NOTES AND CORRESPONDENCE

Universal Multifractals Do Exist!: Comments on ''A Statistical Analysis of Mesoscale Rainfall as a Random Cascade''

D. SCHERTZER

Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, Paris, France

S. LOVEJOY

Physics Department, McGill University, Montreal, Quebec, Canada

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1. Universality and the relevance of multifractals

Since 1985, there have been a series of papers investigating rain and cloud fields understood to be resulting from multifractal cascades; this approach has lead to an increasing number of fruitful applications (for a review, see Lovejoy and Schertzer 1995). More generally, the multifractal approach in meteorology (and particularly in rain) has many fundamental advantages over the usual ones, since it gives dynamic and physical insights at all scales without any ad hoc parameterization or homogeneity hypotheses. This approach not only enables us to overcome the limits of the classical methods, but it also creates a research framework that simultaneously allows for theoretical investigations and empirical analysis based on fundamental physical principals (namely symmetries and invariants). However, mathematically, an infinite number of parameters is generally necessary to specify such a process—for example, hierarchies of singularities and their codimensions (see section 3). Hence, staying on a purely mathematical level (without any physical considerations), such cascades would be unmanageable. Theoretically and empirically, cascades would be irrelevant. Since the mid-1980s, debate has centered on the fundamental physical idea that of this infinite number of parameters only a few might be physically relevant, determining the ''universality'' classes. In the paper by Gupta and Waymire (1993, GW in the following), the authors are also haunted by this issue, since they categorically dismiss the notion of universality in cascade processes as ''untenable'' no less than three times, while devoting only a single sentence[1] to explaining why it is an ''error.'' In this comment, we clarify why this unique error argument is irrelevant and point to what seems to be its precise and misleading origin. The question of universality not only merits debate and clarification, but, due to the growing number of attempts at modeling and analyzing multifractals in rain (and elsewhere), it is becoming central for applications.

In the following, we give a new presentation of universality and universal multifractals. Although it is based on arguments that have been presented under different forms in various papers during the last 10 years, in order to overcome some past misunderstandings, we have taken care not to repeat them in their original form. Although the mathematical propositions concerning universal multifractals are almost trivial—directly analogous as they are to the additive results—we have formalized them in such a way so as to dispel all doubt about the mathematical existence of universality. Let us emphasize that the interesting question is now in the physics: Which routes—if any—to universality are physically relevant?

The associated theoretical and empirical consequences, particularly the fact that the same ''universal'' infinitesimal generator may be responsible for the appearance of the extremes as well as of the mean field (see Fig. 1 for an illustration), underline the necessity of clarifying any misunderstandings or any underestimation of the scope of the basic notion and concept.

2. Universality of additive processes: Random walks

The well-known example of the random walk readily illustrates the idea of universality. Consider the general

Corresponding author address: Dr. Daniel Schertzer, Laboratoire de Modélisation en Mécanique, Boite 162, Université Peirre et Marie Curie, 4 Place Jussieu, F-75252 Paris, Cedex 05, France.
E-mail: schertze@ccr.jussieu.fr

[1] Out of a total of eight sentences devoted to the issue.

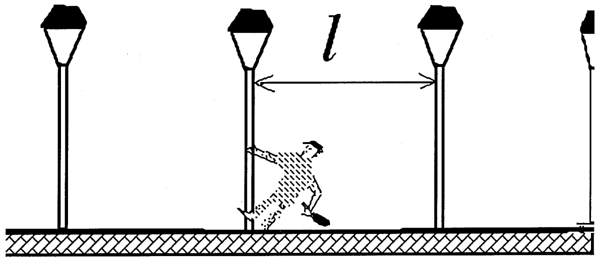

FIG. 1. Scheme showing the drunkard's walk problem.

