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Quasi-geostrophic turbulence and generalized scale invariance, a theoretical reply

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Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



Abstract

Lindborg et al. (2010) claim that the spectrum power law $E(k) \approx k^{-3}$ on scales ≥ 600 km obtained with the help of commercial jetliner trajectory deviations (GASP and Mozaic databases) could not be brought into question (Lovejoy et al., 2009a), because this spectrum corresponds to “a well known theory of quasi-geostrophic turbulence developed by Charney (1971)”. Lindborg et al. (2010) also claim that “limitations [of this theory] have been relaxed in many of the modern models of atmospheric turbulence”. We show that both claims are irrelevant and that generalized scale invariance (GSI) is indispensable to go beyond the quasi-geostrophic limitations, to go in fact from scale analysis to scaling analysis. This enables us to derive vorticity equations in a space of (fractional) dimension $D = 2 + H_z$ ($0 \leq H_z \leq 1$), which seem to be an interesting dynamical alternative to the quasi-geostrophic approximation and turbulence.

1 Introduction

The scale analysis methodology developed by Charney (1948) in his seminal derivation of the quasi-geostrophic (QG) approximation has become standard in meteorology and oceanography (Pedlosky, 1979). Unfortunately, its common and distinguishing features with respect to a scaling analysis have not yet been fully recognized. It is ironical that the source of the present debate on medium and large scale atmospheric dynamics presumably corresponds to the importation into meteorology of two successive techniques from hydrodynamics at two different periods. In order to focus the present paper on this question and therefore on the limitations of the theory of quasi-geostrophic turbulence (QGT, Charney, 1971) and show how to overcome them, let us first reject the second claim of Lindborg et al. (2010), LTNCG hereafter, that models may overcome limitations of a theory because in a very general manner models are obtained by introducing further constraints into a given theoretical framework, e.g. boundary conditions, discretization of partial differential equations, subgrid

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



modeling and other parametrizations. Furthermore, these constraints in the unique quasi-geostrophic model simulation (Tung and Orlando, 2003) cited by LTNCG were such a problem that, as discussed below, they seem to have introduced spurious numerical estimates of the scaling ranges, instead of relaxing limitations of the theory.

Before addressing QGT, we will first discuss a few fundamental features of the derivation of the QG approximation itself, as well as its motivation. We will gradually show that to better understand the fundamentals of atmospheric dynamics a scaling analysis is required instead of a scale analysis, which was the basic technique used by Charney in his derivation. This enables us to obtain straightforwardly a vorticity equation for a turbulence that looks two-dimensional at large scales, three-dimensional at small scales, but is in fact of (fractional) dimension $D = 2 + H_z$ ($0 \leq H_z \leq 1$) at all scales. We believe that this equation may satisfy those who asked for a dynamical alternative to QGT.

2 QG vorticity equation and scale analysis

Before discussing the derivation of the QG approximation, let us highlight why this approximation has remained so much attractive and therefore popular. Historically, it was the first mathematically self-consistent derivation of a closed dynamical system from the diagnostic geostrophic and hydrostatic approximations. Furthermore, it respects the conservation laws of potential temperature and absolute potential vorticity, at least their corresponding first order expressions within the QG approximation. However, one cannot forget that the derivation by Charney (1948) of this approximation was based on a scale analysis, borrowed from aerodynamic boundary layer theory (Charney refers to Goldstein, 1938) to *intentionally* filter out the “meteorologically insignificant wave components” from the “meteorologically significant motions”. The latter were considered to be distinguishable from all other types of atmospheric motion only by a great difference in scales. This question of filtering was essential for the sake of the pioneering development of numerical weather forecasting (Charney et al., 1950) and a part of the

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



paper by Charney (1948) was indeed devoted to checking that this filtering was effective for acoustic waves. Nevertheless, the more general question of which structures were filtered out was not addressed. Another important requirement put forward by Charney was also related to numerical forecasts: the approximation should be (easily) computable. It is worthwhile noting that neither issue is related to the question of obtaining a rather complete description of the atmospheric dynamics.

