NONLINEAR GEODYNAMICAL VARIABILITY: MULTIPLE SINGULARITIES, UNIVERSALITY AND OBSERVABLES

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ABSTRACT. We enlarge on theoretical insights concerning the multiple scaling/multifractal behaviour of geodynamical fields in the space-time domain and the very singular behaviour of their observables which are usually obtained by averaging over scales much greater than that of the homogeneity. We render more direct the link between statistical singularities (divergence of high order statistical moments) and hierarchies of singularities per realization (small scale divergence of densities). In the case of "hard multifractals" (having not only singular realizations but also - contrary to "soft multifractals" - singular statistics), we insist on the importance of the existence of "wild" singularities which although extremely rare, create the statistical divergences, as well as on the need to distinguish between "bare" and "dressed" properties of these multifractal fields.

The bare properties are the properties of a (rather theoretical) process in which nonlinear interactions between scales smaller than the observation scale are filtered out. Conversely, the dressed properties are those of the observables and result from the full hierarchy of nonlinear interactions down to an infinitesimal scale followed by integration over the scale of observation. Both properties involve multiple scaling and hierarchies of dimensions, but the latter introduce statistical divergences, "pseudo-scaling", etc. Observations obtained by averaging over a given dimension therefore "dress" in a drastic manner the "bare" properties of a process. We also underline the fact that in general, multifractals are non-local and hence - contrary to simplistic local multifractal notions - both the scaling exponents and orders of singularities must be understood as statistical exponents, not as point values.

We show that the infinite hierarchies of critical exponents in multifractals may well be very simply determined due to the existence of three-parameter (H, C_1, α) universality classes of the generic multifractal processes. These three fundamental exponents characterize: the degree of non-conservation of flux (H), the deviation of the mean field from homogeneity (C_1), and the deviation of the process from monofractality $(0 \leq \alpha \leq 2)$. We discuss other associated fundamental properties. The five main subclasses of these (H, C_1, α) universal canonical multifractals are outlined with their important theoretical and practical consequences.

A quite different aspect of scaling symmetry is that the scale transformations involved can be strongly anisotropic, nonlinear and even stochastic. This leads us to generalize the idea of scale invariance far beyond the familiar self-similar (or even self-affine) notions. We sharpen the ideas of this nonlinear/stochastic Generalized Scale Invariance, thus introducing an enormous diversity of scaling behaviour.

Beyond the many important theoretical and practical consequences of these findings, we are lead to explore a hidden and unexpected face of multifractal chaos: bare universality under dressed Pandemonium.

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1. INTRODUCTION

1.1. The unification of geophysics ?

An emerging and powerful unifying *problematic* of geophysics is being increasingly recognized: the extreme variability of geophysical phenomena and processes over wide ranges of spatial/temporal scales, which easily cover nine orders of magnitude (earth radius scale/ centimeter scale or e.g. 30 years/second or 10 days/millisecond). Indeed, what has been felt to be a growing and ubiquitous difficulty in geophysics, is more and more perceived as a *fundamental symmetry*: a common behaviour at different scales (*scaling behaviour*). Indeed, this corresponds to *the simplest but also the only symmetry assumption acceptable in the absence of more information or knowledge*. Indeed, we cannot consider the breaking of this symmetry without first exploring its possible manifestations in the largest sense. For instance, the symmetries we will consider are statistical symmetries, *each realization* corresponds in fact to a violation or *a breaking of these symmetries*[1]. The corresponding exponents (dimensions, singularities, ...) are also statistical, not point values.

Since this symmetry is the result of nonlinear interactions -nonlinear (i.e. non proportional) response to a given excitation- between different scales (and/or processes), we are addressing the question of *scaling nonlinear variability*. A general consequence of such variability is that the notion of observables (roughly speaking: what we can observe or measure from a process) is far from trivial, since the details of the process may be overwhelmingly important (due to small scale or high frequency "ultraviolet" divergences or singularities). Unfortunately our observations and measurements are nearly always restricted to resolutions much higher than the scale of the smallest detail, "inner scale" or "scale of homogeneity" which in geophysics is typically of the order of millimeters or less. Full knowledge down to this inner scale is usually out of our scope due to the large number of degrees of freedom involved which can be of the order of *physicists' infinity* such as the Avogadro's number (10^{23}): indeed the number of mm^3 (the number of degrees of freedom) in the atmosphere is of order[2] $10^{10} \times 10^{10} \times 10^7 = 10^{27}$.

This type of unifying problematic is urgently needed in geophysics, since under the heading of "Global Change Research", the geophysical community is tending more and more to address global questions, particularly those pertaining the climate. Unfortunately, up until now we have faced a rather distressing situation: gigabytes of computer codes which are unable to cope with terabyte flows of (often remotely sensed) data, obtained at finer and finer resolution, all because the numerical models work at far larger scales[3]. It would seem to be of doubtful value to try to answer any of the questions raised by Global Change Research without being able to simultaneously think of the global as well as the detailed characteristics of the variability of geophysical fields. Indeed, it would seem fruitless to design sophisticated integrated data acquisition and processing facilities without having a conceptual framework for handling massive high resolution data sets. Indeed, we desperately need to cast order in geophysical chaos, more properly to discover new order in what according to current knowledge is apparently disorder. In other words to master how simplicity can beget complexity.

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[1] As in the the widespread image of a marble rolling on a symmetric bottom of a bottle: each of the experiments will violate the rotational symmetry, yet on the average this symmetry is still respected!

[2] Considering the scale of homogeneity of the order of the millimeter, and the (outer) vertical scale of the order of ten kilometers and the horizontal scale of the order of ten thousand kilometers. In a similar manner the Reynolds number (which is the ratio of the nonlinear to dissipation terms and hence characterize the strenght of nonlinearity) of atmospheric turbulence is usually estimated as $\approx 10^{12}$, taking the ratio of injection (1000 km)/dissipation (1mm) (horizontal) scales as 10^9, since it is the 4/3 power of this ratio.

[3] There is even a growing tendancy of evaluating the performance of models by comparing models with models rather than models with data!
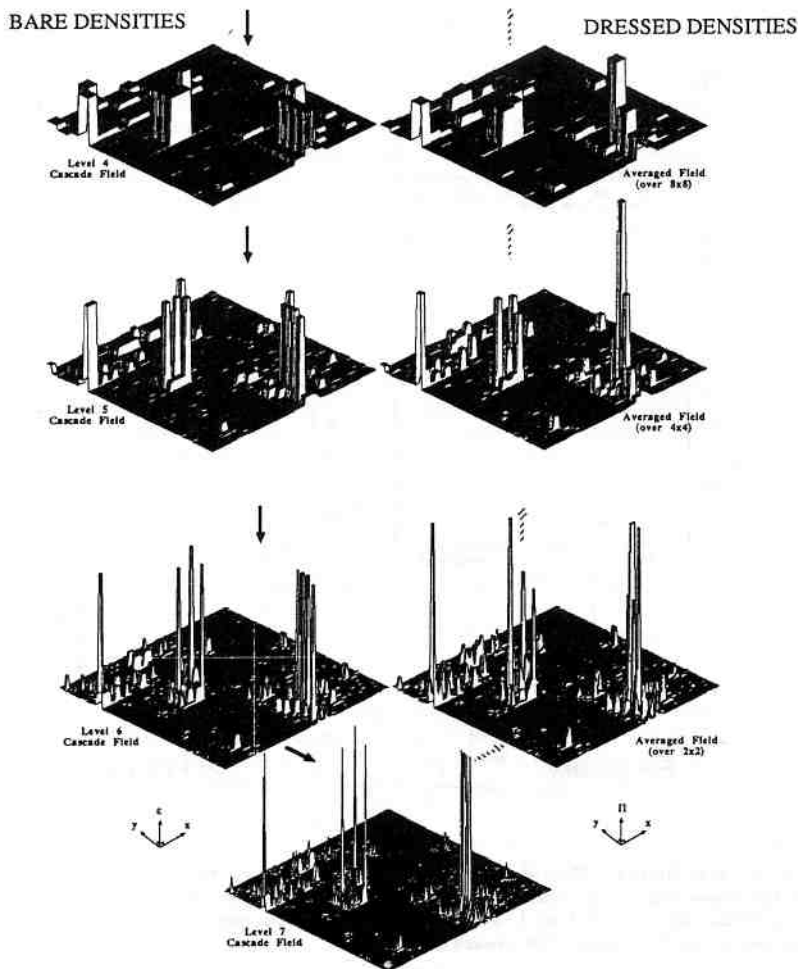
Fig. 1a : Illustration of the "bare"and "dressed" energy flux densities. The left hand side shows the construction step by step of the bare field produced by a multifractal cascade process (the α-model, discussed below) starting with an initially uniform unit density. At each step the homogeneity scale is divided by a constant ratio $\lambda=2$ From top to bottom, the number of cascade steps takes the following values n = 0, 1, 2, 3 and 7, with the corresponding length scale values l = 1, 1/2, 1/4, 1/8, 1/128. When the number of steps n increases, some rare regions of high intensities ("singularities") appear, most of the space becomes inactive. At l =1/8, n=3, one may compare the rather more intense dressed density with the bare density. The sharp contrast arise from the smaller scales singularities, as seen on step n=7, which contribute to high fluctuations of the dressed density.
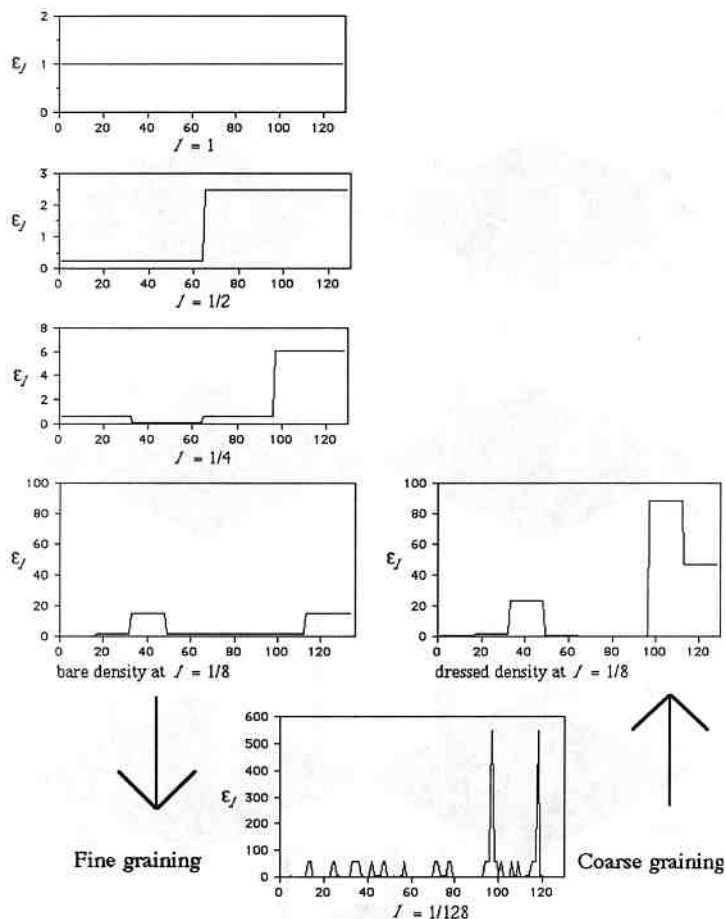
Fig. 1b : as in Figure 1a , illustration of the "bare"and "dressed" energy flux densities, but on a 2 dimensional space. The dressed energy flux densities, obtained by averaging, are presented on the right hand side of the figure. At intermediate scales, level 3 or 4, one may still note the important contributions from smaller scales singularities to high fluctuations of the dressed density.