theme of 1D discrete random walks, illustrated by a drunkard (see Fig. 1) following a straight line for a random distance (according to a fixed probability law), stopping, and then moving either left or right again for a possibly (but not necessarily) random distance. Such a random walk could depend on as large a number of parameters as the theoretician would like to introduce—that is, it could be quite different from just a constant distance left or right (e.g., the distance to the next lamppost in the simplest case). Now consider the ''densification'' of the walk by decreasing the typical length scale l between two consecutive stops. This can be done by replacing a single step by N similar steps as follows:

$$\Delta X = \sum_{i=1}^{N} \frac{\Delta X_i - a(N)}{b(N)}, \tag{1}$$

where $b(N)$ and $a(N)$ are, respectively, the rescaling and the recentering parameters. It is well known that with only the hypothesis of the finite variance of each elementary step, the process converges to Brownian motion (e.g., Feller 1971)—that is, a Gaussian random walk (Fig. 2 displays a 2D example) with $b(N) = N^{1/2}$. Brownian motion is ruled by at most two parameters: the mean of an elementary step (nonzero if the walk is asymmetric) and its variance. Nevertheless—and this not widely enough known—when the hypothesis of finite variance is relaxed, universality still survives! Indeed, Lévy (1925) demonstrated the following generalized central limit theorem.

a. Generalized central limit theorem

The renormalized sums of identically independently distributed (i.i.d.) variables ΔX_i converge toward a Lévy stable law L_α:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{\Delta X_i - a(N)}{b(N)} = L_\alpha. \tag{2}$$

Here, L_α depends only on three parameters;[2] the most important one, the Lévy index, usually denoted by α (≤ 2), in fact describes how the variance diverges. More precisely, it corresponds to the critical order of divergence of moments of the elementary step (ΔX_i):

$$\alpha < 2;\ \forall q \geq \alpha:\ \langle |\Delta X_i|^q \rangle = \infty. \tag{3}$$

[2] Four parameters if we include the almost trivial ''location'' parameter.

b. Comments

1) The Gaussian case corresponds to the limit $\alpha = 2$, although in this case, the critical order of divergence becomes infinite.
2) In the Lévy case, the drunkard will usually move past many lampposts before stopping.
3) The renormalization of the sum by the factor $b(N)$ corresponds to a natural physical requirement, which follows from the original statement of the problem: to replace the large steps by many smaller steps implicitly refers to some common nature of the small and large steps—the commonality is usually taken to be self-similarity (see comment 4)—between the steps of different scales.
4) The limit L_α is a stable fixed point of Eq. (1), in the sense that for any N, ΔX_i ($i = 1, N$) independent and identically distributed as L_α, ΔX [defined by Eq. (1)] has the same distribution. This self-similarity implies a group property for the renormalization factor. More precisely,

$$b(N) = N^{1/\alpha}. \tag{4}$$

5) The renormalization–self-similarity assumption can be relaxed: a wider class of a continuous random walks [including the (strong) universal walks defined by comment 4] is obtained by only requiring that any step must be able to be decomposed in N i.i.d. elementary steps for any N. The corresponding laws, also proposed by Lévy, are therefore called ''infinite divisible laws,'' which clearly include Lévy stable laws as particular cases.
6) We may mention that the second important parameter (usually denoted by β) characterizes the asymmetry of the law; $\beta = 0$ in case of symmetry [a requisite for the $\alpha = 2$ (Gaussian) case]. However, a requisite for a multifractal process with $\alpha < 2$ (Schertzer et al. 1988) is that the law be maximally asymmetric ($|\beta| = 1$), in which case in the divergence of moments, the elementary step (ΔX_i) occurs only on one half-axis (e.g., only for positive steps). Such laws are often called ''extremal Lévy laws.''

Rather than ''densifying'' the stopping points by taking $l \to 0$, we can also obtain the classical Brownian or Lévy limit by simply asking the drunkard to decide how far to move left or right based not only on one random event (trial), but on an increasingly large number of them. For example, the actual distance he moves before stopping may be the sum of many independent processes, which are thus ''mixed.'' Again, the generalized central limit theorem will intervene. As discussed