The scale analysis performed by Charney was based on the common meteorological practice idea that large scale dynamics were quasi-two-dimensional, quasi-geostrophic and quasi-hydrostatic, due to the fact that the rotation $\mathbf{\Omega}$ and the gravitational field \mathbf{g} of the Earth introduce preferential directions in the Navier-Stokes equations:

$$D\mathbf{u}/Dt + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p/\rho + \mathbf{g} + \nu\Delta\mathbf{u} \quad (1)$$

where \mathbf{u} is the velocity, p the pressure, ρ the density, ν the viscosity, D/Dt denotes the material derivative, i.e. the time derivative following the motion ($\partial/\partial t + \mathbf{u} \cdot \nabla$), ∇ denotes the 3-D gradient operator, $\Delta = \nabla^2$ the 3-D Laplace operator. The velocity field is assumed to be solenoidal ($div(\mathbf{u}) = 0$), the mass conservation equation therefore reads:

$$D\rho/Dt = 0 \quad (2)$$

Although there is no fundamental difficulty in dealing with these equations in spherical coordinates, or more generally in local manifold coordinates, it is usual, and in agreement with other approximations done to derive the QG approximation, to approximate Eq. (1) on the tangent space to the Earth at a given point of longitude ϕ_0 and latitude θ_0 , with a vertical unit vector \mathbf{n} :

$$D\mathbf{u}/Dt + f\mathbf{n} \times \mathbf{u} = -\nabla p/\rho + \mathbf{g} + \nu\Delta\mathbf{u} \quad (3)$$

where only the vertical component $f\mathbf{n}$ of the Earth's angular velocity is taken into account, f being the Coriolis parameter at the latitude θ_0 . Furthermore, the variation of

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures



Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



f with respect to the south-north coordinate $y = a(\theta - \theta_0)$ (a being the Earth radius), is usually linearized:

$$f(y) = f_0 + \beta y; (f_0, \beta) = 2\Omega(\sin\theta_0, \cos\theta_0) \quad (4)$$

The scales of interest being much larger than those of the dissipation range, the corresponding term will be omitted in the following (without excluding an effective dissipation at small scales), i.e. the viscosity is considered as infinitely small or the Reynolds number $Re = UL/\nu = ZL^2/\nu$ and the Ekman number $E = gL/\Omega$ as infinitely large, where Z denotes the (relative) vorticity scale of the atmosphere, L the outer scale of the horizontal fluctuations, U the corresponding scale of the (horizontal) velocity. For infinitely small Rossby numbers $Ro = Z/\Omega$, it has been argued that the Navier-Stokes equations reduce to the (diagnostic) geostrophic balance equation with the corresponding 2-D geostrophic solution:

$$2\Omega \times \mathbf{u}_g = -\nabla p/\rho + \mathbf{g} \Rightarrow \mathbf{u}_g = \Omega \times (\mathbf{g} - \nabla p/\rho)/(2\Omega^2) \quad (5)$$

The latter can furthermore be approximated with the help of Eq. (3) and of the stream function ψ as:

$$\mathbf{u}_g = \mathbf{n} \times \nabla \psi = \nabla \times \mathbf{n} \psi; \psi = (\rho/\rho_0 - gz)/f_0 \quad (6)$$

the sub-index 0 of the (barotropic) hydrostatic density ρ_0 , as for other variables, refers to a reference or background vertical profile, which can be understood as a time average of instantaneous profiles. The vertical variation of the stream function corresponds to :

$$\partial \psi / \partial z = \partial(\rho - \rho_0) / \rho_0 f_0 \partial z = -\partial \rho / \rho_0 f_0 \partial z \quad (7)$$

The QG approximation corresponds to introducing a linear three-dimensional ageostrophic perturbation (\mathbf{u}_{ag}, w) of order Ro to the zero-order two-dimensional geostrophic solution $(\mathbf{u}_g, 0)$ and similarly for the relative vorticity ζ

$$\mathbf{u} = (\mathbf{u}_h, w) \approx (\mathbf{u}_g, 0) + (\mathbf{u}_{ag}, w); \zeta = (0, \zeta_g) + \zeta_{ag} \quad (8)$$

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



The material derivative is systematically approximated by its geostrophic approximation (∇_h denotes the horizontal gradient):

$$D_g/Dt = \partial/\partial t + \mathbf{u}_g \cdot \nabla_h \quad (9)$$

which, as emphasized by Charney, is much more manageable than the horizontal material derivative ($D_h/Dt = \partial/\partial t + \mathbf{u}_h \cdot \nabla_h$). In particular, D_g/Dt is easily defined with the help of the stream function ψ (J denotes the 2-D horizontal Jacobian):

$$D_g/Dt = \partial/\partial t + J(\psi, \cdot) \quad (10)$$

To derive the QG approximation, it is convenient to start from the vorticity equation (the curl of Eq. (1), $\boldsymbol{\omega} = 2\boldsymbol{\Omega} + \boldsymbol{\zeta}$ denotes the total vorticity):

$$D\boldsymbol{\omega}/Dt = \mathbf{s} + \mathbf{b}; \quad \mathbf{s} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}; \quad \mathbf{b} = \nabla \rho \times \nabla(\rho)/\rho^2 \quad (11)$$