1.2 The importance of the details

A cornerstone in the early recognition of the importance of the "details" and their appropriate represention seems to be the beginning of our century. For instance, Perrin, in the introduction of his thesis (Perrin, 1913), already pointed out that tangentless curves are the rule rather than the exception, contrary to the academic teaching which tries to render "obvious" a continuous perception of the world. Among the various examples he discussed were flakes and brownian motion. However the example that he stressed was the coastline of Brittany -a question further explored by Richardson (1960) and popularized by Mandelbrot (1982)- even in spite of the fact that our scale dependant representations (maps) are overwhelmingly smooth! He also underscored the conceptual contributions of contemporary mathematicians such as Borel,

who by extending in a discontinuous manner the mathematical notion of measures from the volume-like ones (Lebesgue measures) rendered them at the same time more abstract yet nearer to the discontinuous "real world". It is indeed surprising to discover how contemporary Perrin's discussion remains! Ever since then, in order to deal with these "ultra-violet" divergences mentioned many attempts have been made to define smooth macroscopic "effective" fields from irregular microscopic ones using various techniques ("homogeneization", "renormalization"), the permanent question of "coarse graining" vs. "fine graining".

Concerning fluid dynamics, the question of the small scale singularities became more precise with the work of Leray (1934), and in Von Neuman's review on turbulence (Von Neuman, 1963), but also in the debate between Richardson[1] and Bjkernes on the rather fundamental question: is the characterization of a few large scale singularities (the meteorological fronts) sufficient to forecast the evolution of the weather? More recently, under the theme of the "butterfly effect" Lorenz (1963) gave a stunning image and now popular metaphor for the absolute unpredicibility resulting from the small scale singularities or *sensitive dependence on the initial conditions*: as time passes away, the single (small scale) flutter of a butterfly will introduce large scale disturbances in atmospheric dynamics. The current day debate could be much more precise by dealing with the characterization of hierarchies of scaling singularities. In the following we hope to give clearer insights into this fundamental question with the help of apparently (at first glance) simple models (phenomenological models or "mock geophysics"), which nevertheless possess surprising properties which turn out to be quite general.

The exploration of nonlinear variability was maintained in the restrictive frontiers of geometry for too long a period. This period created some unfortunate attempts to bypass various fundamental problems (among which we may cite several abusive uses of adhoc additive processes). Indeed, the development of concrete analytical methods has tended to show that geometrical frameworks can often be misleading and fractal notions have been most fruitful when divorced from geometry. In particular, the abandonment of the dogma of the uniqueness of fractal dimension (Grassberger, 1983; Hentschel and Proccacia, 1983; Schertzer and Lovejoy, 1983,1984; Parisi and Frisch, 1985; Halsey et al., 1986; Pietronero and Siebesma, 1986; Bialas and Peschanski, 1986; Stanley and Meakin, 1988; Levich and Shtilman, 1989; ...) in favour of hierarchies of dimensions and singularities with their non-geometric generators has been one of the most important recent advances. It is now rather obvious that multiple dimensions and singularities are the rule rather than the exception for fields, hence we are now used to "multiple scaling" or "multifractality" associated with highly intermittent processes in which the weak and intense regions have different scaling behaviour. However, as we will discuss after having left this uniqueness for infinity, the important question of the existence of universality classes gives credence to returning to only few fundamental (dynamic, not geometric) parameters!

The scaling symmetries are rather special when compared to the rotational or parity[2] (i.e. mirror reflection) symmetries, since they are not compact[3] contrary to the latter. One may note also that in its simplest, but very restrictive form (the only one explored by fractal geometry[4] up to 1986!), it is not only space/time invariant, hence global, but also isotropic. These two assumptions are obviously unacceptable in geophysics, since we have to deal with anisotropy in the space/time domain, with rotation, stratification[5] (Schertzer and Lovejoy, 1983, 1984, 1985a) or "texture". This is the reason why we developed some new elements (Schertzer and Lovejoy, 1985b, 1987a, b) for a Generalized Scale Invariance

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[1] Recall that Richardson (1926) didn't hesitate to raise the (sacrilegious?) question "Does the wind have a velocity?" (i.e. are the time derivatives regular?). Indeed, he pointed out the very irregular (fractal) Weierstrass function as a counter-example.

[2] The role of symmetry breaking in parity invariance (helicity) for creating large scale disturbances has been emphasized by Levich and Tzvetkov (1985), Levich and Shtilman (this volume) and Moisseev et al. (1988).

[3] Since unlike angles the scale ratio is unbounded. This lack of compactness has even lead some to doubt wheter the scaling symmetry is respected (Frisch, 1985).

[4] This single fact helps to explain the strangely bitter comments (Mandelbrot, 1986), against some of our earlier papers on scaling anisotropy ...

[5] We proposed to add the second stanza: "Flatter whorls have rounder whorls that feed on instability, and roundish whorls have rounder whorls, and so on to spherocity -in the statistical sense" to Richarsdson's celebrated poem : "Big whorls have little whorls that feed on their velocity, and little whorls have smaller whorls and so on to viscosity -in the molecular sense" (Richarsdson, 1922).

(see section 6). In general, we will need both a local[1] and anisotropic scale transformation and we even consider stochastic scale transformations. This leads us to generalize the notion of scaling: a system may be said to be scaling (or scale invariant) over a range if the small and large scale structures/behaviours are related by a scale changing operation involving only the scale ratio. Hence, scale invariance is not restricted to the familiar self-similar (or even self-affine) notions. It is important to distinguish this idea of local scale transformations from the simplistic multifractal notion of local exponents.

1.3 Geophysical observables

The breathetaking pictures of (geometric) fractal objects often inclined us not to explore the rather immediate question: how will we perceive them with the limited resolution of our eyes if the computing process goes down to a much smaller scale? Contrary to what happens with (geometric) monofractal objects, a drastic *symmetry breaking is caused by the observation* not only by the scale of observation, but also by its dimension. This is the reason why we will insist on the fundamental difference between *"bare" and "dressed" properties* at a given (non-zero) scale i.e. the important differences between a process with a cut-off of small scale interactions and one with these interactions restored (cf. fig. 1a-b for illustrations).

The bare properties are related to fine graining (e.g. the development a cascade) and are the properties of the process with the nonlinear interactions at scales smaller than the observation scale filtered out (i.e. the process is truncated at the scale of observation). The dressed properties are coarse grained, they are the observed properties at a given scale of resolution (i.e. obtained by linear or nonlinear averaging over the smaller details of the same process at the observation scale and with all interactions: the process fully developed down to the smallest scale). In other words, only half the problem has been explored (and even a smaller fraction of the real problem): the "dressed" truth is the one which counts! The terms "bare" and "dressed" are borrowed from renormalization jargon, but here due to the extreme variability, their differences become even more important; not only do they involve a renormalizing factor but also quite different statistical behaviour. This raises immediately the overwhelmingly important question of "wild", singular statistics (divergences of statistical moments (Schertzer and Lovejoy, 1987a,b)) linked to multiple ultraviolet divergences.

2. HOW DOES GOD PLAY DICE?

2.1. Scaling nonlinear variability and "Mock Geophysics"

In a very general manner (Noether's theorem), for every (continuous) symmetry we can associate a conservative quantity. For instance in physics: conservation of energy and momentum for time and space translational invariance, angular momentum for rotational invariance... Here however we are investigating dissipative systems, far from equilibrium. As in turbulence theories, the conserved quantities should therefore rather be the rate of dissipation of energy - more properly speaking the *flux of energy* towards smaller scales, not the energy itself (hence the notion of "quasi-equilibrium" with a constant rate of dissipation or flux of energy). We can already anticipate that the fundamental exponents (H, C_1, α) -that we will show to be sufficient to characterizing universal processes of nonlinear variability- are related to various possible deviations from the simplest hypothesis of conservation of the flux, i.e. homogeneous conservation. Indeed each parameter quantifies a distance from homogeneity, H for the degree of non-conservation of the flux, C_1 for the mean deviation from homogeneity, and α (the Lévy index, $0 \leq \alpha \leq 2$) which indicates how far the process is from monofractality ($\alpha=0$).