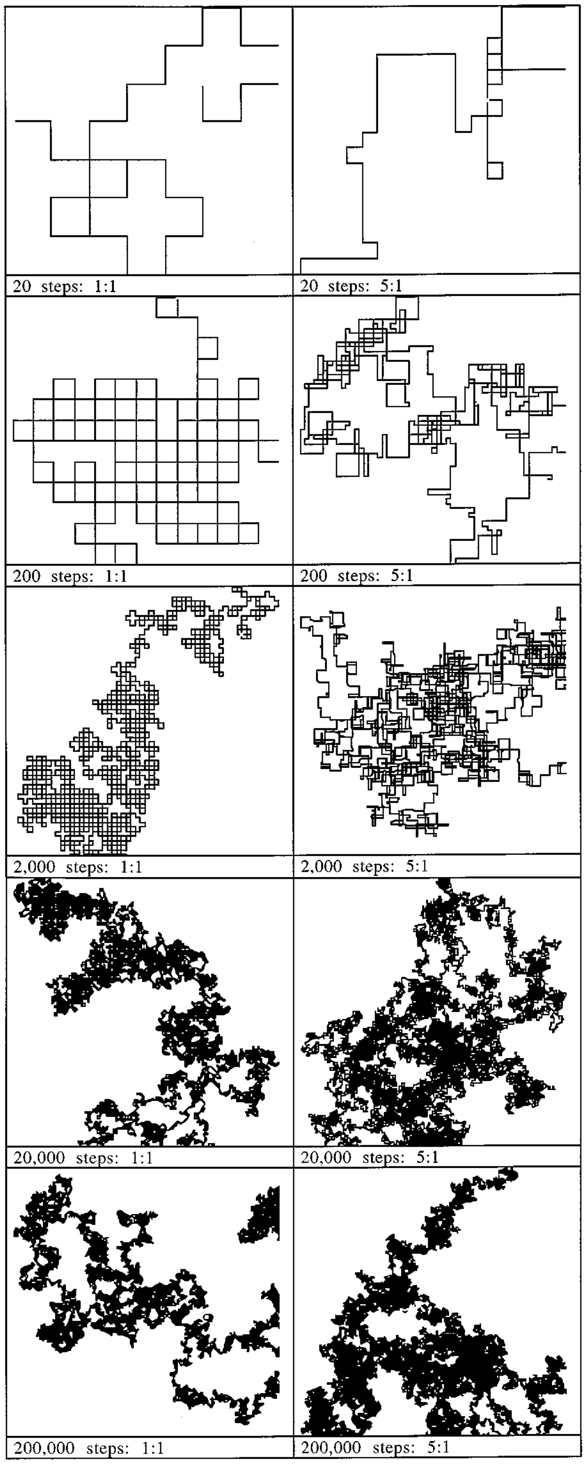

FIG. 2. Universality under the addition of random variables is illustrated here by two random walks. Steps are chosen randomly to be up, down, left, or right with equal probability. On the left of each pair of columns, the steps are all of equal length, whereas on the right, they are occasionally (randomly) five times longer (the lengths have been normalized so that the variances are the same). For a small number of steps, the walks are very different, but for a large enough number, they tend to the same limit and do indeed look similar (we thank C. Hooge).

below, these additive results have direct analogs in multiplicative cascade processes.

3. Universality of multiplicative processes: Multifractals

Multiplicative cascade models of turbulence all work in the following way: a random factor determines the fraction of the rate of energy transferred from one large eddy to one of its subeddies. Figure 3 shows in a 2D cut how the large structures are multiplicatively modulated by smaller ones. We then iterate this construction over the scale ratio $\Lambda = L/l$, where L is the larger scale and l the smallest resolved scale, corresponding to the resolution of our field. For the pedagogical discrete (in scale ratio) cascade models, $\Lambda = \lambda_0^N$, where λ_0 is the (fixed) scale ratio for one step and N is the number of steps. As Λ goes to infinity, we observe singularities: at some points, the field goes to infinity (a singularity), whereas over most of the space, it goes to zero (regularities).