Whereas the baroclinic vector \mathbf{b} is of second order in a quasi-barotropic flow, the stretching vector \mathbf{s} is of first order and non-zero contrary to strictly 2-D fluid motions (including geostrophic motions), where the (vertical) vorticity and the (horizontal) velocity gradient are orthogonal. For QG, the leading term is obtained as the product of the (small) vertical gradient of the velocity by the (large vertical) vorticity component:

$$(\boldsymbol{\zeta} + f\mathbf{n}) \cdot \nabla \mathbf{u} \approx f \partial w / \partial z \approx f_0 \partial w / \partial z \quad (12)$$

The series of approximations involved in Eq. (12) preserves a non-zero term, as in 3-D turbulence and contrary to 2-D turbulence. Nevertheless, this term is no longer a source of nonlinear growth of vorticity. More precisely, the QG vorticity equation corresponds to :

$$D_g(\zeta_g + \beta y)/Dt = f_0 \partial w / \partial z \quad (13)$$

There is a new and important step: the stretching term corresponds in the QG framework to the advection of the quantity q called the pseudo potential vorticity (or QG

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



potential vorticity) in reference to Ertel's theorem. Indeed, it corresponds to its conservation law:

$$D_g q / Dt = 0; q = \zeta + \beta y - g f_0 \partial(\rho / N^2 \rho_0) / \partial z \quad (14)$$

As recalled in the next section, this conservation law is fundamental for the QG dynamics. This law is obtained by eliminating the vertical velocity w from the vorticity equation (Eq. 13) with the help of the the mass conservation approximated by:

$$D_g \rho / Dt - w N^2 \rho_0 / g = 0; N^2 = g d \text{Log}(\theta_0) / dz \quad (15)$$

where N is the (mean) Brunt-Vaissala frequency defined with the help of the vertical profile of the background potential temperature θ_0 , which is assumed to be smooth see (Pedlosky, 1979) for further discussion).

One reason for the enormous success of the QG approximation is that the pseudo potential vorticity (Eq. 14) can be rewritten with only the help of the stream function ψ (due to Eq.7):

$$q = \Delta \psi + \beta y + \partial(f_0^2 \partial \psi / N^2 \partial z) / \partial z \quad (16)$$

which together with Eqs. (10), (14) yields a compact dynamical equation for the stream function:

$$(\partial / \partial t + J(\psi, .))(\Delta \psi + \beta y + \partial(f_0^2 \partial \psi / N^2 \partial z) / \partial z) \quad (17)$$

3 Scaling analysis of the QG approximation and QG turbulence

The fact that the domain of validity of the QG approximation is *a priori* restricted to large scales does not prevent the possibility of studying the scaling behaviour of its solutions on a wider scale range. However, it should not be forgotten that the results obtained could be quite different from a direct scaling analysis of the original equation. This question, which is particularly relevant for a quasi-linear approximation of a nonlinear

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



system like QG, seems to be largely underestimated by LTNCG. Furthermore, this type of approximation may hold over a given range of large scales only if the smaller scale activity does not destroy the conditions of applicability of this approximation. This requires in general a separation of scales that we will discuss below.

5 The fact that the QG vorticity equation (Eq. 13) can be transformed into a conservation law of the pseudo potential vorticity (Eq. 14), similarly to the vorticity of 2-D turbulence, enabled Charney (1971) to argue that QG turbulence (QGT) should display, like 2-D turbulence, both downward enstrophy and upward energy inertial cascades (Fjortoft, 1953; Kraichnan, 1967). More precisely, Charney claimed that there exists
10 a mathematical isomorphism between the 2-D relative vorticity and the 3-D quasi-potential vorticity, although this is not straightforward as soon as the Brunt-Vaisala frequency is variable along the vertical (Herring, 1980) because whereas QG involves a 2-D advection operator, the advected quasi-potential vorticity is 3-D. Nevertheless, this theoretical possibility has been rather confirmed with the help of analytical closures
15 (Herring, 2001, 1980; Salmon and Hendershott, 1978; Salmon, 1978) and numerical simulations (McWilliams et al., 1994; Hua and Haidvogel, 1986; Vallis, 1985; Fu and Flierl, 1978). However, Herring (2001) concluded that many questions remained open. In particular, he mentioned the empirical finding by Lindborg (1999) that the the spectrum $E(k) \approx k^{-5/3}$ thought to be an inverse 2-D energy cascade by Gage (1979) and
20 by Lilly and Paterson (1983) was rather a direct cascade.