The problematic of nonlinear variability over wide ranges of time/space scales, has been considered for a long time with respect to the mysterious turbulent behaviour in fluid dynamics, especially their asymptotic (and universal) behaviour when the dissipation length goes to zero (fully developed turbulence). Conceptual advances occurred using apparently simple models of self-similar cascades, as opposed to the

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[1] One may note localization of scale symmetry has been considered by Weyl (Weyl, 1923) under the name of "local gauge symmetry" in the context of (relativistic) electromagnetism, in the spirit of the (already localized symmetry) of Einstein's theory of gravity.

frustratingly tedious developments of renormalization techniques... which still fail to grasp the intermittency problem. From very general considerations (going back to the famous poem of Richardson (1922)), the phenomenological models of turbulence have become more and more explicit although sometimes in an overly restrictive manner (to quote a few: Novikov and Stewart (1964), Yaglom (1966), Mandelbrot (1974), Frisch et al. (1978), ...; for a review see Monin and Yaglom (1975)). However their common theme of how the energy flux is spread into smaller scales in successive cascade steps (while respecting a scale invariant conservation principle) is far from being restricted to turbulence since spreading into small scales is a general theme in geophysics (from concentration of (passive) substances/scalar fields to the spreading of points on strange attractors). Note that the notion of flux at a given scale or through a scale, can be more precisely understood in Fourier space as the flux through the surface of a sphere of constant wave number radius (the inverse of this wave number being proportional to the corresponding scale). In this sense we can speak of a probability flux of points on a strange attractor, e.g. the flux of points flowing to smaller scales on this strange attractor, hence the "flux dynamics" we will discuss is quite general, paralleling classical thermodynamics, but with very strong divergences .... We will also discuss the related fields which are not constrained to such scale conservation (such as the passive scalar concentration and velocity field ...).

2.2. Pixel worlds and weak measurable properties

Geophysical phenomena (especially when remotely sensed) are more and more often represented with the help of digitized "images", pixel sets. In spite of this the "theoretical" representations of the phenomena are still believed to be of a certain continuous type. Such continuous representations are thought to be rather obvious limits of the pixel representation when the resolution (scale of observation) goes to zero. In particular one would usually associate a function with such an image - a "density" - and consider the digitized field as corresponding to averages of this density over a pixel. Hence from a very rough knowledge of the pixel values, one "naturally" tries to associate a hypothetical function. Such a "natural" hypothesis (already critized by Perrin) is far from being physically obvious: it requires ample (mathematical) regularity constraints which are contrary to the observed strong variability down to smaller scales. Mathematically, it corresponds to very particular measurable properties: one considers only regular measures with respect to the usual line, surface and volume measures, i.e. Lebesgue measures. Indeed, the simplest example of scaling and scale invariance is to consider the (apparently "metric" in fact "measure") idea of dimension of a set of points as it often occurs in geophysics. The intuitive (and essentially correct) definition is that the measure of the "size" of the set n(L) at scale L is given by:

$$n(L) \propto L^D \tag{1}$$

where D is the dimension (e.g the length of a line $\propto L$, the area of a plane, $\propto L^2$... or the number of in situ meteorological measuring stations on the earth in a circle radius $L \propto L^{1.75}$ (Lovejoy et al., 1986a, b), the distribution of raindrops on a piece of blotting paper $\propto L^{1.83}$ (Lovejoy and Schertzer, 1990) and the occurrence of rain during a time period $T \propto T^{0.8}$ (Hubert and Carbonnel, 1988, this volume; Tessier et al., 1989) .... The "volume" (actually the measure of the set) is therefore a simple scaling (power law) function, and the dimension is important precisely because it is scale invariant (independent of L). We recall that the Hausdorff dimension D(A) of a (compact) set A may be defined by the generalization to non-integer D of the divergence rule "the length of a surface is infinite, its volume is zero ..." with the rather straightforward extensions (to non-integer D) of the d-Lebesgue measure (defined for integer d) to the D-dimensional Hausdorff measure. Thus we use the notation $\int d^D x$ for the D-dimensional Hausdorff measure of a (compact) set A and the Hausdorff dimension D of A is hence defined by the divergence rule[1] (see fig. 2 for illustration and Appendix B for more discussion):

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[1] It is easy to check that eq. 1 is consistent with this divergence rule. Indeed, interpreting eq. 1 as the fact that the number of cubes of size $l = L/\lambda$ needed to cover the fractal set will be of the order λ^D and since the D'-volume of an elementary cube is $l^{D'}$, it follows that the sum of their D'-volumes -of the order of the D'-Hausdorff measure-will follow the indicated divergence rule.

$$\int_A d^{D'}x = \infty, \text{ for } D'<D; \qquad \int_A d^{D'}x = 0, \text{ for } D'>D \qquad (2)$$

One may note that the D-measure of A is not necessarily finite and non-zero: some logarithmic corrections (exponents Δ_i on the i-th iterate of the logarithm are "sub-dimensions", c.f. Appendix B) may be needed to obtain finiteness and a precise determination of the Hausdorff dimension (they may give rise to the appearance of 'lacunarity', e.g., Smith et al. (1986)).
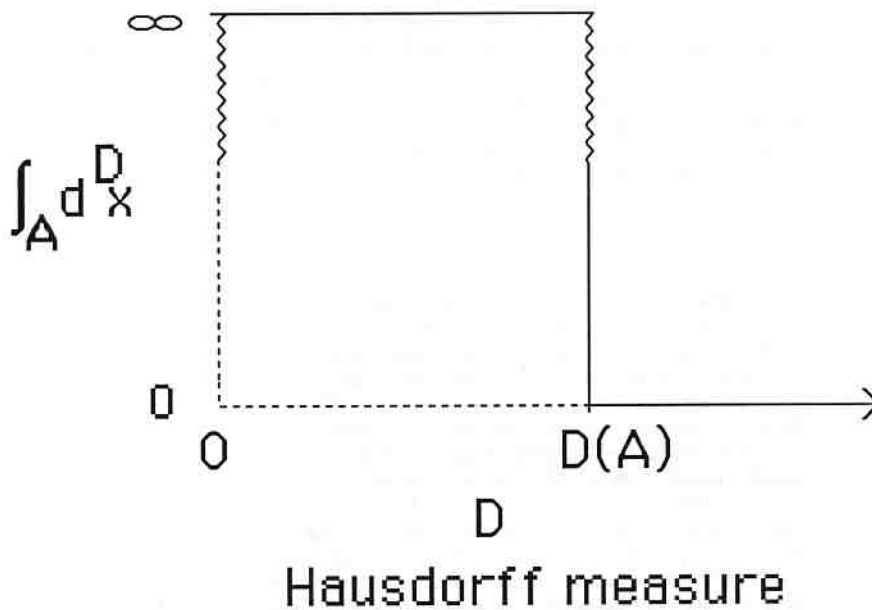

Fig. 2: Illustration of the divergence rule for Hausdorff measures, generalizing the divergence rule "the length of a surface is infinite, its volume is zero...". The transition at D=D(A), from infinity to zero, defines the Hausdorff dimension of the set A.

In other words, the "natural" framework for fields is not functional analysis (nor geometry ...!), but (mathematical) measures. Indeed, the use of functions rather than the (more general) measures is often a purely mathematical artifact. It is unnecessarily restrictive since really what we can empirically measure or describe is not in fact a value at a (geometric) point, but rather a value on "nearly any" (small) set surrounding this point. Such considerations are at the basis of (mathematical) measure theory which renders quite precise the notion of "nearly any" set[1] as small as we wish. Thus geophysics already seems to be more and more associated with singular measures with respect to Lebesgue measures[2]. Going a step further we will be interested in (random) *linear operators acting on measures*, as fundamental tools to study nonlinear variability. Such apparently abstract questions can be concretely addressed by apparently simple-minded geophysical models, but with rather general and non-trivial consequences and properties corresponding to the more abstract tools mentioned above.

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[1] It needs to be member of the "tribe", usually the borelian tribe...
[2] Regular (respectively singular) measures with respect to Lebesgue measures means that (almost everywhere) they correspond to a product of a density (a function) and a Lebesgue measure (resp. they don't).
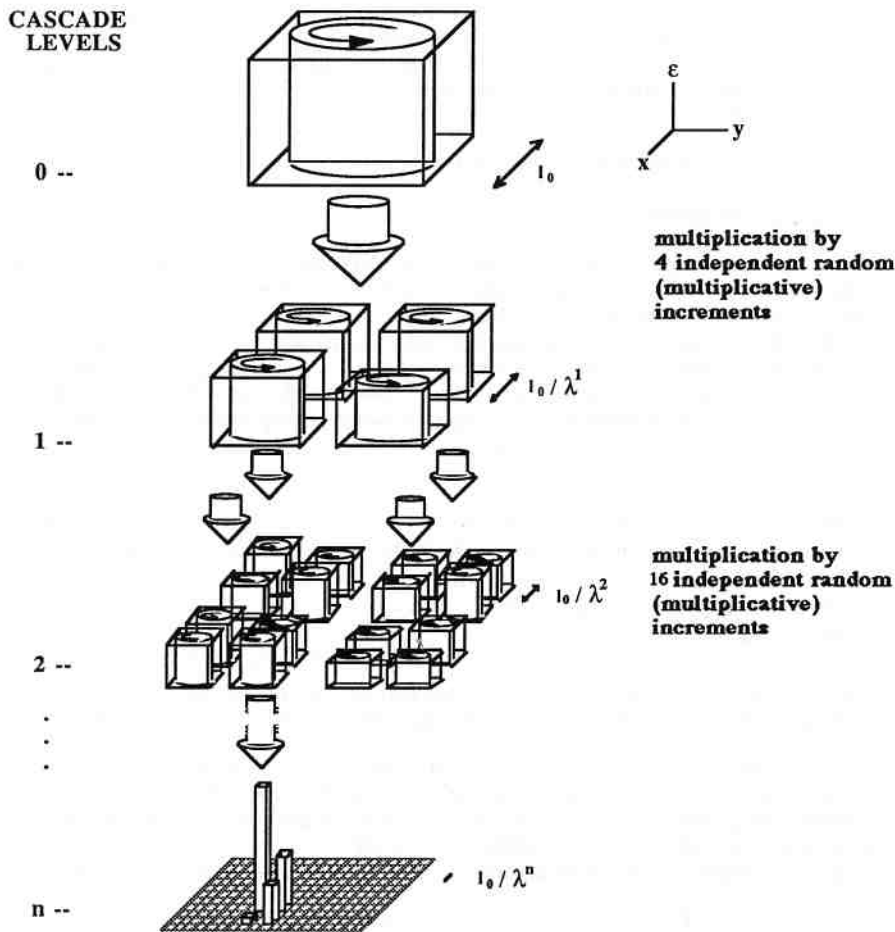
**CASCADE
LEVELS**

Fig. 3: A schematic diagram showing few steps of a discrete multiplicative cascade.process, here the "α-model" with two pure singularity orders γ^- ($>-\infty$) and γ^+ (corresponding to the two values taken by the independent random increments, $\lambda^{\gamma^-}<1$ and $\lambda^{\gamma^+}>1$) leading to the appearance of mixed singularity orders γ ($\gamma^- \leq \gamma \leq \gamma^+$).