The resulting multifractal behavior of the field ε_λ at any intermediate scale ratio λ ($\lambda \leq \Lambda$; in the discrete case, $\lambda = \lambda_0^n$; $n \leq N$) can be determined either by the scaling of its probability distribution, whose exponent is the codimension[3] $c(\gamma)$ of the corresponding singularity γ, as

$$\Pr(\varepsilon_\lambda > \lambda^\gamma) \approx \lambda^{-c(\gamma)}, \tag{5}$$

or (equivalently) by the scaling of its different moments, with corresponding moment scaling function $[K(q)]$, as

$$\langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)}. \tag{6}$$

4. The debate about universality

If we simply iterate the model step by step (e.g., with the fixed ratio of scale λ_0 in a discrete cascade model), we indefinitely increase the overall range of scales Λ. This already poses a nontrivial mathematical problem (a weak limit of random singular measures; see Kahane 1985, 1987). Some antiuniversality prejudice has arisen because of the pervasive use of discrete cascades to help understand cascade properties. Indeed, within the framework of discrete cascades, one is naturally lead to consider the (nontrivial) small-scale limit simultaneously (Yaglom 1966) with the limit of an infinite number of interactions (random variables). However, the convergence of the small-scale limit of ε_Λ implies the divergence of its generator (Γ_Λ)—that is, its logarithm:

$$\varepsilon_\Lambda \approx e^{\Gamma_\Lambda}. \tag{7}$$

This is perhaps easiest to see by considering the second

[3] Here, $c(\gamma)$ is a statistical generalization (e.g., Schertzer and Lovejoy 1992) of the geometric notion of a fractal codimension (c) of a set of dimension D embedded in a d-dimensional space: $c = d - D$.

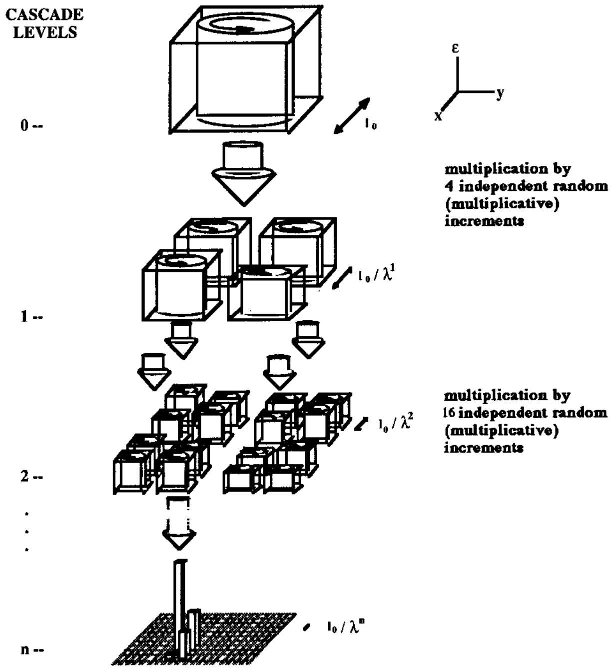

FIG. 3. A schematic diagram showing a two-dimensional cascade process at different levels of its construction to smaller scales. Each eddy is broken up into four subeddies, transferring a part or all its energy flux to the subeddies. In this process, the flux of the field at large-scale multiplicatively modulates the various fluxes at smaller scales; the mechanism of flux redistribution is repeated at each cascade step (self-similarity).

characteristic function $K_\Lambda(q)$ of the generator Γ_Λ, which, in order to satisfy Eq. (6), must behave as $K_\Lambda(q) \approx K(q)\log(\Lambda)$. Therefore, there can be no central limit theorem convergence for the generator due to this logarithmic divergence! Indeed, the particularities of the discrete models (e.g., the α model; see below) remain as a discrete cascade proceeds to its small-scale limit. This has prompted the opposite extreme claim that multiplicative cascades could not admit *any* universal behavior (e.g., Mandelbrot 1989 states that "in the strict sense, there is no universality whatsoever . . . this fact about multifractals is very significant in their theory and must be recognized . . ."). Indeed, GW's "explanation" for the error of universality just repeats the above argument, and they fail to understand the alternatives discussed in Schertzer and Lovejoy (1987, 1991), Lovejoy and Schertzer (1990), and Schmitt et al. (1992).

