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From scalar cascades to lie cascades: joint multifractal analysis of rain and cloud processes

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ABSTRACT There are two primary approaches to modeling rainfall; stochastic modeling and deterministic integration of nonlinear partial differential equations which model the atmospheric dynamics. The statistical advantages of the former could be combined with the physical advantages of the latter by exploiting cascade models based on scale invariant symmetries respected by the equations. Carried to its logical conclusion, this approach involves considering the atmosphere as a space-time multifractal process admitting either a vector, tensor or even only a nonlinear representation. The process is then defined by two groups which respectively specify the rule required to change from one scale to another and the corresponding transforms of fields. Both groups are characterized by their generators, hence by their Lie algebra. We show how to extend existing cascades beyond scalar processes, showing preliminary numerical simulations and data analyses, as well as indicating how to characterize and classify the scale invariant interactions of fields.

1. INTRODUCTION

1.1 The limitations of standard deterministic dynamical and of phenomenological stochastic modeling of rain

Geophysical fields show abundant evidence of nonlinear variability resulting from strong nonlinear interactions between different scales, different structures, and different fields. This variability is quite extreme and is associated with catastrophic events such as earthquakes, tornadoes, flash floods, extreme temperatures, volcanic eruptions. Another fundamental characteristic of this variability is the very large range of scales involved, which often extends from 10,000 km to 1 mm in space, and from geological scales to milliseconds in time. The scale ratio associated with this variability is at least 10^9, and for geophysical flows the corresponding Reynolds number is typically of the order of 10^{12} – so large that without any doubt the dynamics are all turbulent. Recently, a systematic study (Lovejoy et al., 1993) of scaling of cloud radiances at visible and infrared wave lengths (see Fig. 1) has revealed that as suggested by the unified scaling model of atmospheric dynamics (Schertzer & Lovejoy, 1983, 1985) – the scaling holds over at least the range $\approx$4000 km to $\approx$300 m (see also recent dynamics studies (Chigirinskaya et al., 1994, Lazarev et al., 1994)).

Up until recently, there have been two primary approaches to rainfall modeling: phenomenological stochastic modeling favoured by hydrologists, and deterministic dynamical modeling favoured by meteorologists. The former was largely based on ad hoc methods designed to mimic a phenomenology associated notably with a group at the University of Washington (e.g. Austin & Houze, 1972) that is predicated on the assumption that rain processes are qualitatively different a factor two or so in scale. The scientific

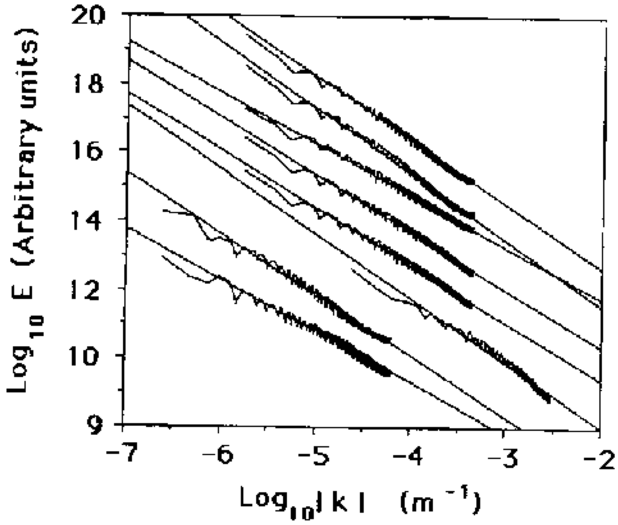


Fig. 1 Average power spectrum for the satellites' images grouped according to the satellite and the frequency range of the images (from bottom to top): LANDSAT (visible) $\beta = 1.7$, METEOSAT (visible) $\beta = 1.4$, METEOSAT (infrared) $\beta = 1.7$, NOAA-9 (channel 1 to 5) $\beta = 1.67, 1.67, 1.49, 1.91, 1.85$ (from Lovejoy et al., 1993).

outcome of relying on this phenomenology has been a series of very complex cluster processes where hierarchies of time and space scales are each assigned 'plausible' variations in rain rate and statistical fluctuations. Perhaps the best known model of this type is the Waymire-Gupta-Rodriguez-Iturbe 1984 (WGR) model which involves a dozen or so empirical parameters, and is at best successful only within the narrow range of time and space scales for which it was calibrated. The variability of the smaller scales not explicitly incorporated into the model yields a behaviour which is unrealistically smooth. The variability of the larger scales yields a behaviour which has unrealistically small variation from storm to storm. A further criticism of this approach that we will outline here is based on the fact that the rain process is coupled to other atmospheric fields in a highly nonlinear way; it cannot be fully treated in isolation. Consequently, rain should really be regarded as a component field of a space-time vector process where each component of the vector represents a different interacting field. Phenomenological stochastic models of the WGR type can be neither trivially nor naturally extended to include these other interacting fields.

In contrast, the deterministic models were developed following the usual methods of geophysical fluid dynamics. They are predicated on the integration of nonlinear partial differential equations which attempt to represent the complex nonlinear dynamics including a hopefully appropriate parametrization of the 'physics'. Because of the limited number of degrees of freedom which can be explicitly modeled, this approach makes drastic scale truncations (studying one scale independently of the others), transforming partial differential equations into ordinary differential equations, arbitrarily hypothesizing the homogeneity of subgrid scale fields, and performing ad hoc and unjustified parametrizations. In summary, both of these traditional approaches are therefore fundamentally limited by their inability to deal adequately with variability spanning many orders of magnitude in scale.

Even if we ignore these (over-) 'simplifying' assumptions, the consequences of such choices (which have increasingly weak links with the real world) are ultimately complex and yield unwieldy numerical codes. The relevance of such codes, obtained after either this long series of butcherings of the initial equations or after a long series of ad hoc attempts to mimic the phenomenology, remains highly questionable. For example, there is an increasing tendency to test deterministic models by making intercomparisons with other models! In contrast, the phenomenological stochastic approach does make closer contact with the data, but is virtually useless outside of the narrow range within which it is calibrated. Moreover, it is not even able to deal with extreme events within the calibration range. Both of these approaches suffer from strong limitations due to their inability to come to grips with the fundamental problem of nonlinear variability. This problem must be overcome if we want eventually to understand the very noisy intermittency of the signals of hydrology and other geophysical systems.

1.2 Cascades and symmetries

An alternative approach to nonlinear variability – first clearly elaborated by Schertzer & Lovejoy (1987) – is based on a fundamental symmetry property of the nonlinear (e.g. Navier-Stokes) equations: scale invariance. Indeed, the simplest way of understanding how geophysical variability occurs over a very large range of scales is to suppose that the same type of elementary process acts at each relevant scale (from the large scale down to the small viscous scale). At first, this began as a fractal approach (even before the word was coined, e.g. Richardson's (1922) celebrated poem on self-similar cascades), then (after 1983) it evolved into a multifractal approach.

These scale invariant multifractal models are superficially quite simple phenomenological 'toy models' (a bit like cellular automata). They give rise to cascades, avalanches, and other exotic phenomena (exotic compared to conventional smooth mathematical descriptions of the real world), but nevertheless have highly nontrivial consequences! For example, as we will see later, simple cascade models already give rise to a fundamental difference between observables and truncated processes, and such a difference is a general property of the wide class of 'hard' multifractal processes (which distinguish between 'dressed' and 'bare' properties respectively). These models produce hierarchies of self-organized random structures, which is also a very general property of (singular) multifractal measures of singularities and of Self-Organized Criticality (Bak et al., 1987). In general, these simple models give us precious hints as to how to cast order in disorder.

Until now, a basic limitation of these cascade processes is that they have been limited to positive scalar fields (such as the energy flux from large to small scales). They are thus even capable of dealing with inverse cascades (negative fluxes; small to large scale transfers), not to mention the more fundamental problem of vector (e.g. wind) or tensor processes (e.g. stress or strain tensors) necessary to deal fully with the nonlinear dynamics. Below, we give a brief review of this scalar cascade theory and then go on to show how it can be generalized to vector, tensor or 'Lie cascades'.

2. THE SCALAR MULTIFRACTAL FRAMEWORK

2.1 Fractals and multifractals: Fractal geometry

Fractal geometry provides the simplest nontrivial example of scale invariance, and is useful for characterizing fractal sets. It can also be useful in producing linear models of rain produced by additive random processes (which involve unique fractal dimensions) such as the (monofractal) simple scaling model of rain tested in Lovejoy (1981) (see also Lovejoy & Mandelbrot, 1985; Lovejoy & Schertzer, 1985). Unfortunately in geophysics we are much more interested in fields and are rarely interested in geometrical sets. However, fractal dimensions can still be useful in counting the occurrences of a given phenomenon over a wide range of scales – as long as we can properly pose this question. If this is the case and the phenomenon is scaling, then the number of occurrences ($N_A(l)$) of an event at resolution scale l (in space and/or time) follows a power law[1]:

$$N_a(l) \approx \left(\frac{l}{L}\right)^{-D_F} \quad (1)$$

D_F being the (unique) fractal dimension, generally not an integer, and L the (fixed) largest scale. For instance, Fig. 2 shows the records of rain events during the last 45 years in Dedougou (Hubert & Carbonel, 1990). These authors show that the occurrence of rainy days during a certain time scale T is fractal, having a dimension $D_F \approx 0.8$, which accounts for the fact that the rain events on the time axis form a Cantor-like set. Amusingly, the wet season is often considered to last 7 months per year, and $0.8 \approx \mathrm{Log}7/\mathrm{Log}12$ (recall that the standard Cantor set is obtained by iteratively removing the (closed) middle section of the unit interval and has a dimension of Log2/Log3).

Numerous similar (mono-) fractal results can be obtained on different fields. However, fields having different levels of intensity do not reduce to the oversimplified binary question of occurrence or nonoccurrence. For instance, in the case of rain we have to address the fundamental question: what is the rain rate at different scales? What is a negligible rain rate? Generalizations of fractal/scale invariance ideas well beyond geometry were desperately needed and appeared in 1983 when the dogma of a unique dimension was finally abandoned (Henstchel & Procaccia, 1983; Grassberger, 1983; Schertzer & Lovejoy, 1983).

However, it is already important to note that the notion of codimension (c) (usually defined by $c = D - D_F$, where D is the dimension of the embedding space) can be considered to be at least as fundamental as the notion of fractal dimension D_F. Indeed, c can be directly defined as measuring the fraction of the space occupied by the fractal set A of dimension D_F. This can be seen by considering that a ball B_l of size l has the following probability of intersecting A:

$$P(B_l \cap A) \approx \frac{N_A(l)}{N(l)} = l^c \quad (2)$$

where $N(l) \approx l^{-D}$ is the number of balls size l necessary to cover a D-dimensional space.

In fact, for multifractal fields, codimensions will be more fundamental and useful than dimensions, since they give intrinsic characterizations of the multifractal process. We will therefore use a codimension formalism (Schertzer & Lovejoy, 1987, 1992) rather than the more popular dimension formalism developed for strange attractors (e.g. Halsey et al., 1986). One may note that recently the need of a codimension formalism has been implicitly acknowledged in Mandelbrot (1991).

2.2 The extension to scalar multifractals

One obtains much more information by looking not at the occurrence of rain, but at the rain rate: a 1 mm daily rain rate is negligible compared to a 150 mm daily rain rate! For instance, Fig. 3 displays the rain rate at Nîmes (France) during a few years, and averaged over varying scales T (from a day to a year). This figure illustrates the great intermittency of rain rates: most of the time it is negligible, while sometimes it reaches 200 mm (228 mm in few hours, for the famous October 1988 catastrophe!) – in comparison the daily average is ≈ 2.1 mm. The variability is so significant in this time series that Ladoy et al. (1991) found some evidence of the

[1] Here and below the sign $\approx$ means equality within slowly varying and constant factors.