Contrary to the simulations by Hua and Haidvogel (1986) that involved a large number of layers, those of Tung and Orlando (2003), TO in the following, were performed with only the help of a two-layer QG model (Welch and Tung, 1998). This model corresponds in fact to two 2-D-QG models weakly interacting through the vertical vector
25 potential gradient: the fundamental variables are the set of values of the velocity potential on the two layers. In any case, with only two layers, not much can be done about either the boundary layer, therefore the boundary conditions, nor the horizontal /vertical anisotropy. Therefore, LTNCG's reference to this study is already questionable. Furthermore, and as emphasized by TO, their goal was quite different: to generate a

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

[Title Page](#)[Abstract](#)[Introduction](#)[Conclusions](#)[References](#)[Tables](#)[Figures](#)[⏪](#)[⏩](#)[◀](#)[▶](#)[Back](#)[Close](#)[Full Screen / Esc](#)[Printer-friendly Version](#)[Interactive Discussion](#)

composite horizontal spectrum of the type $Ak^{-5/3} + Bk^{-3}$ with the help of a model as simple as possible, but with all the necessary elements. This model indeed gave some numerical evidence in the direction of such a composite spectrum. However, this numerical evidence was thoroughly criticized by Smith (2003) who convincingly pointed out that the estimated “meso-scale” spectrum slope $\approx 5/3$ may well be spurious due to an artificial build up of enstrophy (and therefore of energy) at the smallest explicit model scales. This is due to the fact that the hyperviscous dissipation scale is not meridionally resolved (TO used indeed a 10th power of the Laplace operator for small scale dissipation). In any case, TO did not explain which model mechanism could generate a forward energy cascade over the meso-scale, whereas there are two fundamental obstacles to overcome. The first is that the QG approximation is fundamentally irrelevant for the mesoscale range: all the necessary approximations are no longer justified (e.g. the Rossby number becomes much larger than unity). Therefore, it remains unreasonable to hope that some 3-D-like behaviour would occur over the mesoscale range in the model, whereas it might occur in nature. Secondly, the directions of cascades can be theoretically inferred from inviscid statistical equilibria of the systems (Kraichnan, 1971), which yield upward energy and downward enstrophy cascades for both 2-D and QG turbulence. Therefore, we can safely conclude – contrary to LTNCG – that there is neither theoretical argument nor model evidence in favour of two downward cascades. Let us mention that the fact that pressure coordinates are commonly used in meteorology – as advocated by LTNCG – does not prevent them from theoretically introducing biases in statistical analyses because their possible dynamical significance strongly depends on the validity of a number of approximations that we put above into question.

Furthermore, a QG model cannot give insights on the question of the separation of scales that would ensure its own physical relevance. This separation of scales between a 2-D regime and a 3-D regime, if it existed, could be easily destabilized by the vortex stretching mechanism, and we have already discussed the fact that the QG approximation greatly modifies this term in the vorticity equation. More generally, small scale 3-D turbulence cannot be understood as only dissipating large scale structures by eddy

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

[Title Page](#)[Abstract](#)[Introduction](#)[Conclusions](#)[References](#)[Tables](#)[Figures](#)[⏪](#)[⏩](#)[◀](#)[▶](#)[Back](#)[Close](#)[Full Screen / Esc](#)[Printer-friendly Version](#)[Interactive Discussion](#)

viscosity, because it also generates larger scale structures by backscattering (Lesieur and Schertzer, 1978) or renormalized forcing (Fournier and Frisch, 1983). The crucial importance of the separation of scales for numerical weather forecasts was explicitly stated and discussed by Monin (1972) and it explains why the concept of “mesoscale gap”(Van der Hoven, 1957) was so cherished during the early history of weather forecasting.