On the contrary, failure to adequately recognize some of these fundamental properties (such as the bare/dressed distinction) has lead to a simplistic notion of multifractals as mathematical functions (rather than measures) involving properties such as the order of singularities and even dimensions at mathematical points rather than on neighbourhoods of points. This "local" multifractal notion - and its"soft" consequences as discussed in section 4.1 - has been influenced, if not inspired, by an exaggerated emphasis of the geometry of multifractals rather than statistics, and has lead to various frustrating attempts to calculate various local multifractal properties. The difficulties and apparent contradictions which result from local multifractal notions include the apparent existence of negative fractal dimensions, the difficulty of obtaining converging estimates of local orders of singularity. Examples include the use of (Arneodo et al., 1988; Farge and Rabreau, 1988) wavelet analysis (Grossman and Morlet, 1987; Meyer, 1987).

We will try to give two approaches to the weakly[1] measurable scale invariant properties of geodynamics: one which is constructivist (the multiplicative processes) and the other which is non-constructivist ("flux dynamics"). In both cases, the abstract object remains the same: *to study how God plays dice in creating a stochastic chaos in Geophysics!*

3. MULTIPLICATIVE PROCESSES AND FLUX DYNAMICS

3.1 Multiplicative processes

The key assumption in phenomenological models of turbulence (which has recently become more explicit) is that successive steps define (independently) the fraction of the flux of energy distributed over smaller scales. Note that it is clear that the small scales cannot be regarded as adding some energy but can only (multiplicatively) modulate the energy passed down from larger scales (hence in spite of their occasional visual success the lack of relevance of additive processes, e.g. Voss (1983)). Hence bare densities ε_λ, resulting from cascade processes from outer scale l_0 (which will be assumed equal to 1, without loss of generality) to l (the homogeneity scale of ε_λ)=l_0/λ are multiplicatively defined (see fig. 3 for illustration):

$$\varepsilon_{\lambda\lambda'} = T_\lambda (\varepsilon_{\lambda'}) \, \varepsilon_\lambda \qquad (3)$$

T_λ denotes a spatial contraction of ratio λ (>1). In the isotropic case, for any point x; $T_\lambda x = x/\lambda$; for any set A: $T_\lambda(A) = \{T_\lambda x / x \in A\}$; for any function f :$T_\lambda$ [f(x)] = f(λx); for any measure μ and any set A:

$$\int_A d[T_\lambda (\mu)] = \int_{T_\lambda(A)} d\mu$$

and more generally for any function f (i.e. not only for 1_A, the indicator function of the set A) $\int f d[T_\lambda (\mu)) = \int T_\lambda(f) d\mu$. In case of (scaling) anisotropy, more involved contractions of space are required as discussed in the section 6.

Leaving additive (stochastic) processes (which had been used on the purely geometrical grounds of fractal geometry, e.g. fractional brownian motions, for modelling landscapes, etc.) to multiplicative processes, one encounters surprising properties: multiplicity of singularities and dimensions, rather than uniqueness. Let us discuss these properties briefly: a priori a fairly direct consequence of eq. 3 is the existence of a generator for the one parameter multiplicative (semi-) group of the bare densities:

$$\varepsilon_\lambda = e^{\Gamma_\lambda} \qquad (4)$$

where Γ_λ is its generator, still with the homogeneity scale $l = l_0/\lambda$. Γ_λ is a certain operator whose main properties (especially its asymptotic behavior, $l \to 0$ or $\lambda \to \infty$) we will analyze. Γ_λ should in *some sense* (see below) become independent of λ, i.e. approach its limit Γ as the homogeneity scale approaches zero. For positive values γ of Γ_λ, divergence of ε_λ occurs as λ tends to ∞, hence such values correspond to (algebraic) orders (γ) of singularity. Conversely negative values correspond rather to (algebraic) orders of regularity. Nevertheless for brevity, we will frequently keep the expression "singularity" (instead of "regularity") for both cases. Note we are studying a whole family of measures defined by just one density, this is the reason why our notation doesn't reduce to the very specialized notation[2] (α, f(α)) introduced by Halsey et al. (1986), where they refer to a single dimension (the dimension d of the embedding space) and the corresponding specialized measure. Hence, α =d-γ is the order of singularity of the d-dimensional Lebesgue measure, whereas γ is the order of singularity of the corresponding <u>density</u> of the measure, and f(α)=d-c(γ). As soon as this generator does not reduce (Schertzer and Lovejoy, 1983, 1984) to only two values $\gamma^+ > 0$ and $\gamma^- = -\infty$ (the once celebrated "β-model" (Novikov and Stewart, 1964; Mandelbrot,

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[1] "Weak" refers to the type of convergence of the process and of statistical estimators of the process.
[2] Do not confuse this α with Lévy index α used below

1974; Frisch et al., 1978) corresponding to the alternative of dead (λ^{γ}=0) or alive (and $\lambda^{\gamma+}$>1) sub-eddies, the *pure singularity orders* γ^- and γ^+ lead to the appearance of *mixed singularity orders*. In particular, as soon as $\gamma^- > -\infty$ (the "α-model"), mixed singularities of different orders γ, are built up step by step (cf. fig. 3) and are bounded by γ^- and γ^+ ($\gamma^- \leq \gamma \leq \gamma^+$, γ^- and γ^+ corresponding then to the alternative of weak (1>λ^{γ} >0) or strong ($\lambda^{\gamma+}$>1) sub-eddies. In other words, as pointed out by Schertzer and Lovejoy (1983), leaving the simplistic alternative dead or alive ("β-model") for the alternative weak or strong ("α-model") leads to the appearance of a full hierarchy of levels of survival, hence the possibility of a hierarchy of dimensions of the set of survivors for these different levels. In this α-model (as in more elaborate ones) the different orders of singularities (or survival levels) define the multiple scaling of the (one-point) probability distribution :

$$\Pr(\epsilon_\lambda \geq \lambda^\gamma) \approx N_\lambda(\gamma)/N_\lambda \approx \lambda^{-c(\gamma)} \tag{5}$$

where $N_\lambda(\gamma)$ is the number of occurrences of singularities with order greater than γ, N_λ is the total number of events examined. We temporarily postpone discussion on the accuracy of the approximations indicated in eq. 5 -e.g. the sub-multiplicity problem[1] already discussed by Schertzer and Lovejoy (1987a, b) and point out the convenient empirical analysis technique to measure the the probability distribution multiple scaling (PDMS, introduced by Lavallée et al. in this volume) in order to estimate $c(\gamma)$:

$$c(\gamma) \approx -\text{Log}_\lambda \Pr(\ (\text{Log}_\lambda(\epsilon_\lambda) \geq \gamma\ ) \tag{6}$$

Multiple scaling is obviously not restricted to one-point distributions (the latter being incomplete statistical descriptors of a field), indeed we need to know the behaviour of the joint n-point probability distribution (for the n-position vectors $(x_1, x_2, ..x_n)$=x). It suffices to consider a n-dimensional vector $\underline{\gamma}$ =($\gamma_1, \gamma_2, ..\gamma_n$) -instead of a scalar γ- with corresponding codimension $c_n(\underline{\gamma})$, and we should have the n-dimensional multiple scaling of the (n-dimensional) probability distribution:

$$\Pr(\{\epsilon_\lambda(x_i) \geq \lambda^{\gamma_i}\}_{i=1,n}) \approx \lambda^{-c_n(\underline{\gamma})} \tag{7}$$

The hierarchy (n$\rightarrow\infty$)of the codimensions $c_n(\underline{\gamma})$ would be sufficient to assess the statistical behaviour of the field and we will discuss the interpretation of the $\underline{\gamma}$ as extensions of phase portraits in section 4.3. However, the behaviour of $c_n(\underline{\gamma})$ is very sensitive to the distances between the n points x_i and the scale ratio λ. It will be often easier to consider the characteristic functional, as in Schertzer and Lovejoy (1987a, b), which is even more general than the n-dimensional extension considered here (see also Appendix C).

3.2. Wild singularities and the sampling dimension:

We would like to insist on the interest of the above formulae (eq.5-7) in gaining insights into different fundamental aspects of multiple scaling. Obviously singularities will prevent convergence in the usual sense, i.e. even if the ϵ_λ are smooth functions (for a given λ), they do not admit a function as their limit. Indeed, their limit will rather be defined by the limit of the fluxes (i.e. integrals of the density) over different sets. One may note also that N_λ will be proportional to λ^d -the number of boxes, size l_0/λ, required to cover the relevant region of the embedding space (which can be fractal) of dimension d (integer or not)- multiplied by the number (N_i) of realizations (e.g. images) examined. Hence, when $c(\gamma)$ is smaller than d it has a rather immediate meaning of a codimension[2] = d-d(γ); $N_\lambda(\gamma) \approx \lambda^{d(\gamma)}$, where d($\gamma$) (>0) is the dimension of the fraction of the space occupied by the singularities of order greater than γ on "nearly" each realization.

Larger values of $c(\gamma)$ (>d, i.e. "negative dimensions": d(γ)=d-c(γ)<0), which have often been disregarded, correspond to more rare events: singularities of orders which "nearly" never appear on a realization, but nevertheless give overwhelmingly important statistical contributions since they prevent

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[1] This leads to log corrections ignored in Eq. 5, hence the sign $\approx$.
[2] In particular, in the case of the β-model there is a unique codimension c, characterizing the fraction of the space occupied by the alive sub-eddies. The parameter β is λ^{-c}.

convergence of (statistical) moments as shown below! This is the reason we call them "wild singularities". At first glance they seem to correspond to negative dimensions, sometimes mysteriously called "latent dimensions". However, there is no mystery at all, since $c(\gamma)$ still has a meaning of a codimension: no longer in an individual realization, but in the subspace of the (infinite dimensional) probability/phase space that our finite sample size enables us to explore as a finite dimensional cut. Indeed the dimension of this subspace can be estimated as $d + d_S$, where d_S is called the "sampling dimension" (at scale l_0/λ)- and is estimated by writing the number of images (or realizations) N_i as λ^{d_S}. Indeed when $c(\gamma)$ is smaller than $d+d_S$, γ occupies a fraction of the accessible subspace having dimension $d(\gamma) = d + d_S - c(\gamma)$. Of course, increasing the number of images, hence the sampling dimension, allows us to more readily encounter higher order singularities occupying a fraction of the accessible subspace, with well defined dimension $(d(\gamma) = d + d_S - c(\gamma) > 0)$. The corresponding mathematical subtlety underlying the important difference between cases $c(\gamma)<d$ and $c(\gamma)>d$, is the "almost surely" or not properties, the latter do correspond to extremely rare events.