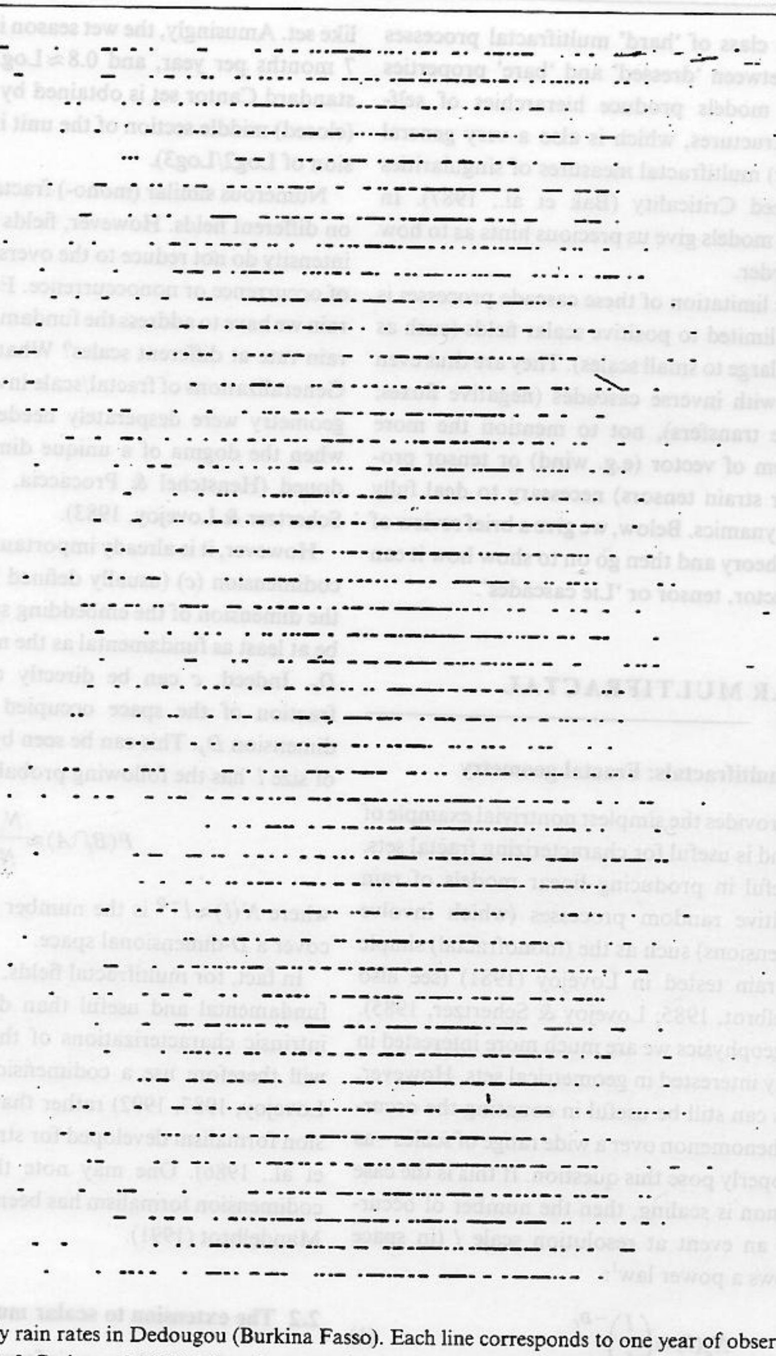
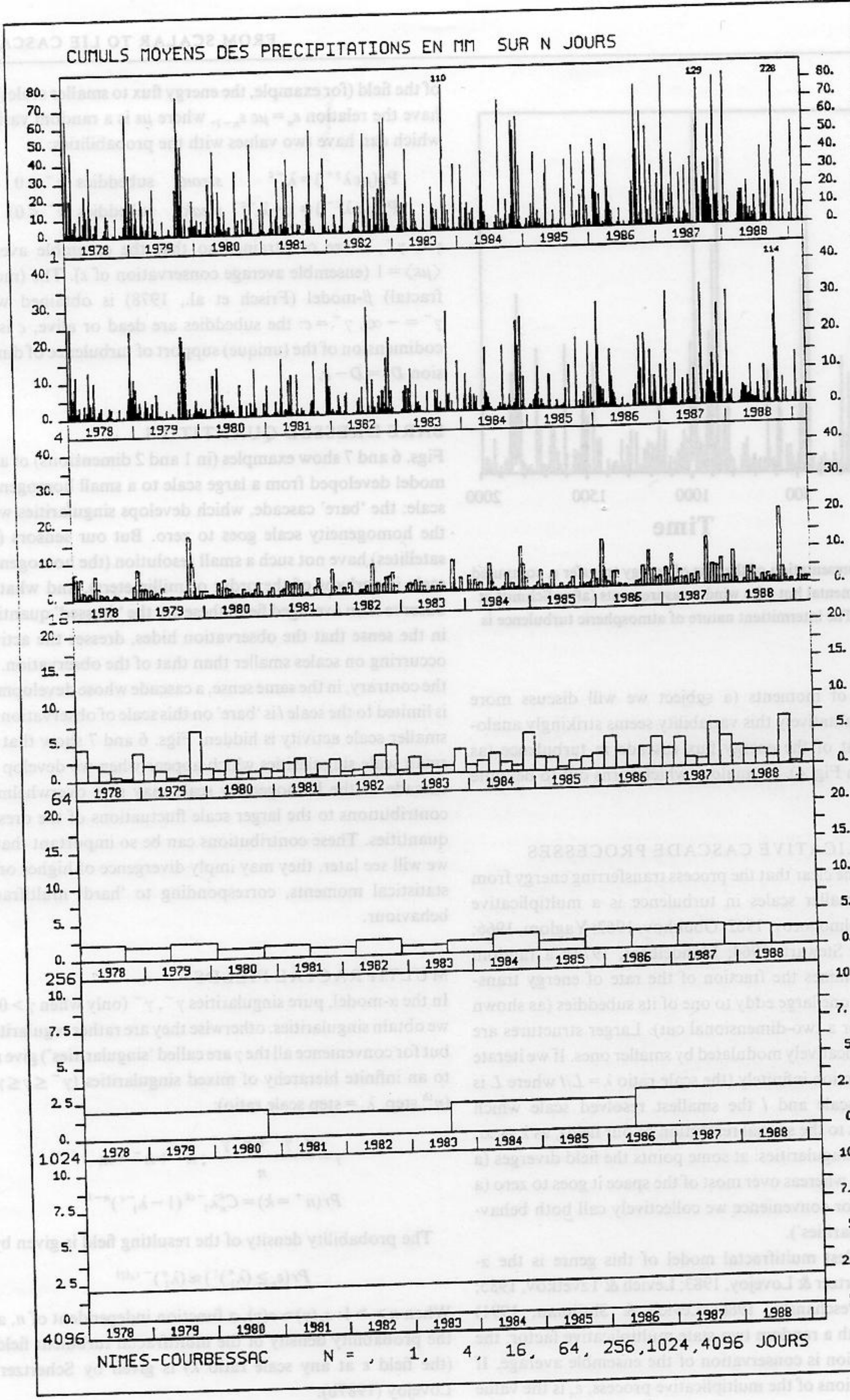


Fig. 2 Picture of 45 years of daily rain rates in Dedougou (Burkina Fasso). Each line corresponds to one year of observation, and each black dot to a rainy day. (Hubert & Carbonnel 1990). The rain events form a Cantor-like set of dimension $D_F \approx 0.8$ (the standard Cantor set is of dimension Log(2)/Log(3)).

Fig. 3 Rain rates in Nîmes (France) during the years 1978–1988, also averaged over 1 day, 4 days, 16 days, 64 days, 256 days, 1024 days and 4096 days respectively (after Ladoy et al., 1993). It illustrates the great intermittency of rain rates: some rare but extreme events of short periods (singularities) gave overwhelming contributions (e.g. the 228 mm which occurred in a few hours) to the October 1988 Nimes catastrophe, which are hardly smoothed over longer periods.


CUMULS MOYENS DES PRECIPITATIONS EN MM SUR N JOURS
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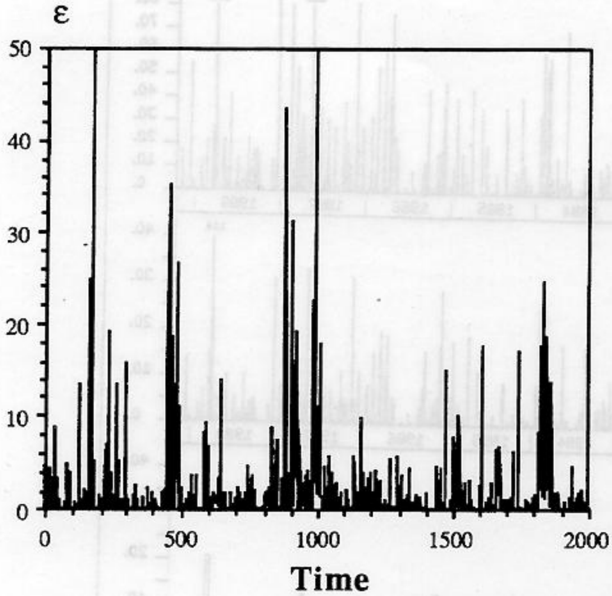


Fig. 4 A representation of the rate of energy transfer ε, computed from experimental hot wire wind measurements (after Schmitt et al., 1992b). The intermittent nature of atmospheric turbulence is obvious.

divergence of moments (a subject we will discuss more below). Qualitatively this variability seems strikingly analogous to that of the energy flux cascade in turbulence (as displayed in Fig. 4), an analogy which turns out to be quite profound.

MULTIPLICATIVE CASCADE PROCESSES

It has become clear that the process transferring energy from larger to smaller scales in turbulence is a multiplicative process (Kolmogorov, 1962; Obukhov, 1962; Yaglom, 1966; Novikov & Stewart, 1964; Mandelbrot, 1974): a random factor determines the fraction of the rate of energy transferred from one large eddy to one of its subeddies (as shown on Fig. 5 for a two-dimensional cut). Larger structures are thus multiplicatively modulated by smaller ones. If we iterate this construction infinitely (the scale ratio $\lambda = L/l$ where L is the larger scale and l the smallest resolved scale which corresponds to the spatial resolution of our field), as $\lambda \to \infty$, we observe singularities: at some points the field diverges (a singularity), whereas over most of the space it goes to zero (a regularity; for convenience we collectively call both behaviours 'singularities').

The simplest multifractal model of this genre is the α-model (Schertzer & Lovejoy, 1983; Levich & Tzvetkov, 1985; Bialas & Peschanski, 1986; Levich & Shtilman, 1991) obtained with a random two-state multiplicative factor: the only restriction is conservation of the ensemble average. If after n iterations of the multiplicative process, ε_n is the value of the field (for example, the energy flux to smaller scales), we have the relation $\varepsilon_n = \mu\varepsilon\, \varepsilon_{n-1}$, where $\mu\varepsilon$ is a random variable which can have two values with the probabilities:

$$\begin{array}{lll} \Pr(\mu\varepsilon\lambda^{\gamma^+}) = \lambda^{-c} & \textit{strong} & \text{subeddies } (\gamma^+ > 0 \\ \Pr(\mu\varepsilon\lambda^{\gamma^-}) = 1-\lambda^{-c} & \textit{weak} & \text{subeddies } (\gamma^- < 0) \end{array} \quad (3)$$

γ^+, γ^-, c are constrained so that the ensemble average $\langle\mu\varepsilon\rangle = 1$ (ensemble average conservation of ε). The (monofractal) β-model (Frisch et al., 1978) is obtained when $\gamma^- = -\infty$, $\gamma^+ = c$: the subeddies are dead or alive, c is the codimension of the (unique) support of turbulence of dimension $D_F = D - c$.