It is remarkable that Charney (1971) – as emphasized by Schertzer and Lovejoy (1985a) – readily admitted the limitations of the QG approximation and expressed the question of separation of scales in terms of temperature gradients that must remain quite moderate. The limitation of scale separation must not be reduced to a problem of boundary layer modeling as suggested by LTNCG, because this problem must be solved over all horizontal levels.

4 Direct scaling analysis and Generalized Scale Invariance

To avoid the limitations induced by the QG approximation, we must proceed to a direct scaling analysis of the Navier-Stokes equations. We first recall how it can be done, when there is no rotation and/or gravitational field, in a formal manner which is both more straightforward and more general than a spectral analysis. This corresponds to analyzing how the equations could be invariant under a space contraction/dilation, i.e. how the different fields (velocity, pressure, etc.) can balance at all scales. This is therefore quite distinct from a scale analysis, where only a few relevant fields at a given scale are preserved. This is obtained by analyzing the effect of a space contraction/dilation on each field, something that is classically done for isotropic space contraction/dilation, whereas stratified and rotating flows require a generalization to strongly anisotropic space contraction/dilations. This can be achieved in the framework of Generalized Scale Invariance (GSI, Schertzer and Lovejoy, 1985b) with space contraction/dilations that scale differently according to space directions. In the present case, we want to reconcile the facts that vorticity at large scales is dominated by Earth’s

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures



Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



rotation and that the nonlinearity of the vorticity equation generates scaling by transferring this large scale input over a wide range of scales. We are therefore led to a generalization of the classical notion of spontaneous symmetry breaking. Let us recall that the latter corresponds to the fact that solutions of a system having a given symmetry, rotational symmetry for the vorticity equations, do not in general respect individually this symmetry, whereas they usually do it statistically. A classical example is the buckling of a cylindrical pillar under an increasing axial load: the pillar finally bends towards some definite direction rather than to being cylindrically deformed. With the help of such anisotropic space transforms, we generate a statistical break up of the statistical rotational symmetry of the vorticity equations to obtain a new set of equations whose symmetry corresponds to anisotropic scaling. As discussed below, to systematically study the effect of space contraction/dilations, it is convenient to use the general notion of a “pullback” transform of a field by these contraction/dilations.

When there is neither rotation nor gravitational field, the Navier-Stokes equations (Eq. 1) and the mass conservation equation (Eq. 2) are rotational invariant, as well as formally invariant under isotropic contraction/dilation T_λ of the space for any arbitrary scale ratio λ ($\lambda > 1$ for a contraction, $\lambda < 1$ for a dilation):

$$\mathbf{x} \mapsto T_\lambda \mathbf{x} = \mathbf{x}/\lambda \quad (18)$$

It suffices indeed to suitably renormalize/rescale the other variables by various powers of this scale ratio λ :

$$t \mapsto t/\lambda^{1-\gamma}; \mathbf{u} \mapsto \mathbf{u}/\lambda^\gamma; \mathbf{f} \mapsto \mathbf{f}/\lambda^{2\gamma-1}; v \mapsto v/\lambda^{1+\gamma}; \rho \mapsto \rho/\lambda^\gamma \quad (19)$$

defined by the singularities γ, γ' of respectively \mathbf{u}, ρ , see (Schertzer and Lovejoy, 2004) for further discussion in particular on the question of the dissipation and forcing terms. For a unique singularity γ , the corresponding power spectrum is a power-law $E(k) \approx k^{-\beta}$ with the spectra slope $\beta = 2\gamma + 1$, because it scales like $u^2 l$. One thus obtains the celebrated Kolmogorov's scaling of the velocity field for the (Kolmogorov's) singularity $\gamma_K = 1/3$, which is defined by considering that the energy flux density ϵ , which scales

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page	
Abstract	Introduction
Conclusions	References
Tables	Figures
⏪	⏩
◀	▶
Back	Close
Full Screen / Esc	
Printer-friendly Version	
Interactive Discussion	



like u^3/l , is homogeneous and strictly scale invariant in the inertial range (Kolmogorov, 1941).