5. The alternative routes to universality

The general theme of alternative routes to universality discussed in the series of papers devoted to universal multifractals cited above is that instead of considering only the iteration of the process down to smaller and smaller scales, one can first consider interactions of this process over a finite range of scales Λ, with *larger and larger numbers of its replicas, and then seek* the limit $\Lambda \to \infty$. Two variations on this general theme have been pointed out, as well as a possible combination:

1) "nonlinear mixing" of these processes, involving multiplication of independent, identically distributed processes over the same range of scales (illustration in Fig. 4a); and
2) "scale densification" of the process,[4] involving introducing more and more intermediate scales—that is, more and more (elementary) multiplication between two fixed scales (illustration in Fig. 4b).

In both cases, multiplying processes corresponds to adding generators. A rather strong and straightforward universal result is obtained if we have *generators* that are *stable* and *attractive* under *addition* via some more or less trivial rescaling and/or recentering.

We now establish two propositions concerning universal multifractals—that is, multifractal processes admitting Lévy or Gaussian generators (often respectively misnamed log-Lévy and lognormal processes[5]). Proposition 1, dealing with nonlinear mixing, is the most straightforward. Schertzer et al. (1991) explicitly shows how the nonlinear mixing of α models[6] in this way leads to a multifractal having a Gaussian generator.

PROPOSITION 1. *The renormalized nonlinear mixing over a finite range of scales of i.i.d. cascade processes converges to a universal multifractal.*

DEMONSTRATION. At each ratio of scale $\lambda < \Lambda$ (for a discrete cascade model), or on any small interval of ratio of scale $[\lambda, \lambda + d\lambda]$ (for a continuous cascade model), the product ε_λ^N of N independent identically distributed random variables $\varepsilon_\lambda^{(i)}$ with generators $\Gamma_\lambda^{(i)}$,

$$\varepsilon_{N,\lambda} = \prod_{i=1}^{i=N} \varepsilon_\lambda^{(i)}, \qquad (8)$$

admits the generator

$$\Gamma_{N,\lambda} = \sum_{i=1}^{N} \Gamma_\lambda^{(i)}. \qquad (9)$$

Taking the power $1/b(N)$ of $\varepsilon_{N,\lambda}$ rescaled by $e^{-a(N)}$ corresponds to rescaling $\Gamma_{N,\lambda}$ by $1/b(N)$ and recentering it by $-a(N)$. The generalized central limit theorem (section 2) then applies to the generator.

The scale densification (Schertzer and Lovejoy 1987,

[4] Using a "test field" argument (Schertzer and Lovejoy 1991), this seems physically based.

[5] They are misnamed because the dressed process (integrated over a large range of scales) is only approximately log Lévy/lognormal for low-order moments or low singularities; see Schertzer and Lovejoy (1997).

[6] An α model is a two-state discrete cascade model originally introduced in Schertzer and Lovejoy (1983) as a multifractal "destabilization" of the monofractal β model.