BARE DRESSED QUANTITIES

Figs. 6 and 7 show examples (in 1 and 2 dimensions) of an α-model developed from a large scale to a small homogeneity scale: the 'bare' cascade, which develops singularities when the homogeneity scale goes to zero. But our sensors (e.g. satellites) have not such a small resolution (the homogeneity scale is perhaps of the order of millimeters), and what we observe is an averaged field; these are the 'dressed' quantities in the sense that the observation hides, dresses the activity occurring on scales smaller than that of the observation. On the contrary, in the same sense, a cascade whose development is limited to the scale l is 'bare' on this scale of observation: no smaller scale activity is hidden. Figs. 6 and 7 show that the small scale singularities which appear when we develop the cascade to the homogeneity scale may give overwhelming contributions to the larger scale fluctuations of the dressed quantities. These contributions can be so important that as we will see later, they may imply divergence of higher order statistical moments, corresponding to 'hard' multifractal behaviour.

MULTIFRACTAL FIELDS

In the α-model, pure singularities γ^-, γ^- (only when $\gamma > 0$ do we obtain singularities, otherwise they are rather regularities, but for convenience all the γ are called 'singularities') give rise to an infinite hierarchy of mixed singularities ($\gamma^- \le \gamma \le \gamma^+$) ($n^{\text{th}}$ step, λ_1 = step scale ratio):

$$\gamma = \frac{n^+\gamma^+ + n^-\gamma^-}{n};\; n^+ + n^- = n; \qquad (4)$$
$$Pr(n^+ = k) = C_n^k \lambda_1^{-ck}(1-\lambda_1^{-c})^{n-k}$$

The probability density of the resulting field is given by:

$$Pr(\varepsilon_n \ge (\lambda_1^n)^\gamma) \approx (\lambda_1^n)^{-c_n(\gamma)} \qquad (5)$$

When $n >> 1$: $c_n(\gamma) \approx c(\gamma)$, a function independent of n, and the probability density of the multifractal turbulent field ε_λ (the field ε at any scale ratio λ) is given by Schertzer & Lovejoy (1987b):

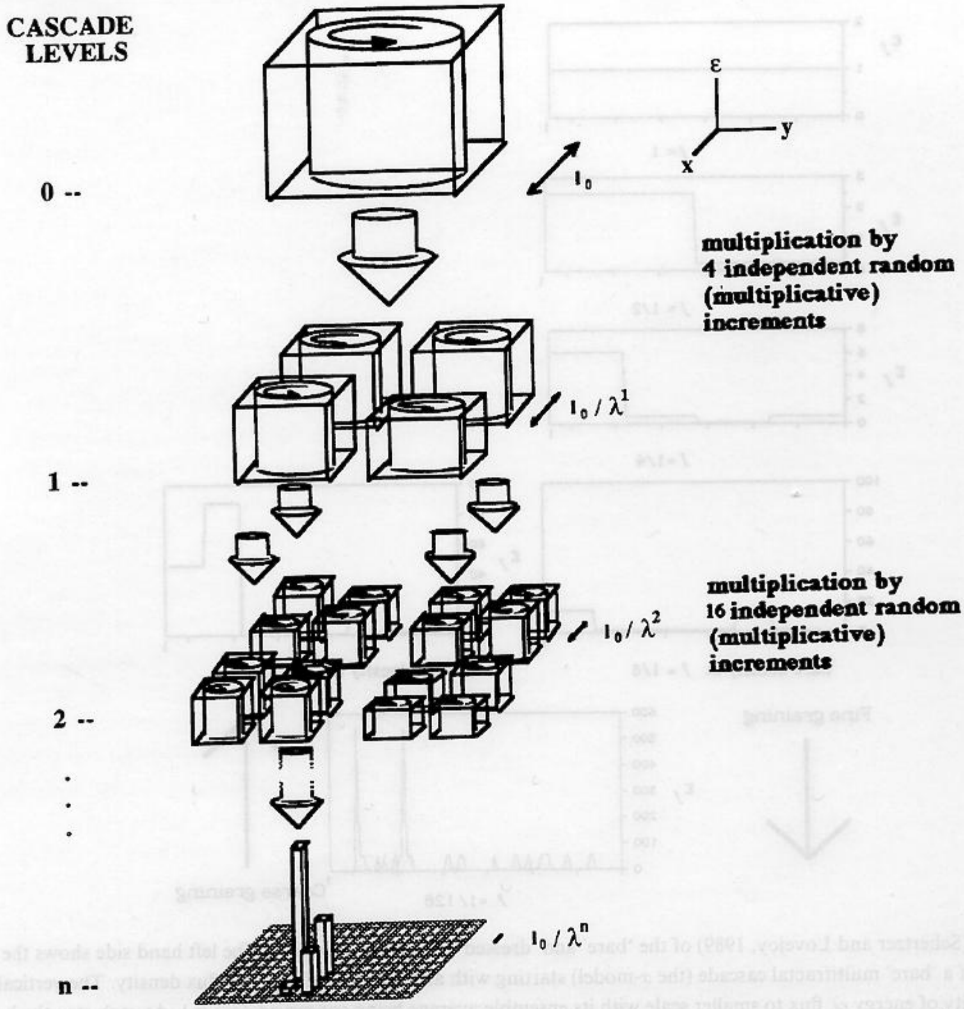


Fig. 5 A schematic diagram showing a two-dimensional cascade process at different levels of its construction to smaller scales. Each eddy is broken up into four subeddies, transferring a part or all its energy flux to the subeddies. In this process the flux of the field at large scale multiplicatively modulates the various fluxes at smaller scales; the mechanism of flux redistribution is repeated at each cascade step (self similarity).

$$Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)} \quad (6)$$

When $c(\gamma) < d$ (d being the dimension of space), as already discussed (eq. (2)), $c(\gamma)$ is the (geometrical) codimension $c(\gamma) = d - D(\gamma)$ corresponding to the (geometrical) fractal dimension $D(\gamma)$ of the support of singularities whose order is greater than γ.

In the most interesting cases $c(\gamma) \geq d$ is associated with a nonobviously negative (!) dimension $D(\gamma)$ (Mandelbrot, 1991). However the function $c(\gamma)$ remains a (finite) codimension on an (infinite dimensional) probability space (see below). The multiple scaling behaviour of this field ε at scale ratio λ can be also characterized by the corresponding law for the statistical moments (via a Laplace transform):

$$\langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)} \quad (7)$$

The relations between the turbulent notation used here and the strange attractor $f_D(\alpha_D)$ and $\tau_D(q)$ notation (the subscript D explicitly emphasizes the dependence of α, f, τ on the dimension of the observing space D) are: $f_D(\alpha_D) = D - c(\gamma)$ and $\tau_D(q) = K(q) - (q-1)D$ with $\alpha_D = (D - \gamma)$. The codimension notation is necessary when dealing with stochastic processes because γ, c, K are intrinsic contrary to α_D, f_D, τ_D which diverge with $D \to \infty$. It also avoids introducing negative ('latent') dimensions when $c(\gamma) > D$.

Just as $f(\alpha)$ is the Legendre transform (Parisi & Frisch, 1985) of $\tau(q)$, so $c(\gamma)$ is the transform of $K(q)$:

$$K(q) = \max_\gamma (q\gamma - c(\gamma)); \quad c(\gamma) = \max_q (q\gamma - K(q)) \quad (8)$$

These relations establish a one-to-one correspondence between orders of singularities and moments ($q = c'(\gamma)$, $\gamma = K'(q)$).

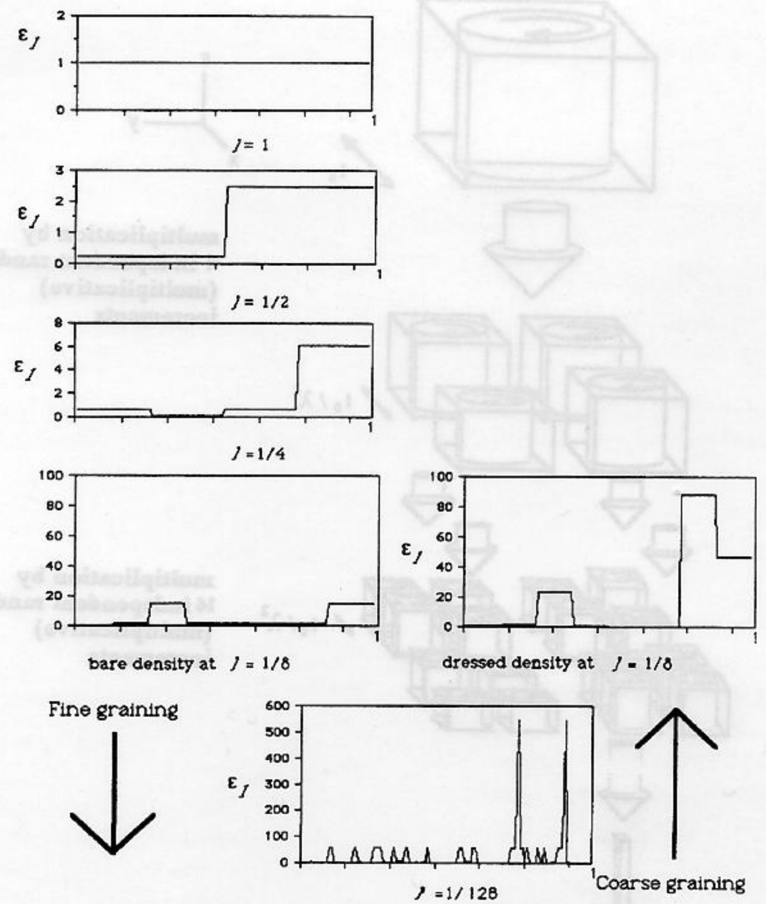


Fig. 6 Illustration (Schertzer and Lovejoy, 1989) of the 'bare' and 'dressed' energy flux densities. The left hand side shows the step by step construction of a 'bare' multifractal cascade (the α-model) starting with an initially uniform unit flux density. The vertical axis represents the density of energy $\varepsilon\lambda$ flux to smaller scale with its ensemble average being conserved $\langle\varepsilon_\lambda\rangle = 1$. At each step the horizontal scale is divided by two. The developing spikes are incipient singularities of various orders (characteristic of multifractal processes).

2.3 Some basic properties of multifractal fields

Multifractal fields, contrary to (mono-) fractal geometry, involve an infinite hierarchy of γs corresponding to the infinite hierarchy of $c(\gamma)$. Indeed, according to eq. (5), the hierarchy of codimensions may be obtained by thresholding the field and computing the fractal codimension of values greater than this threshold λ^{γ} (see Fig. 8). The codimension function $c(\gamma)$ must satisfy only a rather weak constraint (see Fig. 9): not only should it be obviously an increasing function of γ (if $\gamma_1 > \gamma_2$, $\Pr(\varepsilon_\lambda \geq \lambda^{\gamma_1}) \leq \Pr(\varepsilon_\lambda \geq \lambda^{\gamma_2})$, thus $c(\gamma_1) \geq c(\gamma_2)$), but it must also be convex as $K(q)$ (Feller, 1971).