Although extremely rare, the wild singularities will be of overwhelming importance since they will prevent convergence of all moments of (high enough) orders. Indeed, the smoothing introduced by integrating the density over a set A with dimension D (to obtain the flux through A) may be sufficient to ensure the convergence for low order statistics, but not for orders higher than a critical order h_D of divergence. Let us point out this rather immediate consequence of eq. 5, by introducing first the trace (paralleling the definition of the trace of the density operator in Quantum Statistical Mechanics, see below) of the hth power of the flux Π_λ over an (averaging) set A of dimension D (with integration performed with resolution l_0/λ on A_λ, which denotes the set A measured with the same resolution):

$$tr_{A_\lambda} \varepsilon_\lambda^h = \int_{A_\lambda} \varepsilon_\lambda^h \, d^{hD}x \approx \sum_{A_\lambda} \varepsilon_\lambda^h \, \lambda^{-hD} \tag{8}$$

a priori any singularity of order higher than D, may create divergences of the trace but are extremely rare (since their frequency of occurrence tends to zero as $\lambda^{-c(\gamma)}$). One may evaluate the importance of these by considering their statistics (the trace-moments introduced by Schertzer and Lovejoy (1987 a, b)) for an arbitrary singularity of order γ:

$$Tr_{A_\lambda} \varepsilon_\lambda^h = <tr_{A_\lambda} \varepsilon_\lambda^h> \geq N_\lambda(\gamma) \, \lambda^{h\gamma} \lambda^{-hD} = \lambda^{[h\gamma - c(\gamma)] - (h-1)D} \tag{9}$$

which diverges, for some orders of singularity, as soon as:

$$K(h) \geq (h-1)D \text{ or } K_D(h) \geq 0; \quad K_D(h) \equiv K(h) - (h-1)D \tag{10}$$

where:

$$K(h) \equiv \sup_\gamma [h\gamma - c(\gamma)] \quad \{hence: c(\gamma) \equiv \sup_h [h\gamma - K(h)]\}$$

or:

$$h = dc(\gamma)/d\gamma, \, K(h) = h\gamma - c(\gamma) \quad \{\gamma = dK(h)/dh, \, c(\gamma) = h\gamma - K(h)\} \tag{11}$$

On the one hand, eq. 11 corresponds to the Legendre transform of $c(\gamma)$ as pointed out by Parisi and Frisch (1985), Halsey et al. (1986) and as the resulting K(h) does correspond -by the method of steepest descent- to the exponent of the moment of the density of the flux (at least to first order, i.e. omitting logarithmic corrections):

$$<\varepsilon_\lambda^h> = \lambda^{K(h)} <\varepsilon_1^h> = e^{K(h)Log\lambda} <\varepsilon_1^h> \tag{12}$$

The Legendre transform establishes a well defined relation between orders of singularities and orders of moments. It is worth noting that it is straightforward to obtain the n-point statistics for $x = (x_1, x_2,..., x_n)$ by replacing the scalar h by the vector $\mathbf{h} = (h_1, h_2,..., h_n)$ and K(h) by $K_n(\mathbf{h})$:

$$< \prod_{i=1}^{n} \varepsilon_\lambda(x_i)^{h_i}> = \lambda^{K_n(h)} < \prod_{i=1}^{n} \varepsilon_1(x_i)^{h_i}> \tag{13}$$

As for $c_n(\gamma$, $K_n(h)$ will depend sensitively on the distances betweeen the n points x_i and λ. It will far more general (and easier) to consider the characteristic functional $K(f)$ of the generator (see following subsection and appendix C).

Note that conservation in ensemble average of the flux requires conservation of densities ($<\varepsilon_\lambda>=<\varepsilon_1>$) thus $K(1)=0$. On the other hand, as pointed out by Schertzer and Lovejoy (1983), the divergence rule, eq. 10, introduces a hierarchy of critical codimensions $C(h)$, simply defined as:

$$C(h) \, (h-1) = K(h) \tag{14}$$

since the former divergence rule (eq. 8) can be rewritten ($\lambda \to \infty$, $\varepsilon_\lambda \to \varepsilon$):

$$\mathrm{Tr}_A \, \varepsilon^h = \infty \,, \, D<C(h), \text{ i. e. } h>h_D, \, C(h_D) = D \quad \{\gamma_D=dK(h)/dh|_{h_D}\} \tag{15}$$

where h_D is the critical moment order, and γ_D the critical singularity order (the wild singularities correspond to $\gamma>\gamma_D$) at which divergence of trace moments of the flux on the set A of dimension D occurs. For $h_D>1$ it implies the divergence of the usual statistical moments, since:

$$\mathrm{Tr}_{A_\lambda} \, \varepsilon_\lambda^h \leq <\Pi_\lambda^h(A)> \text{ any } h \geq 1 \tag{16}$$

Conversely, as discussed more thoroughly by Schertzer and Lovejoy (1987b), convergence of statistical moments of order h (h>1) is assured by the convergence of the hth trace moment; for h<1 divergence of the trace moment implies degeneracy of the flux (the set A has a so small dimension ($D<C_1 \equiv C(1)$) that almost surely the flux is null). We thus obtain a twin divergence rule for the trace moments (represented in fig. 4) implying non-degeneracy of the flux ($h \leq 1$) and divergence of the flux ($h \geq h_D>1$). Note that non-degeneracy of the flux implies conservation of the ensemble average flux[1]:

$$<\varepsilon_\lambda> = <\varepsilon_1> = 1 \text{ and } D > C_1 \, (\equiv C(1)) \Rightarrow \, <\Pi_\lambda(A)> = <\Pi_1(A)> \, (\equiv \int_A d^D x) \tag{17}$$

Note that C_1 ($\equiv C(1) = K'(1)$, due to eq. 14) is at the same time the codimension of singularities contributing to the average (h=1) and the order of these singularities, since by virtue of Legendre transform it is the fixed point of $c(\gamma)$:

$$c(\gamma) = \gamma \Rightarrow \gamma = C_1 \, (\equiv C(1) \equiv K'(1)) \tag{18}$$

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[1] As it corresponds to a "martingale "property, it assures a "weak measurable" convergence of the process (see Schertzer and Lovejoy 1987b for discussion).
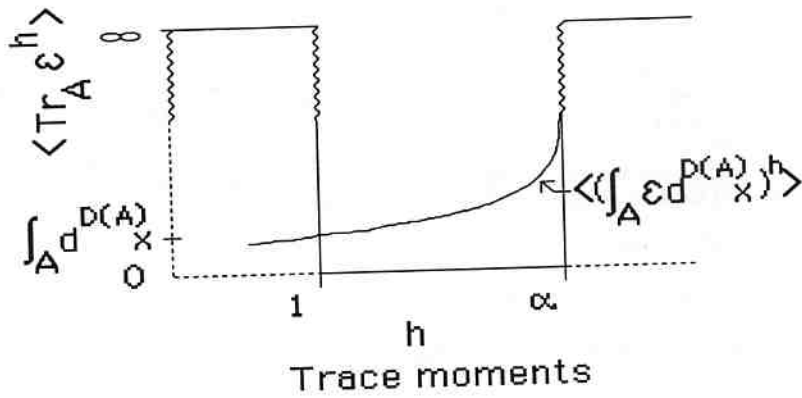
Fig. 4: the twin divergence of trace-moments.

3.2. Characteristic functionals and Fluxdynamics

Multiple scaling (for the statistical moments) corresponds to the fact that unlike the β-model $K(h)$ is no longer linear ($= C_1(h-1)$) and depends on a whole hierarchy of codimensions $C(h)$ ($\neq C_1$, for $h \neq 1$). Since the first (Laplacian) characteristic function (or moment generating function) $Z_\lambda(h)$ and second characteristic function (or cumulant generating function) $K_\lambda(h)$ of the generator Γ_λ, are by definition:

$$Z_\lambda(h) = e^{K_\lambda(h)} = <e^{h\Gamma_\lambda}> \quad (\equiv <\varepsilon_\lambda^h>) \tag{19}$$

multiple scaling corresponds to algebraic divergence ($\lambda \to \infty$) of $Z(h)$ and thus to logarithmic divergence of $K_\lambda(h)$ ($\approx K(h) \log\lambda$, see eq. 12), a fundamental property which we will exploit below. This property is in fact far more general if we consider not only the n-point characteristic function $K_{n,\lambda}$ (eq. 13), but the characteristic functional $K_\lambda(f)$ on "any test" function f of the generator (in eq. 13, we have considered the particular case: $f(x) = \Sigma_{i=1,n} h_i \delta_{x_i}$):

$$Z_\lambda(h) = e^{K_\lambda(f)} = <\exp(\int_A h(x)\Gamma_\lambda(x)d^D x)> \tag{20}$$

Note here, we are dealing with characteristic functions or functionals in the Laplace sense, since $Z_\lambda(h)$ or Z_λ are obtained by Laplace transform (instead of Fourier transform) of the probability distribution. In order to make connections with statistical physics, $-\Gamma_\lambda$ can formally be considered as an Hamiltonian[1] (H_λ) and h as the inverse of temperature ($h=1/T$, the Boltzmann constant being set equal to 1), Z_λ is called a partition function and the "free-energy" (F_λ) would correspond to $K_\lambda(h)/h$. More generally (in Statistical Quantum Mechanics), the "density operators" $\rho_\lambda = e^{-H_\lambda/T}$ (corresponding to $\varepsilon_\lambda^h = e^{h\Gamma_\lambda}$) are considered along with their trace over different sub-spaces of states, each trace corresponding to a partition function. The densities ρ_λ and ε_λ are both defined on a fairly abstract space (e.g., in quantum mechanics the space of wave functions). The trace moments correspond to the average ensemble of the trace of the density operator integrated over an (ordinary or fractal) set A, this integration corresponds to the linear action of an operator generated by N_λ. This is the analogue of the operator for the number of particles; here it is rather the

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[1] As done in random energy models (Derrida and Gardner, 1986; Gardner and Derrida, 1989)

generator of the fractal set A, seen at resolution l_0/λ, i.e. over which the "boxes" where we integrate the flux:

$$\varepsilon_\lambda^h \; (= e^{h\Gamma_\lambda}) = \rho_\lambda \; (= e^{-H_\lambda/T}); \qquad\qquad 1_{A_\lambda} d^D x \approx e^{-N_\lambda};$$

$$Z_\lambda(h) = Tr_A(e^{-N_\lambda/T} e^{-H_\lambda/T}) = Tr_A(\varepsilon_\lambda^h) \qquad\qquad (21)$$

on the space of measures of (compact) supports (for the different sets A, 1_{A_λ} is the indicator function for the set a at resolution λ).