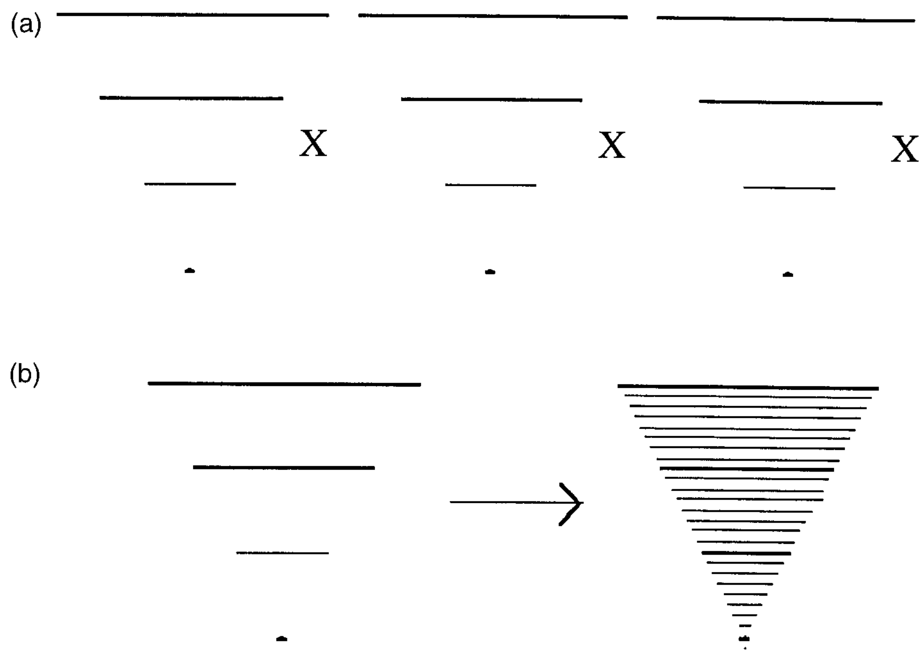

FIG. 4. (a) Schematic of nonlinear mixing: multiplication of independent, identically distributed processes on the same scales, while the total range of scale is kept fixed and finite. (b) Schematic of scale densification introducing intermediate scales, while keeping the total range of scale fixed and finite.

see Wilson et al. 1991 for numerical implementations[7]) corresponds to a slightly different product.

PROPOSITION 2. *The renormalized scale densification over a finite range of scales of a cascade process converges to a universal multifractal.*

DEMONSTRATION. Considering the fixed range of scale of ratio Λ (>1), we will subdivide it into N smaller and smaller subranges, each of ratio of scale $\lambda \downarrow 1$ (or of order 1): $\lambda = \Lambda^{1/N}$. The simplest case corresponds to disjoint subranges. Overlapping subranges correspond in fact to a combination of the previous case with nonlinear mixing. Due to proposition 1, we have only to demonstrate proposition 2 in the case of disjoint subranges. In fact, we are formally back to the mixing case, since the field $\varepsilon_{N,\lambda}$ and its generator $\Gamma_{N,\Lambda}$ over the range of scales Λ are decomposed, respectively, into a product and a sum of i.i.d. elements:

$$\varepsilon_{N,\Lambda} = \prod_{i=1}^{i=N} \varepsilon_\lambda^{(i)} \tag{10}$$

and

$$\Gamma_{N,\Lambda} = \sum_{i=1}^{N} \Gamma_\lambda^{(i)}, \tag{11}$$

although the significance of ε_λ^N and Γ_λ^N is not the same as in proposition 1; the same renormalization of the process (and correspondingly to its generator) leads to the application of the generalized central limit theorem to the generator.

COMMENTS.

1) As for random walks, the renormalization has its importance. Mixing without renormalization corresponds to the larger class of cascades continuous in scale. By analogy with the continuous random walk, their generators are infinitely divisible (Novikov 1994).
2) The failure of Yaglom's (lognormal) conjecture (Yaglom 1966) can be better understood due to the fact that the renormalizing factor diverges as $N^{1/\alpha}$.
3) By densifying over a *finite* range of scales and renormalizing we obtain not only a finite limit for the generator, but also for its exponential—that is, the corresponding field ε.
4) The second characteristic function [$K(q)$] of the limit of the generator (over a fixed finite range of scales) is well defined and should not be considered as an approximation to the moment scaling function of the universal process: $K(q)$ is its moment scaling function without any approximation.

[7] This paper simply explains how to numerically simulate universal multifractals. Contrary to what is implied in GW, it does not introduce any original theoretical arguments concerning universality—the discussion is purely pedagogical. The similarly miscited Lovejoy and Schertzer papers (1990, 1995), the idea of universality is only applied to data analysis. It is curious that GW do not reference the original precise mathematics in appendix A of Schertzer and Lovejoy (1991), which is in the same volume as the Wilson et al. (1991) paper.