THE SAMPLING DIMENSION D_s

Here we point out the utility of the notion of *sampling dimension* D_s. As we are always compelled to analyze finite samples, it is rather obvious that the highest singularities will rarely be present in a given sample. More precisely speaking, some of the singularities will almost surely not be present in a finite sample. Indeed, when we analyze only one sample/realization of the field on a dimension D at resolution λ, the largest singularity γ_s we can reach is given by $c(\gamma_s) = D$. More generally, if we are studying N_s samples, we can introduce the sample dimension $D_s \neq 0$ (at resolution λ) defined as $N_s \approx \lambda^{D_s}$ (Schertzer & Lovejoy, 1989; Lavallée, 1991; Lavallée et al., 1991). This largest singularity increases with D_s, since its order is then given by $c(\gamma_s) \approx D + D_s$ (see Fig. 10), it corresponds to a moment order $q_s = c'(\gamma_s)$ beyond which $K(q)$ becomes spuriously linear. The sampling dimension D_s gives us a quantitative way to describe how larger samples enable us to explore more and more of the probability space, eventually attaining the rare singularities responsible for the wild behaviour of experimental fields.

CLASSIFICATION OF MULTIFRACTAL FIELDS

Most of the theoretical and corresponding empirical studies unfortunately presuppose very restrictive calmness and regularity assumptions on multifractal field. Such a limited view

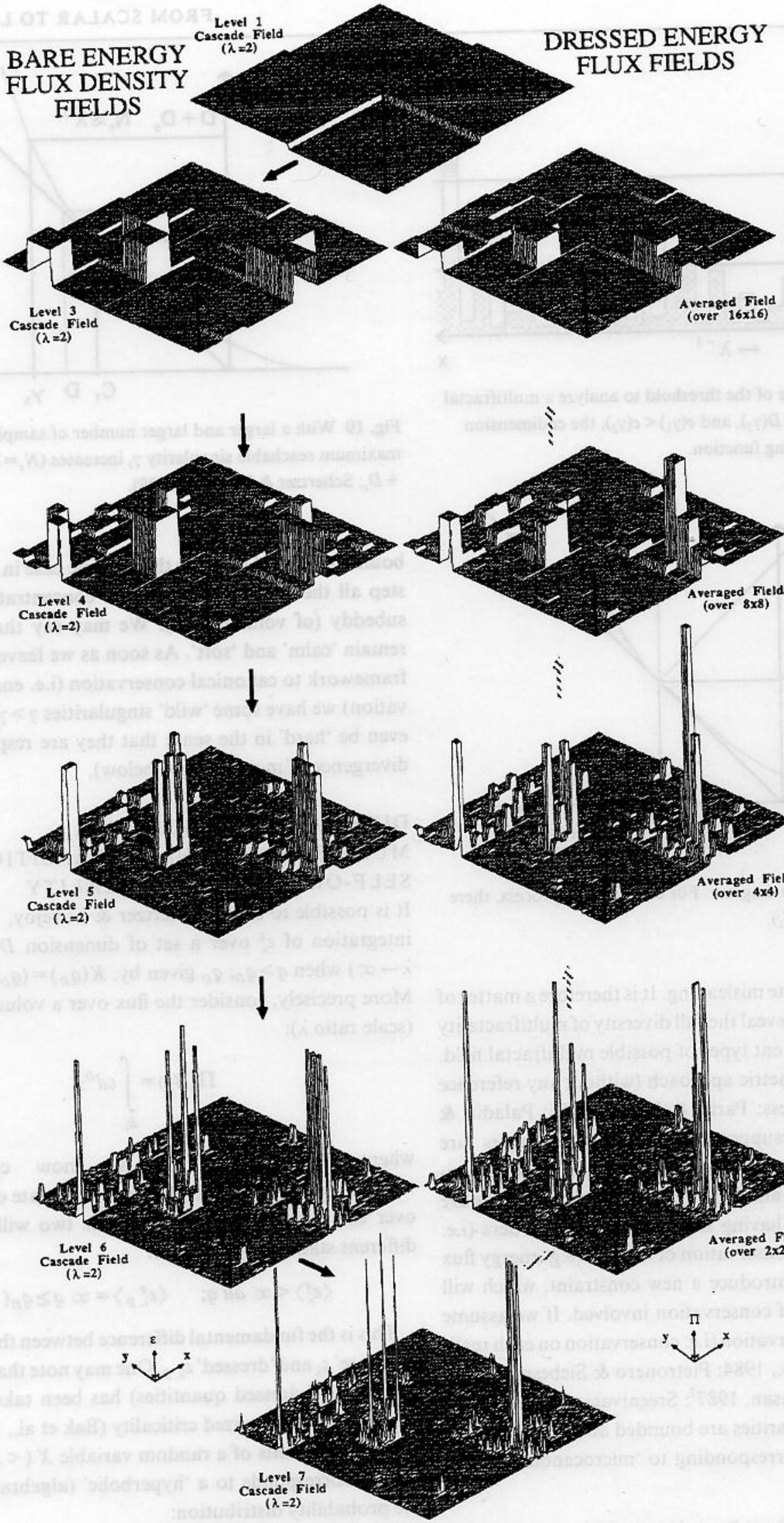


Fig. 7 As in fig. 6, illustration of the 'bare' and 'dressed' energy flux densities, but now for a two-dimensional cut.

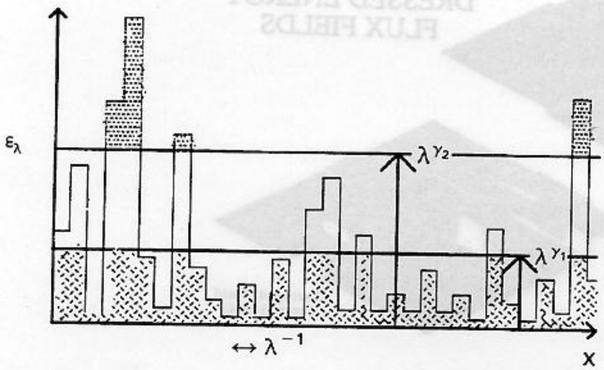


Fig. 8 The importance of the threshold to analyze a multifractal field: if $\gamma_1 < \gamma_2$, $D(\gamma_1) > D(\gamma_2)$, and $c(\gamma_1) < c(\gamma_2)$, the codimension function is an increasing function.

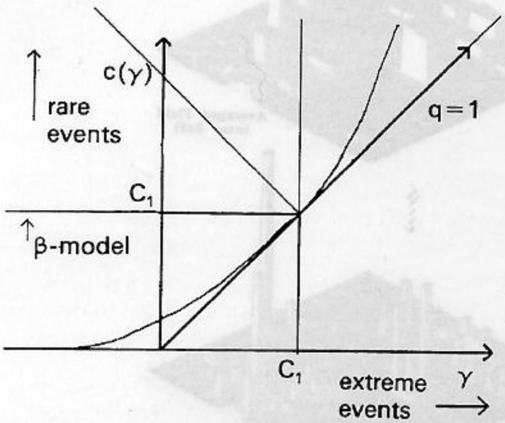


Fig. 9 A schematic $c(\gamma)$ diagram. For conservative process, there is a stationary value (C_1).

of multifractals is quite misleading. It is therefore a matter of some importance to reveal the full diversity of multifractality and classify the different types of possible multifractal field. Indeed a purely geometric approach (without any reference to a stochastic process; Parisi & Frisch, 1985; Paladin & Vulpiani, 1987) presupposes that the singularities are bounded by $\gamma_{\max}^{(g)}$, the upper bound of geometrical singularities, with $c(\gamma_{\max}^{(g)}) = D$ and $\gamma_{\max}^{(g)} < D$. Stochastic processes are generally capable of having singularities of all orders (i.e. $c(\gamma) \geq D$). However, conservation of the flux (e.g. energy flux in turbulence) may introduce a new constraint, which will depend on the type of conservation involved. If we assume microcanonical conservation (i.e. conservation on each realization, see Benzi et al., 1984; Pietronero & Siebesma, 1986; Meneveau & Sreenivasan, 1987[2]; Sreenivasan & Meneveau, 1988), then the singularities are bounded above by $\gamma_{\max}^{(m)} = D$. The superscript m corresponding to 'microcanonical', this

[2] Their celebrated 'p model' is in fact nothing more than a microcanonical restriction of the α-model discussed earlier.

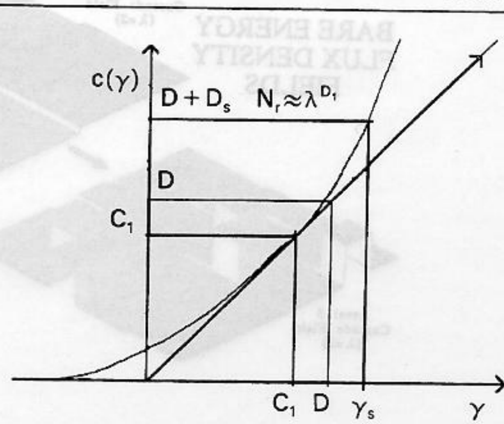


Fig. 10 With a larger and larger number of samples N_s, the maximum reachable singularity γ_s increases ($N_s = \lambda^{D_s}$; $c(\gamma_s) = D + D_s$; Schertzer & Lovejoy, 1989).

bound is reached only for the extreme case in which for each step all the density of the flux is concentrated on a single subeddy (of volume λ^{-D}). We may say that singularities remain 'calm' and 'soft'. As soon as we leave this restricted framework to canonical conservation (i.e. ensemble conservation) we have some 'wild' singularities $\gamma > \gamma_{\max}^{(m)}$, which can even be 'hard' in the sense that they are responsible of the divergence of moments (see below).

DIVERGENCE OF MOMENTS, MULTIFRACTAL PHASE TRANSITIONS AND SELF-ORGANIZED CRITICALITY

It is possible to show (Schertzer & Lovejoy, 1987) that the integration of ε_λ^q over a set of dimension D diverges (for $\lambda \to \infty$) when $q > q_D$; q_D given by: $K(q_D) = (q_D - 1)D$; $q_D > 1$. More precisely, consider the flux over a volume element B_λ (scale ratio λ):

$$\Pi(B_\lambda) = \int_{B_\lambda} \varepsilon d^D x \qquad (9)$$

where $\varepsilon = \lim_{\lambda \to \infty} \varepsilon_\lambda$. If we now consider $\varepsilon_{\lambda,D} = \Pi(B_\lambda)/\mathrm{Volume}(B_\lambda)$ as a (dressed) estimate of the (bare) ε_λ over the D-dimensional ball B_λ; the two will have totally different statistical properties:

$$\langle \varepsilon_\lambda^q \rangle < \infty \; all\; q; \quad \langle \varepsilon_{\lambda,D}^q \rangle = \infty \; q \geq q_D (>1) \qquad (10)$$

This is the fundamental difference between the two quantities 'bare' ε_λ and 'dressed' $\varepsilon_{\lambda,D}$. One may note that the singular statistics (of dressed quantities) has been taken as a basic feature of self organized criticality (Bak et al., 1987). Divergence of moments of a random variable X ($<X^q> = \infty$ for $q > q_D$) corresponds to a 'hyperbolic' (algebraic) fall-off of the probability distribution:

$$Pr(X \geq s) \approx s^{-q_D} (s \gg 1) \Leftrightarrow \langle X^q \rangle = \infty \quad for\; q > q_D \quad (11)$$

The physical significance of divergence of moments is that when $q < q_D$ the dressed moments are macroscopically determined whereas for $q > q_D$ the moments will be microscopically determined depending crucially on the small scale details. It is possible to make a formal[3] analogy between conventional thermodynamics and multifractals; for example, the entropy corresponds to $c(\gamma)$ and the temperature to $1/q$, the Massieu potential (the free energy divided by temperature) to $K(q)$. Therefore, this qualitatively new behaviour for $q > q_D$ (low temperatures) can be considered as discussed in Schertzer & Lovejoy (1992, 1994) and Schertzer et al. (1993); this corresponds to a first order multifractal phase transition, where the thermodynamic potential $K(q)$ has a first order discontinuity at the critical temperature analog q_D^{-1} (Schmitt et al. 1994; Chigirinskaya et al. 1994 for corresponding atmospheric data analysis: $q_D \simeq 7$ for velocity field).