If we consider now the ensemble average of the fluctuations of the operator N_λ itself (i.e. we are averaging over a certain subspace of the random measures of (compact) supports) we define a "grand ensemble" partition function $Z_{G,\lambda}(h)$:

$$Z_{G,\lambda}(h) = Tr(e^{-N_\lambda/T} e^{-H_\lambda/T}) = \lambda^{G_\lambda(T)} \qquad\qquad (22)$$

which makes explicit in a rather formal manner the crucial problem of observations obtained by integration on a scale ($l'=l_0/\lambda'$) much larger than the homogeneity scale of the process ($l=l_0/\lambda$): the possible non-commutation between $N_{\lambda'}$ and H_λ (especially when $\lambda \gg \lambda'$), thus the possible divergence of moments for dressed fluxes ("dressed" by the observation). We may also understand the divergence of moments as a phase transition, i.e. "solidification" by extreme localization in phase space of the contributions (of wild singularities) to high order moments (low temperatures 1/h) The second characteristic function of the trace moment $K_D(h)/h$ is rather the equivalent of a chemical potential and is simply related to K(h) by:

$$K_D(h) = K(h) - (h-1)D = (C(h)-D)(h-1) \qquad\qquad (23)$$

One may note that if the observation sets A are multifractal sets their own nonlinear characteristic function $K_A(h)$ will intervene instead of (h-1)D ($\equiv K_A(h)$ in the monofractal case). On the other hand, since c(γ) characterizes the logarithm of the probability distribution of Γ_λ, it corresponds to the entropy (S_λ) (of the state γ), and indeed the Legendre duality between $K_\lambda(h)$ and c(γ) corresponds to the same (and more familiar) duality between $F_\lambda(T)/T$ and $S_\lambda(E)$ (the conjugate variables being 1/T and the energy E). Let us emphasize that in both cases, this property simply results from the fact that the Laplace transform of the probability distribution[1] (or conversely of the partition function) reduces to a Legendre transform of the exponents. In order to develop a nonconstructivist approach, which we call "fluxdynamics", we consider ε *per se* (the limit ε of the ε_λ, at zero homogeneity scale, λ going to infinity) as a linear operator on the measures (converting the D-volume, D being the dimension of A, integer or not, into the flux over the set A). However, we need to investigate some basic properties of this limit and its generator.

4. SINGULAR STATISTICS, TYPES OF CONSERVATION AND CHAOS

4.1 "Hard" (wild) vs. "soft" multifractality

The divergence of moments is a wild statistical behaviour very far from "soft" statistics (e.g. Gaussianity, quasi-Gaussianity...), and corresponds to "hyperbolic" (algebraic) fall-off of the probability distribution:

$$Pr(|X| \geq s) \approx s^{-\alpha} \; (s \gg 1) \Rightarrow any \; h \geq \alpha : <|X|^h> = \infty \qquad\qquad (24)$$

Among these "hyperbolic" random variables some are rather well defined, since they are mostly (but surprisingly!) generalizations of Gaussian laws. These are the Lévy stable random variables ($0<\alpha<2$) satisfying "generalized central limit theorems", hence involved in additive processes as discussed in subsequent sections and especially with the help of Appendix A which deals with a particular class of them. We used the expression "hyperbolic intermittency" (Schertzer and Lovejoy, 1985) to describe the effect of

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[1] It also implies the convexity of K(h) (or F((T)/T, hence of c(γ) (or S(E)).

this strong variability for a wider range of α (i.e. $\alpha \geq 2$) and we pointed out that this divergence is a general consequence of multiplicative processes and that the corresponding critical order of divergence $\alpha = h_D$ (theoretically, determined by eq. 15) has no absolute bound. Waymire and Gupta (1985) have used the expression "fat-tailed" for such (asymptotically algebraic) distributions, and "long-tailed" for the log-normal law, to distinguish these distributions from standard exponential "thin-tailed" distributions. In the preceding section we showed that hyperbolic behaviour is expected from averaging a multifractal field over a set with too small a dimension D. Its value has been empirically estimated in a variety of meteorological fields: $h_D \approx 5$, for temperature (Lovejoy and Schertzer, 1986a, b; Ladoy et al., 1986), $h_D \approx 1.66$ for changes in storm integrated rainrates (Lovejoy, 1981), $h_D \approx 1.06$ in radar reflectivity factors of rain (Schertzer and Lovejoy, 1987), and respectively $h_D \approx 5$ and 3.33 for wind speed and potential temperatures, $h_D \approx 1$ for the Richardson number (Schertzer and Lovejoy, 1985)...

An important consequence of the divergence of statistical moments is that the usual estimation procedures no longer work efficiently, but rather exhibit spurious (or pseudo-) scaling exponents. Indeed, these methods rely heavily on the law of large numbers which blows up due to the statistical divergences. Fig. 5 shows how the appearance of spurious scaling can be quite misleading. The classical estimation leads to bounded codimensions C(h)... even though the (well understood) simulated field has a linear C(h)! Conversely, clear understanding of spurious scaling can be used to explain most of the behaviour of certain data. For instance we argue (Schertzer and Lovejoy 1983, 1984, 1985; Lovejoy and Schertzer, 1986a) that the presumed critical moment order $h_D \approx 5$ for wind speed, may well explain the overall behaviour of the observed scaling exponents of the structure functions of the velocity field collected by Anselmet et al. (1984)!
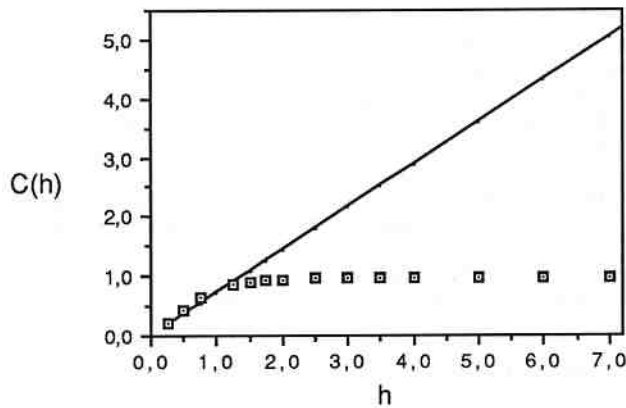

Fig. 5 : Illustration of the consequences of spurious scaling of the "dressed" quantities for the estimated C(h), that stay bounded for large values of h, in disagreement with the linear behaviour of the theoretical C(h) given by the continuous curve The estimated C(h) is obtained by trace moment analysis of 2000 independent samples of density fields induced by log-normal ($\alpha = 2$) multiplicative cascade process, the scale ratio $1 = 2^{10}$.

A direct consequence of the hyperbolic behaviour of the dressed densities $\varepsilon_{\lambda,D}$ (obtained by D-dimensional averaging, at scale l/λ) is that their singularity codimensions $c_D(\gamma)$ are quite different from their bare counterparts $c(\gamma)$, since they become linear for orders greater than the critical singularity order γ_D:

$$c_D(\gamma) \approx c(\gamma_D) + h_D(\gamma - \gamma_D), \ \gamma \geq \gamma_D \qquad (25)$$

this is an immediate consequence of hyperbolic behaviour as described by eq. 24, as well as from the corresponding divergence of the characteristic function ($K_D(h) = \infty$, $h \geq h_D$) and the fact that the Legendre

transform breaks down[1] for linear functions. Conversely, for the same reasons, $K(h)$ becomes linear as soon as there is an upper bound (γ_0) of the singularity order:

$$c(\gamma) \to \infty, \text{ when } \gamma \to \gamma_0 \Leftrightarrow K(h) = \gamma_0 h \ (h \gg 1), \text{ hence } C_\infty \equiv \lim_{h \to \infty} \{K(h)/(h-1)\} = \gamma_0.$$
(26)

4.2 Canonical vs. microcanonical conservation

Hyperbolic behaviour is expected only for singularities of order greater than the dimension of the averaging set A. It obviously can't occur if we are imposing a much more strict conservation than conservation of ensemble average such as a strict conservation on A of the flux in each realization. This follows from the fact that in the latter case we have:

$$\varepsilon_\lambda \, \lambda^{-D} \leq \Pi_\lambda(A) = \Pi_1(A)$$
(27)

Paralleling, once again classical thermodynamics, one can speak respectively of canonical conservation (or cascade) in the former case, and micro-canonical conservation (or cascade) in the latter (see for instance[2] Benzi et al. (1984), Pietronero and Siebesma (1986), Sreenivasan and Meneveau (1988)). The micro-canonical conservation assumption has many defects: not only are we usually dealing with open systems (as in thermodynamics), but this assumption turns out to be quite demanding and restrictive. In the framework of scale invariance, it requires *strict conservation at every scale*, so we can even speak of a "*pico-canonical*" assumption: strict conservation is implied not only on the largest scale of A, but on the smallest scale due to scaling behaviour of the process! Hence, we have rather sharp distinctions between *hard (wild) multifractality* allowed by canonical conservation, and *soft multifractality* implied by micro- (in fact pico-) canonical conservation. Furthermore, one may note that micro-canonical conservation refers to a given dimension: it no longer holds on sets with dimensions smaller than the characteristic dimension of the micro-canonical conservation. Hence, micro-canonical conservation is at the same time too precise and too vague. In contrast, realizations of a canonical process can be understood as low dimensional cuts of very high dimensional processes (the strict canonical case corresponds to phase spaces of infinite dimension).