We are now looking for a generalization of this scaling analysis for anisotropic equations with the help of generalized contraction/dilation operators T_λ , which still form a one-parameter multiplicative group ($T_{\lambda'} \circ T_\lambda = T_{\lambda'\lambda}$), like in the isotropic case, and have therefore a generator G . In the framework of linear GSI, which is also known under the name of “operator scaling”, G and T_λ are matrices:

$$T_\lambda = \exp(-G \text{Log} \lambda) = \lambda^{-G} \quad (20)$$

Obviously, $G = Id$ (Id being the identity matrix) corresponds to the classical, scalar scaling (Eq. 18). It is now very convenient to use the notion of a “pullback” transform (or “composition operator” (Shapiro, 1993)) of a field u by a given space transform, here a contraction/dilation T_λ . It is so general that it is often passed over without mention and it can be just seen as a convenient and compact notation for the scaling of a given field. As illustrated by Fig. 5, the pullback T_λ^* corresponds to a straightforward generalization to (infinite dimensional) functional spaces of the (contravariant) change of coordinates on finite vector spaces:

$$\forall \mathbf{x} : T_\lambda^*(u)(\mathbf{x}) = u(T_\lambda \mathbf{x}) \quad (21)$$

the composed function $u(T_\lambda)$ pulls back the field u from the coordinates \mathbf{y} to \mathbf{x} , with the “change of coordinates” $\mathbf{y} = T_\lambda(\mathbf{x})$ (for further discussion see (Schertzer et al., 2010) although with an emphasis on the dual “pushforward” transform). This transform can be extended for differential operators D :

$$\forall f : T_\lambda^*(D)T_\lambda^*(f) = T_\lambda^*(Df) \quad (22)$$

For instance, the pullback of the gradient operator ∇ is:

$$T_\lambda^*(\nabla) = T_\lambda^{-1} \nabla \quad (23)$$

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



due to the fact that the transform T_λ is linear and is therefore its own Jacobian matrix. To study the (possibly) anisotropy of atmospheric dynamics, it is sufficient to consider a diagonal generator under the following form:

$$G = \text{diag}(g_i); g_1 = g_2 = 1; g_3 = H_z = 1 - h \quad (24)$$

with an exponent $0 \leq H_z \leq 1$ that defines the anisotropy of T_λ . For instance, $H_z = 0$ corresponds to a (scaling) 2-D flow, $H_z = 1$ to a 3-D (isotropic scaling) flow, and more generally intermediate values to a $(2 + H_z)$ – dimensional (anisotropic) scaling flow. Now, a generalized scaling of the field $\mathbf{u}(\mathbf{x}, t)$ is defined by the fact that its pullback transform T_λ^* corresponds to a (possible random) generalized contraction with a generator Γ :

$$T_\lambda^* \mathbf{u} = \lambda^\Gamma \mathbf{u} \quad (25)$$

The simplest case corresponds to the fact that both generators G and Γ commute, i.e. are diagonal on the same vector base. We consider the following particular case:

$$\Gamma = \text{diag}(\gamma_i); \gamma_1 = \gamma_2 = \gamma; \gamma_3 = \gamma + h \quad (26)$$

where γ is a given (scalar) singularity. To give a concrete example of interest, let us consider the scaling model of stratified atmosphere based on the conservation (i.e. strict scale invariance) of the energy flux along the horizontal and of the buoyancy force variance flux along the vertical (Schertzer and Lovejoy, 1984, 1985b). These conservation laws single out respectively the Kolmogorov singularity $\gamma_K = 1/3$ and the Bolgiano-Obukhov singularity $\gamma_{BO} = 3/5$. The latter is obtained similarly to the former, but for a flux that scales like u^3/l^5 (Bolgiano, 1959; Obukhov, 1962). Note that these scaling exponents were both obtained assuming homogeneous and isotropic fluxes, whereas Schertzer and Lovejoy considered heterogeneous and anisotropic fluxes. In this anisotropic framework the theoretical value $H_z = 5/9$ of the exponent of anisotropy merely results from the correspondence between the singularities of Kolmogorov and Bolgiano-Obukhov: $\gamma_K = 1/3 = H_z \cdot \gamma_{BO}$.

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



The choice of having the same h in Eqs. (24), (26) makes, independently of its value, not only scale invariant the incompressibility condition, but the advection term and the material derivative have furthermore the same scalar scaling:

$$T_\lambda^*(\mathbf{u} \cdot \nabla) = \lambda^{\gamma+1} \mathbf{u} \cdot \nabla; T_\lambda^*(D/Dt) = \lambda^{\gamma+1} D/Dt \quad (27)$$