2.4 The three fundamental exponents: H, C_1, α

It is already important to note that three parameters are sufficient to characterize *locally* (around the mean singularity) the infinite hierarchy of fractal codimensions $c(\gamma)$. Furthermore, this characterization turns out to be global under certain general hypotheses of universality we discuss in the next section. The three fundamental exponents are the following:

- H describes the *deviation from conservation* of the flux: $\langle \varepsilon_\lambda \rangle \approx \lambda^{-H}$. $H = 0$ for conservative fields (for instance the energy flux in turbulence, $\langle \varepsilon_\lambda \rangle$ independent of λ) whereas according to the Kolmogorov relation in real space $\Delta v_\lambda \approx \varepsilon_\lambda^{1/3} \lambda^{-1/3}$ (where Δv_λ is the wind shear amplitude $|v(x+\lambda^{-1}) - v(x)|$ at scale ratio λ), the wind shear is a nonconservative field ($H = 1/3$).
- C_1 describes the *mean inhomogeneity*: it is the codimension of the mean singularity: $C_1 = c(C_1 - H)$, in the case of conservative fluxes it is also the order of the mean singularity (and simultaneously the fixed point of $c(\gamma)$).
- α represents the *degree of multifractality* measured by the convexity of $c(C_1)$ around the mean singularity $(C_1 - H)$ measured by the radius of curvature: $R_c(\gamma = C_1 - H) = 2^{3/2} \alpha C_1$ which increases with the range of singularities (starting from zero with the monofractal β-model). As shown below, in the case of universal multifractals, α is also the Levy index of the generator and $0 \le \alpha \le 2$.

2.5 Universality by mixing of multifractal processes

The particularities of the discrete models (e.g. α-model) remain as the cascade proceeds to its small scale limit. If we simply iterate the model step by step with a fixed ratio of scale λ_1 for the elementary step, we indefinitely increase the range

[3] Formal since we here are considering systems out of equilibrium.

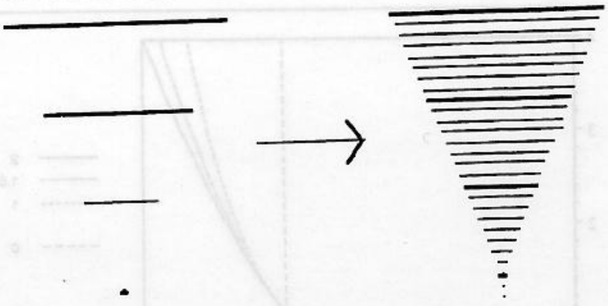


Fig. 11 Keeping the total range of scale fixed and finite, introducing intermediate scales and then seeking the limit of an infinite range of scales leads to universality.

of scales $\Lambda \to \infty$ which already poses a nontrivial mathematical problem (weak limit of random measures, see Kahane, 1985). On the contrary, by *keeping* the total *range of scale fixed and finite*, *mixing independent processes* of the *same type* (by multiplying them, preserving certain characteristics, e.g. variance of the generator) and *then* seeking the limit $\Lambda \to \infty$, *a totally different limiting problem is obtained!* For instance, this may correspond to *densifying* the excited scales by introducing more and more intermediate scales (see Fig. 11), and seeking thus the limit of continuous scales of the cascade model. Alternatively, we may also consider the limit of multiplications of i.i.d. discrete cascades models.

In both cases, multiplying processes corresponds to adding generators: $\varepsilon_\lambda \approx e^{\Gamma_\lambda}$ where ε_λ is the process and Γ_λ is the generator. If we seek *generators* which are *stable* and *attractive* under *addition* (using the results on the second Laplace characteristic function $K(q)$ equivalent to the free energy), we must consider (Schertzer & Lovejoy, 1987, 1989; Schertzer et al., 1988; Fan, 1989) stable extremal Lévy noises with $1/f$ spectra, which are characterized by a *Lévy index* α: $Pr(-\Gamma \ge s) \approx s^{-\alpha}$ $(s \gg 1)$ $\Rightarrow$ any $q \ge \alpha$: $\langle (-\Gamma)^q \rangle = \infty$. Except for the Gaussian exception $\alpha = 2$, α is the order of divergence of moments of the generator. These generators yield the following *universal* expressions for the scaling function of the moments of the field $K(q)$ and of the codimension function $c(\gamma - H)$:

$$c(\gamma - H) = C_1 \left(\frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha} \right)^{\alpha'} \qquad \alpha \neq 1$$
$$c(\gamma - H) = C_1 \exp\left(\frac{\gamma}{C_1} - 1 \right) \qquad \alpha = 1 \tag{12}$$

$$K(q) + Hq = \frac{C_1}{\alpha - 1}(q^\alpha - q) \qquad \alpha \neq 1$$
$$K(q) + Hq = C_1 q Log(q) \qquad \alpha = 1 \tag{13}$$

where $(1/\alpha + 1/\alpha' = 1$, and for $q = dc/d\gamma > 0)$ and C_1 is related to the coefficient C of the canonical Lévy measure dF by:

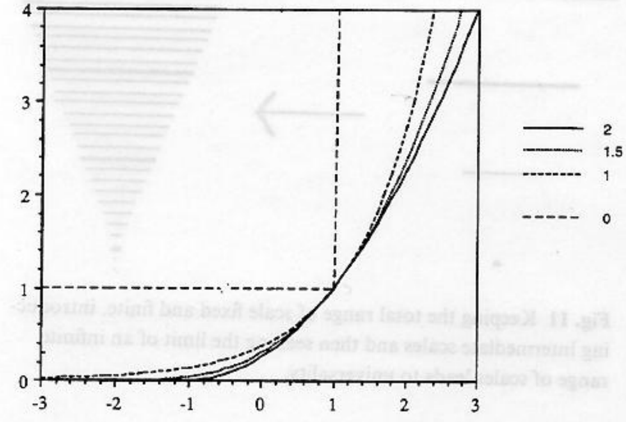


Fig. 12 Universal (bare) singularities codimension $c(\gamma)/C_1$ corresponding to the five classes; here $\alpha = 2, 1.5, 1, 0$.

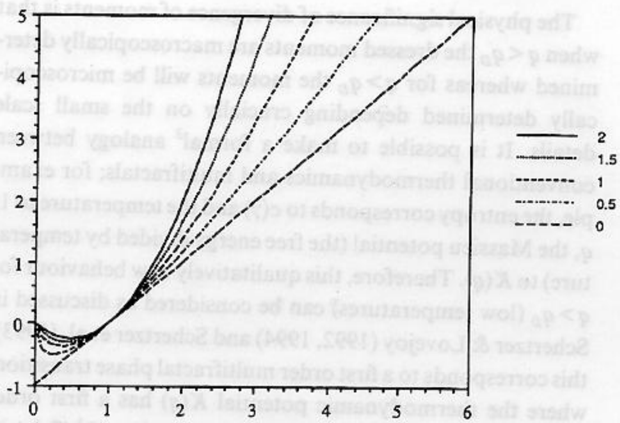


Fig. 13 Universal (bare) singularities codimension $c(\gamma)/C_1$ corresponding to the five classes; here $\alpha = 2, 1.5, 1, 0.5, 0$.

$$C_1 = C\Gamma(3-\alpha)/\alpha; \qquad dF = 1_{x>0}C(2-\alpha)x^{-\alpha}dx/x \qquad (14)$$

(here Γ is the usual gamma function and should not be confused with the generator. Figs. 12 and 13 show universal $K(q)$ and $c(\gamma)$ curves, α varying from 0 to 2.

The two functions $K(q)$ and $c(\gamma)$ are analytic, and depend only on the three parameters H, C_1, and α. The knowledge (either by measurements or from theoretical considerations) of these parameters is then enough to compute all the statistical properties of the field. The implicit hypothesis is that this field results from a universal process, hence these parameters are universal. The first, H, is often known theoretically and experimentally, and is therefore already recognized to be universal for many fields. The second, C_1, may perhaps fluctuate slightly with time and location (e.g. Tessier et al., 1993). In fact the most important parameter, the Levy index α, which is fundamental for the classification of the fields (see Tables 1 and 2) is the most likely to be universal. Some experimental results tend to confirm this assumption: at least for the temporal rain rate: five different experiments (Hubert et al., 1993) have (independently) estimated the different time periods, geographical locations, and for both rain gauge accumulations and radar measurements the value $\alpha = 0.5 \pm 0.05$ (see also Lovejoy & Schertzer, 1991, 1992, 1995).

2.6 Scaling anisotropy and generalized scale invariance (GSI)

The standard picture of atmospheric dynamics is that of an isotropic 2-D large scale and an isotropic 3-D small scale, the two separated by a 'meso-scale gap'. Mounting evidence now suggests that, on the contrary, atmospheric fields, while strongly anisotropic, are nevertheless scale invariant right through the meso-scale. The idea of generalized scale invariance (GSI) is to leave the artificial 2-D/3-D dichotomy and to postulate first scale invariance and then study the (unusual) remaining symmetries.

The specification of GSI requires a generator (G) which can be a nonlinear function (varying from point to point): data sets with very large ranges of scale will be needed, and even then, some simplifying approximations will be necessary. As a result of these difficulties the first empirical tests were studies of the compression (stratification) part of GSI associated with the trace of the generator (the elliptical dimension d_{el}). The studies have specifically avoided the difficult differential rotation problem (see below) by concentrating on the vertical stratification Schertzer & Lovejoy 1983, 1985) who estimated $D_{el} = 23/9 = 2.555\ldots$ for the horizontal wind field, $D_{el} = 2.22 \pm 0.07$ for the vertical stratification of rain, and $D_{el} = 2.5 \pm 0.3$ in space/time for the rainfield (Lovejoy et al., 1987; Lovejoy & Schertzer, 1991).

To go from one scale to another, we only need to specify the scale ratios (see Fig. 14, which shows anisotropic scale invariance). We can here define a (semi) group of scale changing operators $T_\lambda = \lambda^{-G}$ (G being the generator) which reduces the scale of vectors by the scale ratio λ: $B_\lambda = T_\lambda(B_1)$ is the ball of all vectors at scale λ (where the unit 'ball' B_1 defines all the unit vectors). Virtually the only other restriction on T_λ is that the B_λ are strictly decreasing ($B_\lambda \supset B_{\lambda'}$; $\lambda < \lambda'$), hence that the real parts of the (generalized) eigenvalues of G are all > 0.

Approximating G by a matrix leads to linear GSI: when there are no off-diagonal elements we obtain only differential stratification, 'self-affine' (multi-)fractals. Off-diagonal elements are associated with differential rotation and can be empirically estimated on scanned cloud satellite images with the help of the Monte Carlo Differential Rotation Technique (Pflug et al., 1993) or (better) by the Scale Invariant Generator Technique (Lewis et al., 1995).