4.3 Stochastic chaos vs. deterministic chaos

Considering the n-points statistics at points $(x_1, x_1,...,x_n)=x$, via the n-dimensional singularity vectors $\gamma = (\gamma_1, \gamma_2,..., \gamma_n)$ with conjugate vectors of moments order $h = (h_1, h_2,..., h_n)$, we are in fact exploring the (stochastic) phase space by using "phase portraits". Such protraits have been often considered in the very particular framework of deterministic chaos (e.g., Grassberger and Procaccia (1983)) for time series $\{\varepsilon(t), \varepsilon(t+\tau), \varepsilon(t+2\tau),..., \varepsilon(t+n\tau)\}$ where the time lag τ is of no fundamental significance if scaling is observed. In this approach, finite dimensionality D of the attractor is infered if:

$$n \gg 1 \quad n - c_n(\gamma) \approx D$$
(28)

Obviously the condition that n should be large is very demanding because we need an enormous number N of points ($N \gg n$) to obtain a reasonable estimate of $c_n(\gamma)$, hence it may easily require a prohibitive number of samples (see, for instance, the theoretical estimates of Essex, in this volume). Indeed, numerous doubts have been raised (Grassberger (1986), see also Nerenberg et al. and Viswanathan et al. in this volume) concerning some preliminary results (with n of the order 10, and N of the order of several hundred) which reported very low dimensionality ($D \approx 3$) of climatic and other geophysical attractors (e.g., Nicolis and

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[1] This is a direct consequence of the geometrical interpretation of the Legendre transform as the envelope of the tangents.
[2] In fact a micro-canonical version of the α-model is often called a "random β-model" (Benzi et al., 1984) or "p model" (Sreenivasan et al., 1988). The former expression (which refers to the fact that the fraction of the space occupied by active sub-eddies is randomly chosen) is somewhat misleading, since the "β-model" is already a random model...

Nicolis (1983)). These results could be profitably reexamined with a straightforward generalization of sampling dimension for singularity vectors.

It is important to call attention to two basic facts. On the one hand, low values of D do not imply deterministic chaos (even if the converse is by definition true!). Indeed, an obvious counter-example is the brownian motion which in any space (of dimension $d \geq 2$) will be concentrated on a stochastic attractor of dimension 2. On the other hand, and more fundamentally, deterministic chaos (as the result of a set of coupled ordinary differential equations) was originally studied as an oversimplified version of the chaos generated by partial differential equations (Lorenz 1963), which a priori has an infinite dimensional phase space. It is then not logically consistant to take too literally what the simplified model tells us, otherwise we may be lead to simplistic conclusions. Indeed, this image and the "real world" share the common property that only a very small fraction of the phase space is active, thus corresponding to nonzero codimensions $c_n(\gamma)$ for any n. In this respect, eq. 28 is too strong a requirement. Let us briefly point out one of the major differences between the deterministic image and the stochastic one presented by multiplicative processes: in the latter, time is treated on an equal footing with space (even if we introduce scaling anisotropy in the space-time domain, hence scaling anisotropic space/time transformations) in the sense that the variability in time and space will be of the same type, whereas in the former case the process will be regular with respect to time (i.e. the time variable is very smooth, contrary to the space variables).

In our opinion, the little empirical evidence collected to date indicates no more than that only a small fraction of the phase space is active. And due to the preceding arguments, it seems more natural to interpret time series as low dimensional cuts of a very high dimensional process (in the canonical case discussed in the preceding section), i.e. stochastic chaos.

5. (H, C_1, α) UNIVERSAL MULTIFRACTALS

5.1 Characterization of the generator

Since a multiplicative process is a one-parameter group, the characterization of its generator is fundamental. Corresponding to the definition of the Γ_λ (eq. 4), we have - at least formally - for the limit Γ of the Γ_λ ($\lambda \to \infty$):

$$\varepsilon \equiv \text{limit } _{\lambda \to \infty}(\varepsilon_\lambda) = e^{\Gamma} \tag{29}$$

One may also note that we have (corresponding to eq. 4) the following "dynamical" (and somewhat formal) equation for the ε_λ:

$$\partial \varepsilon_\lambda / \partial \lambda = \gamma_\lambda \, \varepsilon_\lambda \; ; \gamma_\lambda = \partial \Gamma_\lambda / \partial \lambda \tag{30}$$

where γ_λ is the infinitesimal generator. This equation gives the cascade dynamical sense when we are studying a cascade on a space-time domain[1]. As discussed by Schertzer and Lovejoy (1987a, b) and expanded here in Appendix C, the generator must satisfy four main properties:

i) Γ is a random noise, with infinite band-width $[1, \infty]$. The bare or (finite resolution) generator Γ_λ should rather be understood as the corresponding filtered noise restricted to the wave-number band $[1, \lambda]$ of Γ.

ii) the second characteristic function (or cumulant generating function) $K_\lambda(h)$ of the (bare) generator Γ_λ has a logarithmic divergence ($\lambda \to 0$) in order to assure multiple scaling, i.e.:

$$K_\lambda(h) \approx \text{Log}(\lambda) \, K(h) \tag{31}$$

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[1] On the space-time domain, the scaling is usually strongly anisotropic as we will discuss more thoroughly in section 6.

iii) in order to obtain some finite moments of positive order ($Z_\lambda(h)$ and $K_\lambda(h)<\infty$ for $h>0$), the probability distribution of positive fluctuations of the bare generators Γ_λ must fall off more quickly than exponentially.

iv) the generator needs to be normalized ($K_\lambda(1)=0$) in order to assure (canonical) conservation of the flux.

It turns out that properties i) and ii) correspond to the fact that the (generalized) spectrum $E_\Gamma(k)$ of the generator should be then proportional to the inverse of the wave-number:

$$E_\Gamma(k) \propto k^{-1} \tag{32}$$

since the characteristic function will correspond to its integral and we will have the appropriate log divergence. Such noises are often called "1/f noises" or "pink noises". Usually, one considers only Gaussian noises or quasi-Gaussian noises. We have already indicated that there is no fundamental reason to restrict our attention to quasi-Gaussianity, and thus we consider hyperbolic noises. Indeed, among the hyperbolic noises, Lévy stable noises ($0<\alpha<2$) are particularly important, since they define a family of universal generators as we will discuss latter. However, the third property, which is due to the fact the moments of ε_λ are Laplace transforms of the probability density of Γ_λ, lead us to restrict our attention to extremely unsymmetric hyperbolic noises, since we can accept a hyperbolic fall-off of the probability distribution only for the negative fluctuations of Γ_λ. Considering Lévy-stable noises (or hyperbolic noises with $0<\alpha<2$), one has to generalize the notion of spectrum (the usual spectrum diverges, since it is a second order moment) as discussed by Schertzer and Lovejoy (1987a, b). The fourth property is easy to satisfy since if Γ'_λ is not yet normalized, we can obtain a normalized generator Γ_λ by:

$$e^{\Gamma_\lambda} = e^{\Gamma'_\lambda} / <e^{\Gamma'_\lambda}> \tag{33}$$

however, as we will later insist this method of normalization is not unique, since a fractional integration may achieve the same result.

Note that the properties of the generators stressed above are on the one hand quite different from the usual properties of Hamiltonians, since they have a 1/f spectrum and the equivalent of negative energies (the positive singularities). On the other hand, they give a precise definition of multiple scaling, especially by specifying the necessary and sufficient properties of the second characteristic functional, which might be called the "free flux" since it is the analogue of the free energy (as outlined in section 3.2).

5.2 The conservative $(0, C_1, \alpha)$ universal canonical multifractals

5.2.1. The universal generators obtained by densification of scales. In this subsection we will concentrate our attention on conservative fluxes, and we will show that they indeed depend only on two fundamental parameters (C_1, α). In the following section we will consider generalizations to nonconservative fields (such as the temperature and velocity fields, ...).

Just as in additive processes, one may look for universality classes in the sense that whatever generator is used (here the flux generator, the infinitesimal increment in additive processes) under repeated iteration - through (renormalized) multiplication or addition - it may converge to a well defined limit which depends on relatively few of its characteristics. Appendix A first recalls the classical (but not well enough known) results for additive processes associated with the generalized central limit theorem, here the classes and "basins of attraction" are primarily[1] defined by the Lévy index α, the critical order of moments of the increments (moments of order $h>\alpha$ diverge).

One has to be careful about the definitions of convergence and universality, since it has been obscured by some misplaced claims (Mandelbrot, 1989) that such universality cannot exist in multifractal processes. Indeed, it is easy to check that repeated multiplication corresponding to a process with fixed discretization

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[1] There are two subsidiary Lévy parameters which are fixed in our case: the 'location parameter' (fixed by the normalization constraint) and the 'skewness' (set to its extremal value -1 by the condition iii, as explained below). The third subsidiary parameter, the 'scale parameter' is defined by C_1.

(i.e. a fixed elementary ratio of scale $\lambda_0 > 1$) fails to create a simplifying convergence to universal generators (e.g. the α-model remains an α-model), and it seems that this is the reason why Kolmogorov (1962) postulated a lognormal behaviour, without claiming convergence[1] to it. However, if we are discussing continuous cascade processes, i.e. processes which have an infinite number of cascade steps over any *finite* range of scales (i.e. the elementary ratio of scales $\lambda_0 \rightarrow 1_+$), we are facing quite a different limiting procedure, namely the *densification of scales at fixed overall scale range*, instead of *increasing the range at fixed density of scales*. Indeed, the continuous processes are obtained from a discrete model (finite number of discrete steps over the given ratio of scales) by introducing more and more steps up to an infinity of infinitesimal ones and while fixing some properties (e.g. the variance of the generator on this given scale ratio). This densification of scales can be done explicitly either with Fourier techniques (Schertzer and Lovejoy, 1987a, 1988, Wilson et al. , this volume) or with wavelets[2]. Obviously while such properties are mathematically best studied directly on the generator, we should also establish the physical relevance of doing so. Indeed, -generalizing the test field method introduced in homogeneous turbulence by Kraichnan (1971)- we may introduce new intermediate scales first as rather passive components, advected by the others, and then include them in the whole set of "active" scales. In this respect, the passive scalar example (studied by Schertzer and Lovejoy (1987a,1988), Wilson et al., (this volume)), is illustrative: the density of the flux (φ) controlling the passive scalar diffusion is a product of powers of densities of the energy flux (ε) and the scalar scalar variance flux (χ) -mainly from dimensional arguments, we have: $\varphi = \chi^{3/2} \varepsilon^{-1/2}$. In the first step, χ (corresponding to ε on the new intermediate scales) and ε can be considered as independent but of the same type. In the second step we identify φ as a more complete ε. Hence, we are multiplying densities by densities, or simply adding generators to generators...