5 The pullback is more involved for the vorticity ($\nabla = (\partial_1, \partial_2, \partial_3)$, $\mathbf{u} = (u^1, u^2, u^3)$):

$$T_\lambda^*(\boldsymbol{\zeta}) = \lambda^{\gamma+1} \begin{pmatrix} \lambda^h \partial_2 u^3 - \lambda^{-h} \partial_3 u^2 \\ \lambda^{-h} \partial_3 u^1 - \lambda^h \partial_1 u^3 \\ \partial_1 u^2 - \partial_2 u^1 \end{pmatrix} \quad (28)$$

which in turn yields the pullback of the stretching vector:

$$T_\lambda^*(\mathbf{s}) = \lambda^{2(\gamma+1)} \begin{pmatrix} \lambda^h (\partial_2 u^3 \partial_1 u^1 - \partial_1 u^3 \partial_2 u^1) + \lambda^{-h} (\partial_1 u^2 \partial_3 u^1 - \partial_3 u^2 \partial_1 u^1) \\ \lambda^h (\partial_2 u^3 \partial_1 u^2 - \partial_1 u^3 \partial_2 u^2) + \lambda^{-h} (\partial_3 u^1 \partial_2 u^2 - \partial_2 u^1 \partial_3 u^2) \\ \partial_1 u^2 \partial_3 u^3 - \partial_3 u^2 \partial_1 u^3 + \partial_3 u^1 \partial_2 u^3 - \partial_2 u^1 \partial_3 u^3 \end{pmatrix} \quad (29)$$

10 Due to the fact that the vorticity equations should be satisfied for any λ , which intervenes with three distinct exponents, we obtain the following set of dynamical equations (not yet taking into account the baroclinic vector) :

$$\begin{aligned} D\partial_2 u^3 / Dt &= \partial_2 u^3 \partial_1 u^1 - \partial_1 u^3 \partial_2 u^1 \\ -D\partial_3 u^2 / Dt &= \partial_1 u^2 \partial_3 u^1 - \partial_3 u^2 \partial_1 u^1 \\ -D\partial_1 u^3 / Dt &= \partial_2 u^3 \partial_1 u^2 - \partial_1 u^3 \partial_2 u^2 \\ D\partial_3 u^1 / Dt &= \partial_3 u^1 \partial_2 u^2 - \partial_2 u^1 \partial_3 u^2 \\ D(\partial_1 u^2 - \partial_2 u^1) / Dt &= \partial_1 u^2 \partial_3 u^3 - \partial_3 u^2 \partial_1 u^1 + \partial_3 u^1 \partial_2 u^3 - \partial_2 u^1 \partial_3 u^3 \end{aligned} \quad (30)$$

which corresponds to a partial decoupling of the 3-D vorticity equations that read:

$$\begin{aligned} D(\partial_2 u^3 - \partial_3 u^2) / Dt &= \partial_2 u^3 \partial_1 u^1 - \partial_1 u^3 \partial_2 u^1 + \partial_1 u^2 \partial_3 u^1 - \partial_3 u^2 \partial_1 u^1 \\ D(\partial_3 u^1 - \partial_1 u^3) / Dt &= \partial_3 u^1 \partial_2 u^2 - \partial_2 u^1 \partial_3 u^2 + \partial_2 u^3 \partial_1 u^2 - \partial_1 u^3 \partial_2 u^2 \\ D(\partial_1 u^2 - \partial_2 u^1) / Dt &= \partial_1 u^2 \partial_3 u^3 - \partial_3 u^2 \partial_1 u^1 + \partial_3 u^1 \partial_2 u^3 - \partial_2 u^1 \partial_3 u^3 \end{aligned} \quad (31)$$

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion

i.e. the anisotropic scaling splits the dynamical equation of each horizontal vorticity component into two dynamical equations of a velocity gradient. This corresponds to the introduction of new constraints, i.e. solutions of Eq. (30) are, as they should be, a subset of solutions of Eq. (31) that are invariant under the pullback transforms defined by Eqs. (24), (26). Overall, solutions of Eq. (30) are the solutions of Eq. (31) that statistically respect the anisotropic scaling prescribed by Eq. (24), (26) and statistically break the isotropic scaling of Eq. (31). These solutions corresponds to an alternative to QGT: they share the common boundary condition that vorticity is dominated by Earth's rotation at large scales, but contrary to QG the transfer of the vorticity to smaller scales is obtained with the help of a nonlinear stretching term, contrary to the QG approximation (Eq. 12).