Table 1. *Values of the universal multifractal parameters as estimated by different authors on different data sets.*

	Gauge, daily accumulation	Gauge, 6 minutes resolution	Gauge, daily accumulation	Gauge, daily accumulation	Gauge, 15 minutes resolution
Location	Global network	Réunion Islands (France)	Nîmes (France)	Dédougou (Burkina Faso)	Alps (France)
Stations	4000	1	1	1	28
Duration	1 year (scaling regime up to 16 days)	1 year (scaling regime up to 30 days)	40 years (scaling regime up to 16 days)	45 years	4 years
α	0.5	0.5	0.45	0.59	0.50
C_1	0.6	0.20	0.6	0.32	0.47
Reference		Hubert et al. (1993)	Ladoy et al. (1993)	Hubert et al. (1993)	Desurosne et al. (1995)

Table 2. *The estimated universal multifractal parameters for each group of satellite pictures studied in Tessier et al. (1993). The accuracy on the values of the parameters is about ± 0.1.*

Satellite	Sensor	Wavelength	Scaling range	α	C_1	H
NOAA 9	AVHRR channel 1	0.5 to 0.7 μm	1 to 512 km	1.13	0.09	0.4
NOAA 9	AVHRR channel 2	0.7 to 1.0 μm	1 to 512 km	1.10	0.09	0.4
NOAA 9	AVHRR channel 3	3.6 to 3.9 μm	1 to 512 km	1.11	0.07	0.3
NOAA 9	AVHRR channel 4	10.4 to 11.1 μm	1 to 512 km	1.35	0.10	0.5
NOAA 9	AVHRR channel 5	11.5 to 12.2 μm	1 to 512 km	1.35	0.10	0.5
METEOSAT	VIS	0.4 to 1.1 μm	8 km to 4000 km	1.35	0.10	0.3
METEOSAT	IR	10.5 to 12.5 μm	8 km to 4000 km	1.21	0.09	0.4
LANDSAT	MSS	0.49 to 0.61 μm	166 m to 83 km	1.23	0.07	0.4

3. BEYOND SCALAR MODELING

3.1 Motivations

Until now the multifractal modeling of rain has relied on the simplifying hypothesis that the interaction between rain and the dynamics can be reduced to a scalar relationship (namely between their respective fluxes). This is fundamentally the reason why until now, multifractal results have always been expressed in terms of scalar fields. Theoretically, however, even in the simplest case of passive advection this relation is vectorial (the velocity field coupled with the concentration field via the gradient of the latter[4]). This situation is in a way paradoxical: classical methods, such as those used in GCM modeling, deal easily with this vectorial interaction but on a very limited range of scales, whereas scaling models deal easily with an infinite range of scales but avoid treating this vectorial interaction.

[4] And not by a scalar relationship between their respective fluxes, as simplified in multifractal scalar cloud modeling (Wilson et al., 1991; Pecknold et al., 1993).

Below, we develop a rather general framework of 'Lie cascades' in order to analyze and generate multiplicative processes for vectorial and tensorial fields. More generally we study the rather abstract fields admitting a Lie group of symmetries. This framework opens a scaling *and* vectorial alternative to GCM techniques, since then we may consider the generator of the joint field $(v, R, I, \ldots)$, (= velocity, rain rate, radiance, etc.) which generates not only each component field, but also their (vectorial, tensorial, etc.) interrelations.

Are the scalar cascade processes in fact restricted to positive scalar fields? If such was the case, then their relevance to turbulence could be quite questionable. Indeed, the energy flux density ε_λ (from larger to smaller scales L/l, $\lambda = [1, \infty]$) in turbulence is not always positive. Using analytical closure schemes (Lesieur & Schertzer, 1976), or the Renormalization Group approach (Forster et al., 1977; Herring et al., 1982), the essential backwards contribution of the flux has been shown to result in a 'beating term' or 'renormalized forces' due to nonlocal interactions. Turbu-

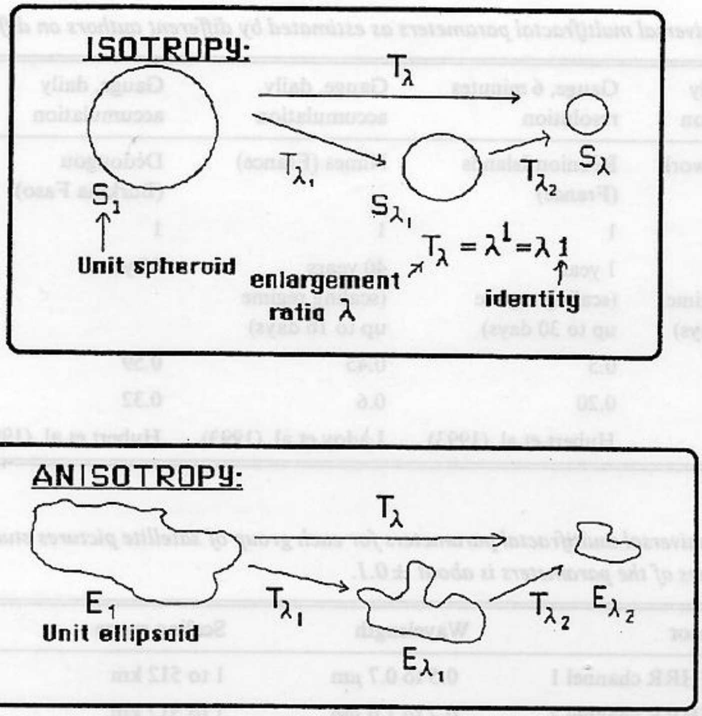


Fig. 14 An illustration of a self-invariant operator giving isotropy as compared to a generalized scale-invariant operator giving anisotropy.

lence clearly cannot be reduced to an eddy-viscosity and a one-way downward energy flux, the balance is somewhat more subtle.

Another fundamental reason to look for extensions of cascade processes is that turbulence is fundamentally vectorial as recalled by Kraichnan (1958). Even more generally mechanics is at least tensorial! This is clearly seen by considering the fundamental continua mechanics equation relating the acceleration $\boldsymbol{a}$ to the stress-tensor $\boldsymbol{\sigma}$ and the field of forces $\boldsymbol{F}$:

$$\rho \boldsymbol{a} = \mathrm{div}(\boldsymbol{\sigma}) + \boldsymbol{F} \qquad (15)$$

The relationship between the strain tensor $\boldsymbol{D}$ and $\boldsymbol{\sigma}$ generally involves the fourth order tensor of elasticity. And if we want to investigate the similarities and differences between (fluid) turbulence and seismicity understood as the 'turbulence of solids' (Kagan, 1992), tensorial cascades will be indispensable.

Before solving the problem, let us point out the difficulty. The main problem with a real cascade with alternating signs is that the set of real numbers is not algebraically closed, i.e. it doesn't satisfy d'Alembert's theorem; in particular positive numbers have 2 square roots, negatives none. A rule related to this is the sign of products: products of the same sign give positive numbers, while products of two opposite signs give negative results. As a consequence there are obviously some nontrivial problems in renormalizing a discrete real cascade by a factor 2 and conversely to introducing intermediate scales. More fundamentally, and especially when one considers a continuous process, a series of multiplications corresponds to an exponentiation of a sum, unfortunately an exponential of any real number is a positive number!

3.2 Complexification of a cascade as an example

By considering the algebraic closure of the real numbers (i.e. the set $\boldsymbol{C}$ of the complex numbers) we should already be able to solve the above mentioned difficulties. For instance, the image of $\boldsymbol{C}$ under exponentiation is $\boldsymbol{C}$ itself. On the other hand, complex multiplication (with $v(=x_1+ix_2)$) corresponds to a particular linear transformation on $\mathcal{R}^2$, i.e. the conformal transformation which is a particular subgroup of $L(\mathcal{R}^2, \mathcal{R}^2)$ (the set of all linear transformations on $\mathcal{R}^2$) which can be identified with the product of rotation (angle θ) and dilation (ratio r) ($v = re^{i\theta}$).

$$v_\lambda = \exp(\Gamma_\lambda)v_1 \quad \textit{where} \quad v_\lambda, v_1 \in \boldsymbol{C}; \Gamma_\lambda \in \boldsymbol{C} \qquad (16)$$

The significance of $\Gamma_{R,\lambda} = \mathrm{Re}(\Gamma_\lambda)$ and $\Gamma_{I,\lambda} = \mathrm{Im}(\Gamma_\lambda)$ is obvious: $\Gamma_{R,\lambda}$ generates a nonnegative cascade process which modulates the amplitude of the modulus of v_λ, whereas $\Gamma_{I,\lambda}$

gives the rotation of v_λ, hence the sign of its real part. We may focus ourselves on the special case where $\Gamma_{R,\lambda}$ and $i\Gamma_{I,\lambda}$ are independent stochastic processes with corresponding characteristic functions $K_R(q)$, $K_I(q)$:

$$\langle v_\lambda^q \rangle = \langle \exp(q\Gamma_{R,\lambda}) \rangle \langle \exp(iq\Gamma_{I,\lambda}) \rangle \langle v_1^q \rangle = \lambda^{K_R(q)} \lambda^{K_I(q)} = \lambda^{K(q)} \quad (17)$$

The characteristic function $K(q)$ of the complex process is therefore simply:

$$K(q) = K_R(q) + K_I(q) \quad (18)$$

It is important to note that whereas $K_R(q)$ is real for any real q, $K_I(q)$ is complex, being in general neither real nor pure imaginary. The condition of conservation ($\langle v_\lambda \rangle = 1$) still corresponds to $K(1)=0$, but not to $K_I(1)=0$, i.e. Γ_R generates a nonconservative process for the vector modulus. Let us consider as an example (Brethenoux et al., 1992) and as an illustration the discrete lognormal case (Gaussian generator; λ_0 being the fixed step scale ratio). The real and imaginary exponential increments $\Gamma_{R,\lambda 0}$ and $\Gamma_{I,\lambda 0}$ respectively will be a Gaussian variable of variance σ_R^2 and mean m_R (resp. σ_I^2 and m_I) which lead to a generalization of the scalar universal scaling function (eq. (13) with $\alpha=2$):

$$\begin{aligned} &K_R(q) = C_{1,R}(q^2-q) - H_R q; \quad C_{1,R} = \sigma_R^2/2; \quad H_R = C_{1,R} - m_R \\ &K_I(q) = -C_{1,I}(q^2-q) - H_I q; \quad C_{1,I} = \sigma_I^2/2; \quad H_I = C_{1,I} - im_I \\ &K(q) = C_1(q^2-q) - H_q; \quad C_1 = C_{1,R} - C_{1,I}; \quad H = H_R + H_I \end{aligned} \quad (19)$$

A conservative field is obtained with $m_R = -C_1$ (i.e. $\neq -C_{1,R}$, as required to obtain a conservative cascade of modulus), $m_I = 0$. Fig. 15(a)–(e) gives the first steps of the corresponding complex cascade.

One may note that $K(q)$ remains of the standard universal form even for complex q. Similar properties hold for Lévy processes when $\Gamma_{R,\lambda}$ and $\Gamma_{I,\lambda}$ are independently identically distributed. However, $\Gamma_{R,\lambda}$ and $\Gamma_{I,\lambda}$ do not necessarily need to have the same α and there is no longer the requirement that $\Gamma_{I,\lambda}$ should correspond to an extremal Lévy process, since $K_I(q)$ for real q is the Fourier characteristic function of $\Gamma_{I,\lambda}$ whereas $K_R(q)$ remains the Laplace characteristic function of $\Gamma_{R,\lambda}$, and admits the usual scalar universal form (eq. (13), with respectively H_R, $C_{1,R}$, α_R instead of H, C_1, α). The rather more general universal form[5] of $K_I(q)$ is defined for all q (the $\pm$ is the sign of q). Note that β is the asymmetry parameter of the Lévy process $\Gamma_{I,\lambda}$ and $\beta=-1$ for an extremal Lévy process such as $\Gamma_{R,\lambda}$:

$$K_I(q) + H_I q = -\frac{C_{\pm 1,I}}{\alpha - 1}(|q|^{\alpha'} - a) \quad (\alpha \neq 1)$$

$$K_I(q) + H_I q = -C_{\pm 1,I} q |Log q| \quad (\alpha = 1) \quad (20)$$

$$C_{\pm 1,I} = C_I \{\cos(\pi\alpha/2) \pm i\beta \sin(\pi\alpha/2)\} \Gamma(3-\alpha_I)\alpha_I \quad (\pm = \mathrm{sgn}(q))$$

with C_I being the coefficient of the canonical Lévy measure dF (cf. eq. (14)) defining $\Gamma_{I,\lambda}$. Fig. 16 displays the complex scaling analysis for a visible and infrared satellite image pair ($v = I_V + iI_R$, I_V and I_R being the visible and infrared radiances respectively).