As a consequence of either case, GW's claim that "the error in [our] arguments is that infinite convolutions of nonidentical distributions do not need to lie in the domain of attraction of Gaussian or other Lévy stable distribution," is irrelevant because of the following:

1) In the case of nonlinear mixing, at *each level* of the cascade, we multiply (proposition 1) i.i.d. fields or add their corresponding generators with equal weights.
2) In the case of scale densification, we indeed consider (proposition 2) equal scale ratios—that is, sums of i.i.d. random variables with equal weights.
3) The same comment applies to possible combinations of nonlinear mixing with densification.

These elements directly clarify the circumstances that lead to the existence of universality for discrete cascades and, therefore (by proposition 2), indirectly for those with continuous cascades as their limit. The next section (which can be skipped without any prejudice for the rest of the text) discusses this question directly in the framework of continuous cascades, since GW's claim seems to stem from a misleading confusion between weights of generators and weights of their Fourier components for continuous cascades.

6. The weights of the generator

In order for the generalized central limit theorem to hold, a necessary condition is that the independent random variables in the sum [Eq. (1)] should be identically distributed—that is, not only from the same basic type of distribution, but also with equal weights/amplitudes. Contrary to GW's claim that we require unequal weights, in this section we demonstrate that the generators do have equal (real) weights in spite of the fact that their Fourier components must not have equal weights. Furthermore, we have the following proposition.

PROPOSITION 3. (a) *Unequal weights for the Fourier components of the generator—the generator is a colored and not a white noise—are a requisite condition for obtaining equal weights for the generator over the N spherical shells of a given scale ratio step λ_0 obtained by dividing the cascade into N steps, where $\lambda_0^N = \Lambda$. The corresponding N spherical shells are as follows in the physical space:*

$$S_n = \left\{ r \left| \frac{L}{\lambda_0^{n+1}} \leq r < \frac{L}{\lambda_0^n} \right. \right\}; \; n = 0, N-1; \quad (12)$$

or in the Fourier space, they are

$$\hat{S}_n = \left\{ k \left| \frac{\lambda_0^n}{L} \leq k < \frac{\lambda_0^{n+1}}{L} \right. \right\}; \; n = 0, N-1. \quad (13)$$

(b) *More precisely, the generator of a universal cascade process is a pink Lévy noise, obtained by a D-dimensional (or C-codimensional) fractional integration over a white noise (γ_0, called the "subgenerator"), with*

$$C = \frac{d}{\alpha} \Leftrightarrow D = \frac{d}{\alpha'}\left(D \equiv d - C; \frac{1}{\alpha} + \frac{1}{\alpha'} \equiv 1\right). \quad (14)$$

COMMENTS.

1) Gupta and Waymire (1993) confuse the weights of the Fourier components with the weights of the spherical shells (i.e., of the integral of the Fourier components of a shell). Equality of the (real space) generator weights requires inequality of the Fourier weights.
2) Mixing corresponds to considering sums of i.i.d. generators having identical wavenumbers; hence, the question of weights is irrelevant for proposition 1. For a given wavenumber, the "coloring" introduces only a (constant) factor.
3) In the case of the scale densification route (whether or not in combination with mixing), the colored nature of the generator is, on the contrary, central.

DEMONSTRATION. Following these remarks, we concentrate on the less obvious case of pure scale densification. Specifically, we decompose a cascade over a (fixed) finite range of scales Λ into N steps of scale ratio λ_0 ($\lambda_0^N = \Lambda$) with corresponding spherical shells S_n [Eq. (12)] or $\hat{S}_n$ [Eq. (13)] in the Fourier space. The link between continuous cascades and discrete cascades is the following: at the cascade resolution Λ (i.e., at the scale $l = L/\Lambda$), a white Lévy noise ($\gamma_{0,\Lambda}$) over a set A corresponds to a sequence of i.i.d. Lévy variables supported by pixels of cascade resolution size, covering[8] A. The integration of the noise over A corresponds to summing the corresponding Lévy variables. The number N_n of these pixels covering S_n varies with scale as $N_n \approx (\Lambda/\lambda^n)^d$; therefore, in order that the sum over each spherical shell S_n has the same weight (i.e., independently of n), the Lévy variables on S_n have to be renormalized [due to Eq. (1)] by the factor b_n, which scales as

$$b_n \approx b(N_n) = N_n^{1/\alpha} \approx \left(\frac{\Lambda}{\lambda^n}\right)^{d/\alpha}. \quad (15)$$

The weighted Lévy noise $\gamma_{0,\Lambda}/b_n$ has individually unequal weights, but, indeed, their sum/integral over any S_n does have identical weights.