With respect to Eq. (30), there are obviously many questions to be explored, e.g.: do their solutions exist for any value of the exponent H_z , does the theoretical value $H_z = 5/9$ play a special role? However, these questions are beyond the scope of our reply to LTNCG's arguments. Presently, let us mention that the baroclinicity can be taken into account at all scales in Eq. (30). Indeed, the baroclinic vector has the following pullback:

$$T_\lambda^*(\mathbf{b}) = \lambda^{2(\nu+1)}(\lambda^{-h}b_1, \lambda^{-h}b_2, b_3) \quad (32)$$

which has an opposite behaviour to that of the potential vector $\boldsymbol{\psi}$:

$$T_\lambda^*(\boldsymbol{\psi}) = \lambda^{\nu-1}(\lambda^h\psi_1, \lambda^h\psi_2, \psi_3) \quad (33)$$

which corresponds to the expected phenomenology: to be aligned along the vertical for large scales, whereas it is no longer the case for smaller scales.

5 Conclusions

We believe that the critical analyses of the scale analysis leading to the QG approximation (Sect. 2) and of the scaling analysis of the resulting QGT (Sect. 3) rule out

Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures



Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



LTNCG's claim that the scaling laws of atmospheric dynamics could not be put into question because they should correspond to QGT predictions. Beyond this preliminary clarification, we furthermore put forward an alternative to the QG approximation that reconciles the facts that Earth's rotation dominates vorticity at large scales and that the vorticity stretching nonlinearly transfers this large scale input over a wide range of scales in a scaling manner. Indeed, by introducing a statistical anisotropic break of the statistical isotropic scaling of the vorticity equation, we obtain a new set of equations whose solutions correspond to this alternative. Whereas much remains to be done about these equations, the already known statistical behaviour of their solutions would correspond to an atmospheric turbulence of dimension $D = 2 + H_z$ over a wide range of scales, which seems to result from the application of 'Occam's Razor' applied to today large quantity of unbiased atmospheric data. In this direction, we need to emphasize that - contrary to LTNCG to subsume potential with kinetic energy does not solve the problem of addressing the vertical statistics. Indeed, the naive scaling exponents, i.e. without including intermittency effects or logarithmic corrections (Kraichnan, 1967) are obtained by arguments *à la* Kolmogorov (1941) that are ultimately based on dimensional analysis of the relevant fluxes. Therefore in order to get different horizontal and vertical scaling exponents (scaling anisotropy) one needs to consider a vertical turbulent flux with physical dimensions different from energy. The theoretical choice of the buoyancy force variance flux (Schertzer and Lovejoy, 1984, 1985b) still seems reasonable, although it was a bit bold when first proposed. Indeed, since the pioneering studies of Adelfang (1971) and Endlich et al. (1969) and especially since the 1980s, there has been a growing body of evidence that the scaling is indeed anisotropic with at least roughly the predicted exponents, see the numerous references cited in our previous comment (Lovejoy et al., 2009b).

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Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures



Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion

1423, 1967. 3308, 3316

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Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

Title Page

Abstract

Introduction

Conclusions

References

Tables

Figures

⏪

⏩

◀

▶

Back

Close

Full Screen / Esc

Printer-friendly Version

Interactive Discussion



Schertzer, D. and Lovejoy, S.: Generalised scale invariance in turbulent phenomena, Physico-Chemical Hydrodynamics Journal, 6, 623–635, 1985b. 3310, 3313, 3316

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Quasi-geostrophic turbulence and generalized scale invariance

D. Schertzer et al.

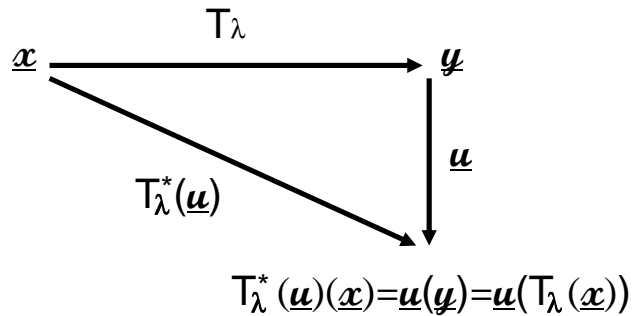


Fig. 1. Scheme of the “pullback” T_{λ}^* that pulls back the field u from the coordinates y to x with the help of the (anisotropic) space contraction/dilation T_{λ} .

Title Page	
Abstract	Introduction
Conclusions	References
Tables	Figures
◀	▶
◀	▶
Back	Close
Full Screen / Esc	
Printer-friendly Version	
Interactive Discussion	