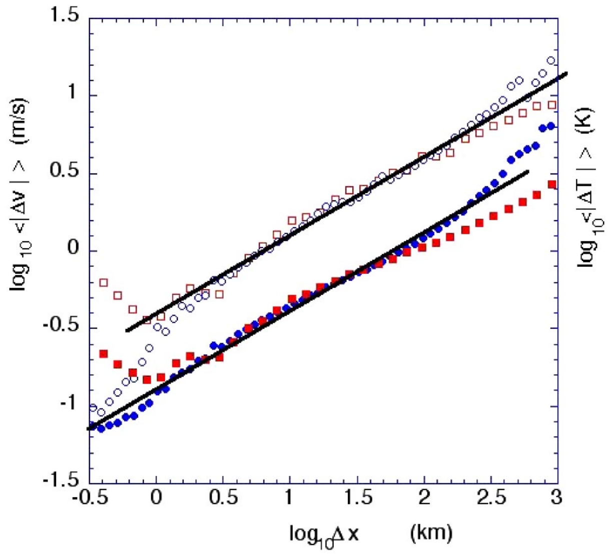

FIG. 3. The mean first-order (transverse) velocity and temperature structure functions. The hollow symbols are for v, the solid for T. Circles indicate functions of Δx, squares indicate functions of the theoretically predicted compensated vertical scale $l_s(\langle|\Delta Z(\Delta x)|\rangle/l_s)^{(1/H_z)}$ with $H_z=5/9$, $l_s=4$ cm. The reference lines have slopes $1/2$.

$=\langle|\Delta Z(\Delta x)|\rangle$, we calculated the moments for $q\neq 1$. We found that $\zeta(q)$ is nonlinear, indicating that the fields are multifractal. However, the horizontal-to-vertical exponent ratio was always close to H_z as expected.

Before continuing, let us briefly comment on the empirical value of l_s. First, it is not a constant, but varies from trajectory to trajectory (by at least an order of magnitude for the cases studied here); this is not surprising since from Eq. (5), we see that it depends on two highly variable (multifractal, intermittent) fluxes. The mean value 4 cm is in the range of direct measurements of vertical cross sections of passive scalars obtained from lidar data [17]. Larger structures are flattened in the horizontal, smaller ones elongated in the vertical. The fact that l_s is the same for both the velocity and temperature follows because, dimensionally, the two fluxes ε, ϕ yield a unique scale l_s [Eq. (5)].

We can now consider the statistics of aircraft flying in a linear trajectory over distance Δx, slope s, i.e., $\Delta z=s\Delta x$. From Eq. (5) we see that there will be critical slope $s_c=(\Delta x/l_s)^{H_z-1}$. If $s<s_c$, then we obtain $\|\Delta r\|\approx\Delta x$, if $s>s_c$ $\|\Delta r\|\propto\Delta z\propto\Delta x^{1/H_z}$. With $\Delta x_i=3$ km, $\Delta x_f=300$ km, $l_s=4$ cm, we find $s_i=6.8\times10^{-3}$, $s_f=8.9\times10^{-4}$. The interpretation is that up to 3 km, the aircraft inertia keeps it flat to within $\approx 0.4°$ leading to (sufficiently) smooth trajectories and $H=1/3$ for v. At scales >300 km, the linear ≈ 1 m/km rise ($\approx s_f$) dominates the turbulence leading again to a linear trajectory, with v, T exponents $(1/3)/H_z=3/5$ (see Fig. 2). The intermediate fractal range has a randomly varying $s: s_i<s(\Delta x)<s_f$. In this way the degree to which the autopilot can keep a flat course influences the extent of the fractal regime.

Over the fractal scale range, $\zeta_v(1)$, $\zeta_T(1)$ are both close to $H=1/2$, which suggests that it ought to be possible to explain them on the basis of some simple physical argument combined with dimensional analyses. Knowing that the control algorithm introduces correlations between v and T [Eq. (1)], over the same scale range (and for $0<q<2$), we calculated the "Yaglom moments" $\langle(\Delta T^2|\Delta v|)^q\rangle$ finding they were nearly proportional to $\langle(\Delta T)^{3q}\rangle$. This implies that Δv, ΔT are indeed highly correlated (this is also confirmed by "cross-extended self-similarity" analyses). This high correlation may be the key to explaining the nonclassical exponent $1/2$.

IV. CONCLUSIONS

Aircraft data are used in many branches of atmospheric science and a knowledge of the dimension of their trajectories is fundamental, classically D=1. However, if the turbulence is scaling, then we expect aircraft trajectories to have long-range structures, which imply fractal trajectories and nonclassical exponents; we find D≈14/9. In comparing our results to the two main tropospheric campaigns—"GASP" [7] and more recently "MOZAIC" [22]—both of which used large numbers of commercial aircraft trajectories, several differences should be borne in mind. First, the altitude: both Kolmogorov and Bolgiano-Obukhov statistics have been reported in the stratosphere and troposphere (see the review [20]). Theoretically, we expect the main difference to be weaker stratospheric buoyancy forces. However, this does not imply a change in dynamical mechanism (or H_z). Second, due to the far greater mass of commercial aircraft, the inner scale of the trajectory (below which the wind follows Kolmogorov statistics) may be significantly larger than the 3 km found for the single seat ER-2. Third, commercial aircraft travel at around Ma=0.9 and use a different control law; hence, we may expect to find different nonclassical spectral exponents. Both of these predictions are indeed observed in the GASP and MOZAIC statistics. Both show Kolmogorov statistics for scales <200 km, and <50 km, respectively. For larger scales, the MOZAIC data has $H\approx 3/5$ a result that our theory explains very readily; for example, if $\Delta x_f=50$ km, for $l_s=4$ cm, this corresponds to an average slope of $s_f\approx 2$ m/km, which seems a plausible average for these short-haul European flights. Also relevant is [8] who found Kolmogorov statistics on horizontal legs of 12 km collected by an instrumented IL-18 D aircraft (heavier than the ER-2).

Although we did not have a quantitative explanation for the nonclassical value $\zeta(1)\approx 1/2$ (v, and T), by comparing statistics in the horizontal and vertical, we argued that these results can be understood if there exists a physical-scale function characterized by exponent $H_z=5/9$. We tested this directly by showing that the horizontal and vertical statistics of v, T are the same if we replace Δx by $l_s(\langle|\Delta Z(\Delta x)|\rangle/l_s)^{(1/H_z)}$ (with $l_s\approx 4$ cm). If these results are confirmed in other studies (including a reanalysis of MOZAIC data), then we may need to reappraise aircraft measurements.

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