3.3 Vectorial processes

In the previous subsection, we extended scalar cascades to two component vector cascades by complexifying the cascade. More generally, we may consider nonpositive cascades as being components of more or less straightforward vectorial extensions of positive real processes:

$$v_\lambda = \exp(\Gamma_\lambda) v_1; \quad v_\lambda, v_1 \in \mathcal{R}^d; \Gamma_\lambda \in L(\mathcal{R}^d, \mathcal{R}^d) \quad (21)$$

the vs being vectorial fields from $\mathcal{R}^d$ to $\mathcal{R}^d$, v_1 being a homogeneous vectorial field (e.g. in the strictest sense: $\forall y \in \mathcal{R}^d v_1(x+y) = v_1(x)$). Just as in the positive scalar case, in order to obtain multiple scaling of the moments Γ_λ should be some band limited $1/f$ noise although now we have a tensor scaling function $K(q)$:

$$\forall \lambda > 1: \langle \exp(q\Gamma_\lambda) \rangle \approx \lambda^{K(q)}; \quad K(q) \in L(\mathcal{R}^d, \mathcal{R}^d) \quad (22)$$

Introducing furthermore the vectorial singularities γ and their codimensions $c(\gamma)$:

$$\forall \gamma \in \mathcal{R}^d, S_\lambda(\gamma) = \{v \in \mathcal{R}^d, v_i \geq \lambda^{\gamma_i}\} \quad Pr(v_\lambda \in S_\lambda(\gamma)) \approx \lambda^{-c(\gamma)} \quad (23)$$

For conservative processes, we still have the same type of conservation law[6]:

$$\langle v_\lambda \rangle = \langle v_1 \rangle; \quad i.e. \quad K(1) = 0 \quad (24)$$

3.4 Lie groups and their Lie algebra of generators

In fact, *independently* of the representation of the v field and of the B_λ balls (as discussed above), we are only using the (multiplicative) group properties related to the basic fact that scale ratios simply multiply. Using $l = l_1 l_2$ we obtain:

$$T_\lambda = T_{\lambda_1} T_{\lambda_2}; \quad \tau_\lambda = \tau_{\lambda_1} \tau_{\lambda_2} \quad (25)$$

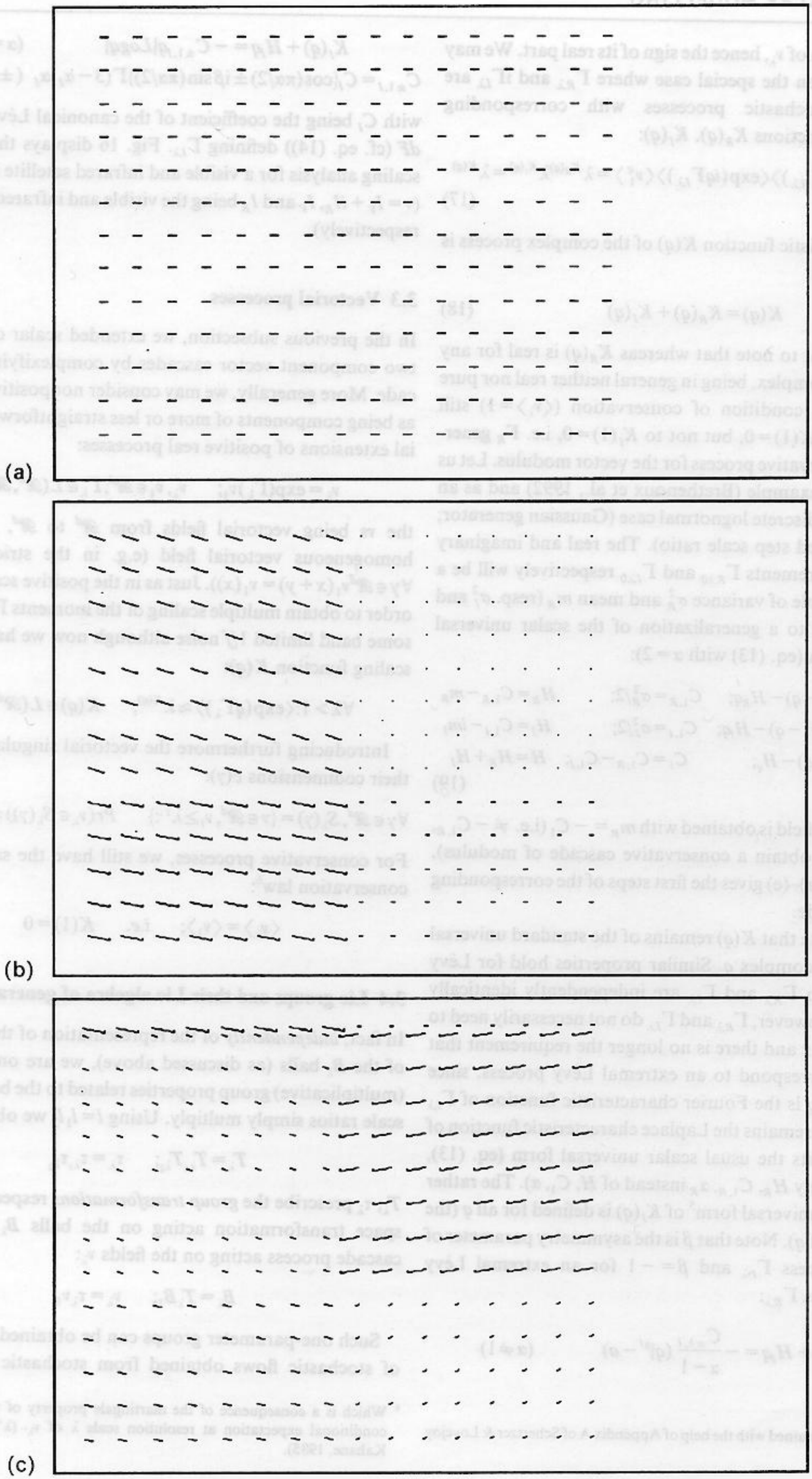
T_λ, τ_λ prescribe the *group transformations* respectively of the space transformation acting on the balls B_λ and of the cascade process acting on the fields v_λ:

$$B_\lambda = T_\lambda B_1; \quad v_\lambda = \tau_\lambda v_1 \quad (26)$$

Such one-parameter groups can be obtained as the result of stochastic flows obtained from stochastic integrations

[5] Which can be obtained with the help of Appendix A of Schertzer & Lovejoy (1991).

[6] Which is a consequence of the martingale property of the process (the conditional expectation at resolution scale λ of $v_{\lambda'}$ ($\lambda' > \lambda$) is v_λ) (see Kahane, 1985).


(a)
(b)
(c)

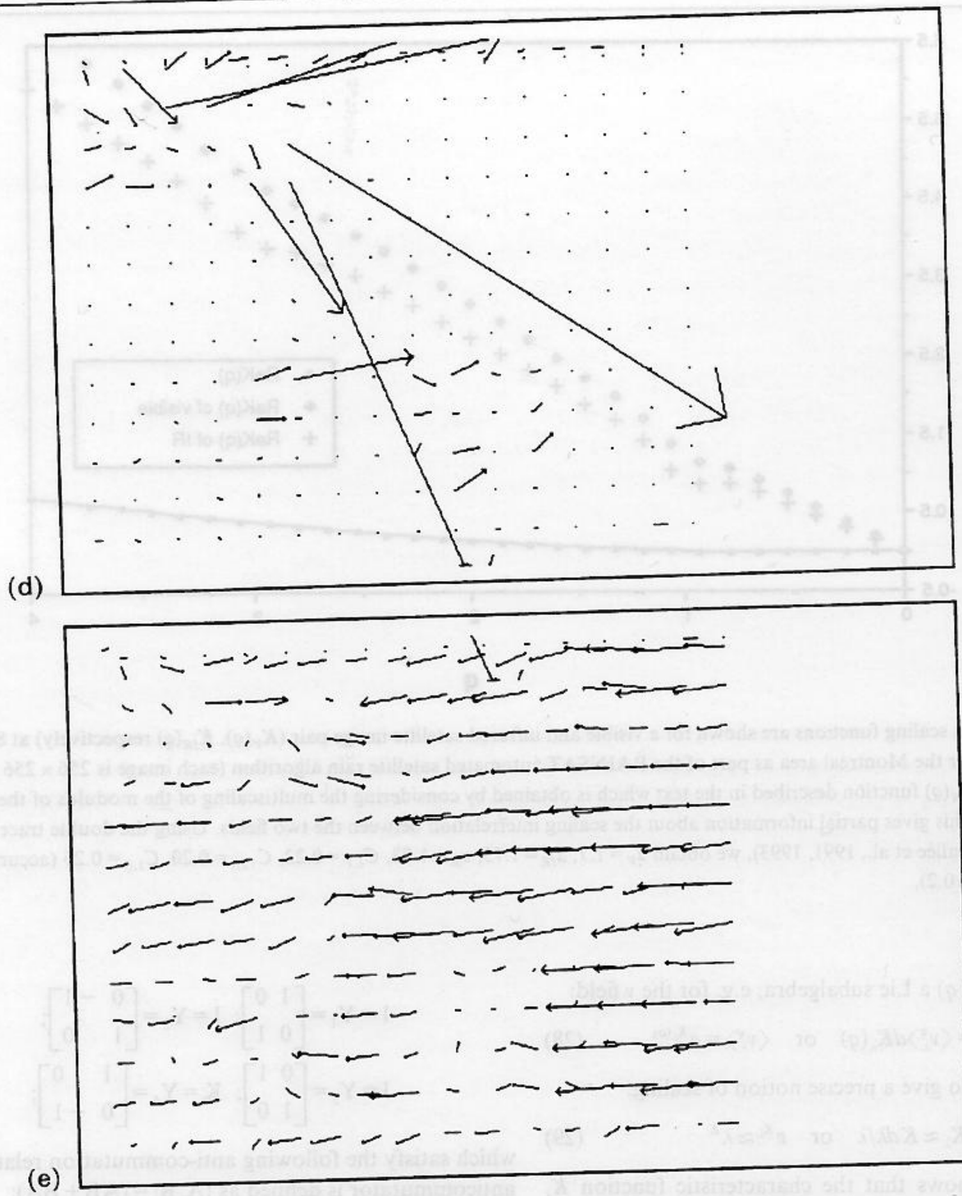


Fig. 15 An example of a complex cascade with a Gaussian generator ($\alpha = 2$). (a) Initial homogeneous complex field, (b) the bare field obtained at the first step, (c) the corresponding complex singularities, (d) the bare field obtained at the third step, (e) the corresponding complex singularities.

(more precisely from Stratatovich integrations as discussed[7] in Schertzer & Lovejoy 1995) over infinitesimal (random) generators $d\Gamma$ and dG:

$$d\varepsilon_{\lambda} = \varepsilon_{\lambda} d\Gamma_{\lambda}; \qquad dT_{\lambda} = T_{\lambda} dG_{\lambda} \tag{27}$$

Originally (Schertzer & Lovejoy, 1991) such an integration was proposed only on T_{λ} in the case of the so-called nonlinear (random) GSI (generalized scale invariance). The solutions of the above equations will be denoted as $e^{\Gamma_{\lambda}}$ and $e^{G_{\lambda}}$ respectively.