The second part of the proposition is demonstrated by considering the limit $\lambda_0 \downarrow 1$ ($N \uparrow \infty$), the renormalization on each sphere S_n by the factor b_n, which corresponds to a convolution by the scaling of Green's function $G(x) \propto |x|^{-C}$; $C = d/\alpha$ (a multiplication by its

[8] If they are not disjoint, the pixels should have a negligible intersection.

Fourier transform $\hat{G}(k) \propto |k|^{-D}$; $D = d - D$ in Fourier space). Therefore, the generator (at any cascade resolution) is obtained by fractional integration of order D:

$$\Gamma_\Lambda(\mathbf{x}) = \gamma_{0,\Lambda} * |\mathbf{x}|^{-C} = \int \gamma_{0,\Lambda}(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{-C} d^d\mathbf{x}', \quad (16)$$

or equivalently in Fourier space,

$$\hat{\Gamma}_\Lambda(\mathbf{k}) = k^{-D}\gamma_{0,\Lambda}(\mathbf{k}). \quad (17)$$

7. Strong versus weak universality

If for any physical reason the ''strong'' universality based on renormalization as described above fails to hold, weaker types of universality may still prevail (Schertzer et al. 1995). These will involve relationships between the generator and a few of its iterates that are looser than rescaling and/or recentering. These looser relations correspond to a different subclass of infinitely divisible generators, since, as already mentioned, any continuous cascade generator is infinitely divisible. For instance (as noted by She and Waymire 1995), the log-Poisson statistics considered by She and Leveque (1994) and Dubrulle (1994) provide a particularly simple example, which turns out to be a (nonrenormalized) continuous limit of the α model (Schertzer et al. 1995)![9] More generally, as an infinitely divisible law (e.g., Feller 1971) corresponds to a random (Poisson) sum of jumps, there is indeed a kind of weak universality if the distribution[10] of jumps is defined by only a limited number of parameters.

Although both types of universality exist mathematically, strong universality is physically more appealing—it corresponds to a renormalization property of the generator. A priori, one must look for weak universality only in the case of a failure of strong universality. In any case, it seems that strong universality is supported by the experimental evidence in the atmosphere, including in rain (Lovejoy and Schertzer 1995).

8. Conclusions

The multifractal approach yields a convenient framework for the analysis and simulation of highly nonlinear meteorological fields over a wide range of scales and intensities. A priori, this approach requires the determination of an infinite hierarchy of singularities and their associated codimensions. Fortunately, thanks to the general physical notion of universality, we emphasized that it is possible that this hierarchy could be fully determinable by a small number of fundamental exponents. We have given some details about two possible scenarios as well as about their possible combination, each of which allows us to reach a strong type of universality in a straightforward manner. These routes to universality rely only on a decomposition of the generator into (possibly infinitesimal) steps of equal weights, whereas GW's counter argument presupposed a decomposition based on unequal weights. The origin of this misunderstanding was traced to an erroneous interpretation of the significance of the numerical recipe pedagogically discussed in the cited Wilson et al. (1991) paper. While each generator is a noise with decreasing Fourier amplitudes, universality is still obtained by summing (i.i.d.) generators having equal weight on spherical shells of identical scale ratio and, hence, does not involve ''convolutions of independent but nonidentical distributions'' at all. In short, in spite of some confusion, the mathematical existence of universality is now rather straightforward; the real question is what the physically relevant mechanisms are. On the whole, we are convinced that this well-founded notion of multifractal universality is the key for the rapid development and further understanding of meteorological fields.

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[9] The renormalized limit being the (misnamed) ''lognormal'' multifractal (Schertzer et al. 1995).

[10] This ''Lévy canonical measure'' need not even be a probability measure.

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