Corresponding to the group property of the transformation of the field or of the space, there is a Lie algebra structure for the generators, i.e. there is a skew and distributive product [.], called the Lie bracket. The group properties of the statistical moments of the field or scale transformations correspond to the fact that the second characteristic function (cumulant generating function) $K_{\lambda}(q)$ generates (for the

[7] Contrary to the most popular stochastic integration, i.e the Ito integration, the Stratatovich integration corresponds to a centred integration (e.g. Kunita, 1990).

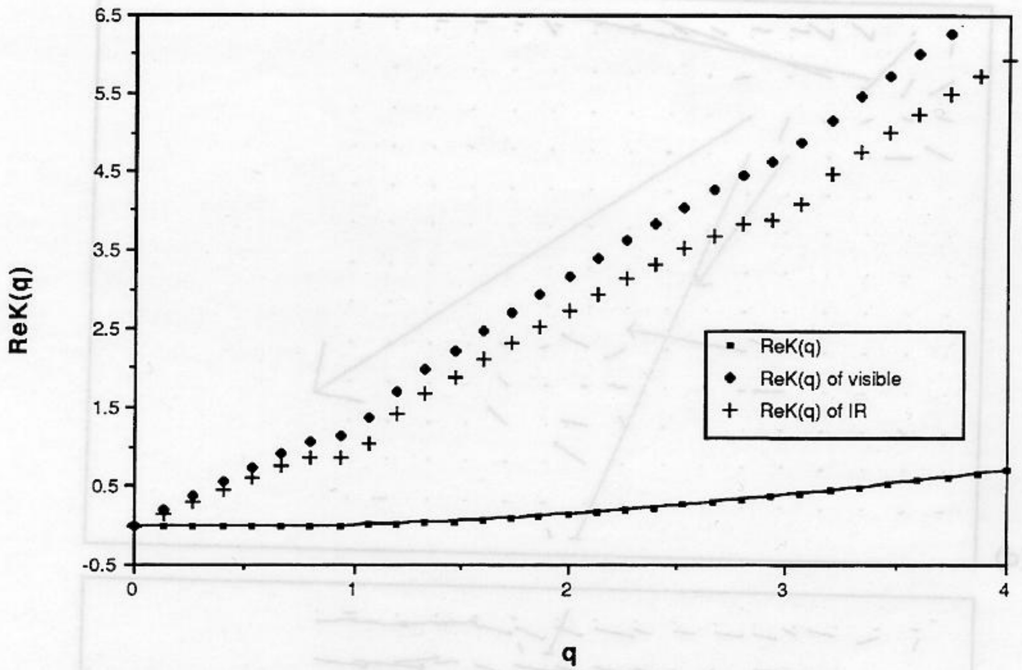


Fig. 16 The moment scaling functions are shown for a visible and infrared satellite image pair ($K_V(q)$, $K_{IR}(q)$ respectively) at 8 km resolution taken over the Montreal area as part of the RAINSAT automated satellite rain algorithm (each image is 256 × 256 pixels). Also shown is the $K_R(q)$ function described in the text which is obtained by considering the multiscaling of the modulus of the vector (visible, infrared). This gives partial information about the scaling interrelation between the two fields. Using the double trace moment (Lavallée, 1991; Lavallée et al., 1991, 1993), we obtain $\alpha_V = 1.7$, $\alpha_{IR} = 1.75$, $\alpha_R = 1.73$, $C_{1,V} = 0.22$, $C_{1,IR} = 0.20$, $C_{1,R} = 0.25$ (accuracy of the estimates is about ±0.2).

different values of q) a Lie subalgebra, e.g. for the v field:

$$d\langle v_\lambda^q\rangle = \langle v_\lambda^q\rangle dK_\lambda(q) \quad \text{or} \quad \langle v_\lambda^q\rangle = e^{K_\lambda(q)} \tag{28}$$

We are now able to give a precise notion of scaling:

$$dK_\lambda \approx K d\lambda/\lambda \quad \text{or} \quad e^{K_\lambda} \approx \lambda^K \tag{29}$$

This already shows that the characteristic function K_λ should be log divergent in the scale ratio λ just as in the framework of the scalar cascade. It can further be shown (Schertzer & Lovejoy, 1993) that the log divergence of K_λ still corresponds to the fact that the generator is a 'pink noise', i.e. having a (generalized) spectrum being exactly inverse to the wave number.

3.5 A quaternion-like representation of $L(\mathcal{R}^2, \mathcal{R}^2)$ as a preliminary example

Considering the linear transformations of the plane, we need not restrict our attention to conformal ones, which correspond to complex multiplications. Indeed, one basis $\{Y_i\}$ of the matrix representation of $L(\mathcal{R}^2, \mathcal{R}^2)$ is given by the 4 following matrices:

$$\begin{aligned} 1 = Y_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad I = Y_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \\ J = Y_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad K = Y_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \end{aligned} \tag{30}$$

which satisfy the following anti-commutation relations (the anticommutator is defined as $\{A, B\} = (AB + BA)$:

$$\{Y_2, Y_3\} = 0, \quad \{Y_2, Y_4\} = 0, \quad \{Y_3, Y_4\} = 0 \tag{31}$$

Compared to the complexification example (Section 3.2) this has a richer structure due to the addition of the 'parity' operator J and the 'conjugation' operator K. With real and independent pink noises Γ^i, $\Gamma = \Gamma^i Y_i$ is a scaling tensor field (satisfying eq. 29). Its characteristic function $K_\Gamma(q)$ will be simply the sum of the characteristic functions $K_i(q)$ of the Γ^i. Below we see that when quaternions and Clifford algebra are considered an even richer structure is obtained.

3.6 Classification and factorization of the Lie processes

Bearing in mind the very general Lie framework, we now can study a system which is invariant under different symmetries, not only scaling symmetries. We are naturally led to look for a kind of classification of the possible algebra. The answer is rather classical and half-positive: any Lie algebra can be decomposed[8] into a (semi-direct[9]) sum into a 'semi-simple algebra' σ and its radical R. Whereas there exists a universal Cartan classification (e.g. Sattinger & Weaver, 1985) of the (real or complex) semi-simple algebra (e.g. the celebrated $so(n)$...), no such result exists for radicals, which not only are totally different from semi-simple algebras, but correspond, as pointed out below, to scale symmetries. It means we are entering a particular world of symmetries which has not yet been explored. Indeed, whereas a *semi-simple* algebra is one having no abelian ideals[10] (other than 0), the radical R contains abelian ideals being defined as the maximal solvable ideal[11]. A trivial but fundamental example of an abelian ideal is the ideal generated by the identity $\{\lambda 1 | \lambda \in \mathcal{R} \text{ or } \in C\}$, hence the trivial scaling, it corresponds to the stronger property of nilpotency[12].

On the one hand, the solvability of the radical prevents it respecting a simple universal classification, on the other hand, it yields a simple generalization of factorization. Indeed, for a generator defined on an abelian ideal s, then the field merely factorizes as a product of fields generated by a element of the basis X_i ($\Gamma = \Gamma^i X_i$) of s ($[X_1, X_2, \ldots, X_n]$ spanning s):

$$\varepsilon_{\lambda} = \prod_{i=1}^{n} \lambda^{\Gamma^i X_i} \tag{32}$$

where the Γ_i are pink noises. Such a factorization can be extended to the whole radical of the algebra thanks to the Lie theorem on solvable algebra and a Yamato-Kunita theorem (Kunita, 1990). Indeed, the factorization still holds in the following sense: $\{X_1, X_2, \ldots, X_i\}$ which spans the increasing ideals $g_i([R, g_{i-1}] = g_i$; $R = g_n \supset g_{n-1} \supset \ldots \supset g_1 \supset 0$, $\dim(g_i) = i$.), then the Γ^i are replaced in eq. (32) by N^i which are sums, or products, or integrals, or exponentials of the Γ^j. In the case of nilpotency[13], which may be relevant for scaling, exponentials do not intervene, hence we are back to a rather simple factorization.

3.7 Quaternions and Clifford algebra as examples

Quaternions (for the dimension 4) and Clifford algebra C_n (for the dimensions $2^n, n > 2$) are the real linear Lie algebra (a subalgebra of $L(C^n, C^n)$) defined by the following anti-commutators relations corresponding to a (Pauli) factorization of the Laplacian[14]:

$$\{\alpha^{\mu}, \alpha^{\nu}\} = 2\delta^{\mu\nu} \tag{33}$$

This algebra is generated by $1, \alpha_1, \ldots, \alpha_n$ and all their products $\alpha_i.\alpha_j \ldots \alpha_l$ (which can be always be ordered, since $\alpha_i.\alpha_j = -\alpha_j.\alpha_i, \alpha_i^2 = 1$), thus contains 2^n elements. For the quaternions:

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \alpha_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{34}$$

and they can be expressed with the help of the Pauli matrices σ_i:

$$\sigma_1 = \alpha_1; \quad \sigma_2 = -\alpha_2; \quad \sigma_3 = -\alpha_1.\alpha_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{35}$$

For $n = 4$, we have:

$$\alpha_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}; \alpha_2 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}; \alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}; \alpha_4 = \begin{bmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{bmatrix} \tag{36}$$

More generally $n = 2m$ or $2m + 1$, and for any $j \le m$; the α_j can be expressed on $(C^2)^{\otimes n}$ (the spinors space) and for $n = 2m + 1$; $\alpha_{2m+1} = \sigma_3^{\otimes 2m+1}$. Not only do these well-known examples allow us to generalize rather straightforwardly the result of complex cascades to $n > 1$ (simulations as well as data analysis, e.g. multichannel radiance field will be developed in subsequent works) but also, as mentioned earlier they correspond in a given representation to well-defined equations. This shows that the Lie algebra of the generators of symmetries might well be the indispensable tool necessary to bridge the gap conclusively between stochastic models and deterministic equations.

4. CONCLUSIONS

In Schertzer & Lovejoy (1987), it was proposed that in many geophysical applications scaling symmetries can be used as dynamical constraints instead of coupled nonlinear partial differential equations. This is the usual physics notion that a system is totally determined once all its symmetries are known. Since scaling arguments are so general it has become urgent to develop a formalism for handling scaling for coupled multifractal processes (e.g. vector cascades) as well as for restricting the generator of the scaling symmetries using additional symmetries. Both goals require the formalism of the Lie cascades developed here.

[8] Via a Levi decomposition.

[9] The sum of of two subalgebras a and b is direct when the two commute ($[a, b] = 0$), it is semi-direct when $a \supset [a, b]$.

[10] A subalgebra s is an ideal of g if $s \supset [s, g]$ (i.e. not only $[s, s]$).

[11] I.e. the largest ideal leading by a nested sequence of ideals to an abelian ideal.

[12] The limit of a nested sequence of ideals is not only an abelian ideal, but it commutes with the whole algebra.

[13] Nilpotency is a slightly stronger property than solvability.

[14] $(\alpha^{\mu}\partial_{\mu})^2 = \partial_{\mu}^2$.

We have not only shown there is no fundamental reason to restrict cascade processes to positive (real) quantities, but there are also very wide possible generalizations to rather abstract processes. As a consequence, a potentially wider unity of geophysics is restated in a new way. At the same time a quantitative way to classify the wide diversity of phenomena which occur on a large range of scales is pointed out with the help of the classification of the corresponding Lie algebra. This classification will also enable us to discover new types of nonlinear interactions.

Immediate applications of the ideas discussed here include the simulation of vector multifractals and the scale invariant characterization of the interrelations of rain, cloud radiance and other fields. We gave a first example of the latter by analyzing the multiscaling of the vector moments of the joint visible and infrared cloud radiance fields from GOES satellite data. When this is extended to radar reflectivities of rain and rain gauge measurements, the resolution independent characterization of their interrelation can form the basis of resolution independent satellite rain algorithms, as well as for the resolution independent calibration of radars from rain gauges.

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