Now, we have to investigate which classes of generator are stable and attractive under addition and such that for the corresponding density ε_λ will at least converge for some positive order moments (i. e. the probability density of the generator admits a Laplace transform as already discussed). Either we examine those Lévy stables -usually studied in a Fourier framework (e.g. Lévy (1924, 1925, 1954), Gnedenko and Kolmogorov (1954), Gnedenko (1969), Feller (1971), Zolotarev (1986), Gupta and Waymire (1990)- which also satisfy a Laplace transform or we directly study (as done in Appendix[3] A) the generalized central limit theorem in the Laplace framework which is much more immediate and natural.

In any case, it is clear that the restriction imposed by Laplace transforms is that we require (as condition (iii) already discussed) a steeper than an algebraic fall-off of the probability distribution for the (positive) orders of singularities, hence with the exception of the Gaussian case ($\alpha = 2$), we have to employ strongly asymmetric, "extremal" Lévy laws, as explicitly emphasized by Schertzer et al. (1988). In our case, we are not considering random variables but noises, however the same characterization is relevant (characteristic functionals are involved rather than characteristic functions).

Let us examine the classes of universal generators (ranging from $\alpha = 2$ down to $\alpha = 0$), recalling that the corresponding characteristic functional $K(h)$ and codimension functions $c(\gamma)$ estimated by Legendre transform, are (Schertzer and Lovejoy, 1987a,b), since h^α / α and $\gamma^{\alpha'} / \alpha'$ are Legendre dual with $1/\alpha + 1/\alpha' = 1$, $0 \leq \alpha \leq 2$, $-\infty < \alpha' \leq 0$ or $2 \leq \alpha' < \infty$):

$$\alpha \neq 1: K(h) = \frac{C_1 \alpha'}{\alpha} (h^\alpha - h) \quad \text{(only for } h \geq 0 \text{ when } \alpha < 2; = \infty \text{ for } h < 0\text{)};$$

$$\alpha = 1: K(h) = C_1 \, h \, Log(h) \tag{34}$$

and (restricted to increasing branches when $\alpha < 2$, since $dc/d\gamma = h$):

$$\alpha \neq 1: c(\gamma) = C_1 \left(\frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha} \right)^{\alpha'} \quad (dc/d\gamma > 0 \text{ when } \alpha < 2)$$

$$\alpha = 1: c(\gamma) = C_1 \exp(\frac{\gamma}{C_1} - 1) \tag{35}$$

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[1] Yaglom (1966) seems to be less cautious on this point.
[2] Work is in progress on this, in collaboration with P. Brenier.
[3] One may note that classically only the case $0 < \alpha < 1$ is treated by Laplace transforms, Appendix A extends the result for $1 \leq \alpha \leq 2$. See also Schertzer and Lovejoy (1990) for discussion of this point.

We recall that C_1 ($\equiv C(1) = K'(1)$) is the fixed point of $c(\gamma)$, being simultaneously the codimension of singularities contributing to the average and the order of these singularities (see eq. 18). We may introduce another convenient characteristic order of singularity :

$$\gamma_0 = -\frac{C_1\alpha'}{\alpha} \tag{36}$$

γ_0 is[1] either the lower bound ($\alpha>1$, $\alpha'>2$) of fractal singularities ($c(\gamma)=0$, i.e. singularities occupying all the space) or ($\alpha<1$, $\alpha'<0$) the upper bound of singularities ($c(\gamma)=\infty$, i.e. unreachable singularities). It is then also the slope of the tangent at the origin of K(h) ($\gamma_0=K'(0)$; $\alpha>1$) or of the asymptote ($\gamma_0=K'(\infty)$; $\alpha<1$). We may then rewrite eq. 35 ($\gamma\geq\gamma_0$ when $2\geq\alpha>1$; $\gamma<\gamma_0$ when $\alpha<1$) as:

$$\alpha\neq1,\ \gamma_0 = -\frac{C_1\alpha'}{\alpha},\ c_0=c(0):\ c(\gamma) = c_0\left(1-\frac{\gamma}{\gamma_0}\right)^{\alpha'} \tag{37}$$

One may note that the $c(\gamma)$ introduced here corresponds rather to the probability density, instead of the probability distribution. Both are equal when $c(\gamma)$ is increasing (e.g. for extreme singularities: $\gamma>>0$), but obviously decreasing $c(\gamma)$ (e.g. for extreme regularities: $\gamma<<0$) of the probability is offset for the probability distribution by its minimum value (see below the role of γ_0 for the Gaussian case). On the other hand, the $c(\gamma)$ don't coincide with the log of the probability density due to (at least!) some logarithmic terms (corresponding to sub-codimensions) which are missed by the Legendre transform, but are of no fundamental importance (as easily seen by considering the exact log of the probability density).

5.2.2. The five main universality classes. Let us review briefly the properties of the five main classes (α going from 2 to 0, hence α' going from 2 to ∞, then from $-\infty$ to 0), ranging from the Gaussian generator to the β-model, crossing three Lévy cases (see the corresponding fig. 6a and fig. 6b):

i) $\underline{\alpha=\alpha'=2}$: the Gaussian generator is almost everywhere (almost surely) continuous. K(h) and $c(\gamma)$ are parabolas, $c(\gamma)$ is tangent on the γ axis at $\gamma_0=-C_1$, C(h) is linear (= C_1 h).

ii) $\underline{2>\alpha>1\ (2<\alpha'<\infty)}$: the Lévy generator is almost everywhere (almost surely) discontinuous and is extremely asymmetric. The lower bound γ_0 (=-C_1 α'/α) of fractal singularities decreases from -C_1 to -∞, as α decreases from 2 to 1. The corresponding $c(\gamma)$ of the probability distribution, will remain on the γ axis for $\gamma\leq\gamma_0$, these singularities are space-filling (distributed on "fat fractals" which involve only sub-codimensions). It is also strongly asymmetric (even for the probability density, since $K_\lambda(h)=\infty$ for h<0). The wild singularities ($\gamma>>1$) give rise to a steeper algebraic branch than before ($c(\gamma) \propto \gamma^\alpha$, $\alpha'>2$).

iii) $\underline{1>\alpha>0\ (-\infty<\alpha'<0)}$ the generator is everywhere (almost surely) discontinuous, and is obtained in fact by a one-sided unnormalized generator hence the orders of singularities are bounded by γ_0 (thus decreasing, with α, from -∞ to C_1), which thus defines a vertical asymptote, and now the algebraic asymptote intervenes for the large orders of regularity ($\gamma\rightarrow -\infty$, $c(\gamma) \propto (\gamma - \gamma_0)^{-|\alpha'|}$). As the singularities are bounded, the same occurs for the hierarchy of critical codimensions C(h) of the different moments, since we can now always smooth out the highest singularity on a set A of high enough dimension D. Indeed γ_0 bounds also C(h) (see eq. 26), hence to obtain convergence of every (positive order) moment of the flux it suffices that: $D>C_\infty=\gamma_0=C_1|\alpha'|/\alpha$. We thus leave hard (wild) multifractality for soft multifractality.

iv) $\underline{\alpha=1\ (\alpha'=\pm\infty!)}$: it is the special in-between case ("extremal Cauchy[2]"), associated with the ambiguity on α' (note that γ_0 has opposite limits: $\gamma_0=\pm\infty$), this corresponds in fact to a special case of quasi-stability (or not strict stability) briefly outlined in Appendix A. Note that the curves K(h) and $c(\gamma)$ are nevertheless the limits ($\alpha\rightarrow 1\pm$, $\alpha'\rightarrow\pm -\infty$) of the two preceding cases, especially the former algebraic asymptotes of $c(\gamma)$ tend to exponential behaviour since: $(x/\alpha'+1)^\alpha \rightarrow e^x$ when $\alpha'\rightarrow\infty$.

v) $\underline{\alpha=0+\ (\alpha'=0-)}$: this limiting case corresponds to divergence of every statistical moment of the generator and seems at first glance very strange, but one of its representations is none other than the once

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[1] This is the negative of the former γ_0 introduced by Schertzer and Lovejoy (1987a, b). The change of sign is required to obtain directly the bounds of singularities/regularities as explained.
[2] The usual Cauchy variable is the symmetric stable Levy variable, with $\alpha=1$.

celebrated β-model ($\gamma^- = -\infty$, $\gamma^+ = C_1$ = limit $\alpha \to 0+(\gamma_0)$)! This fact in turn, shows clearly the peculiarities of the β-model, once thought to be a more or less crude approximation of intermittency ...
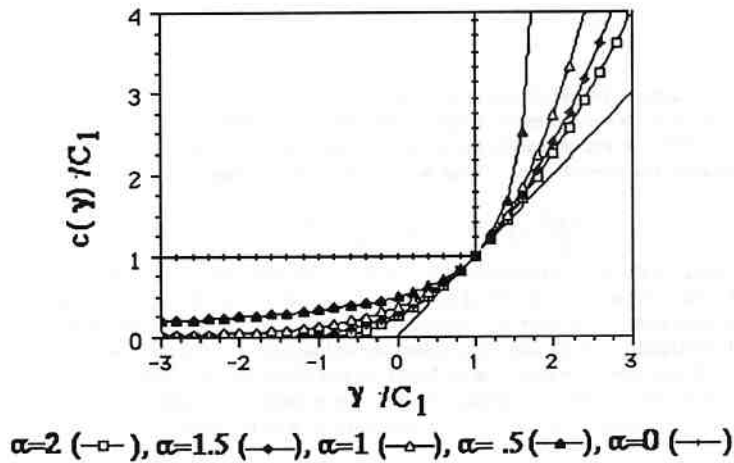

$\alpha = 2$ (—□—), $\alpha = 1.5$ (—•—), $\alpha = 1$ (—◇—), $\alpha = .5$ (—▲—), $\alpha = 0$ (———)

Fig. 6a: universal (bare) singularities codimension $c(\gamma)/C_1$ corresponding to the five classes; here $\alpha = 2, 1.5, 1, .5, 0$.
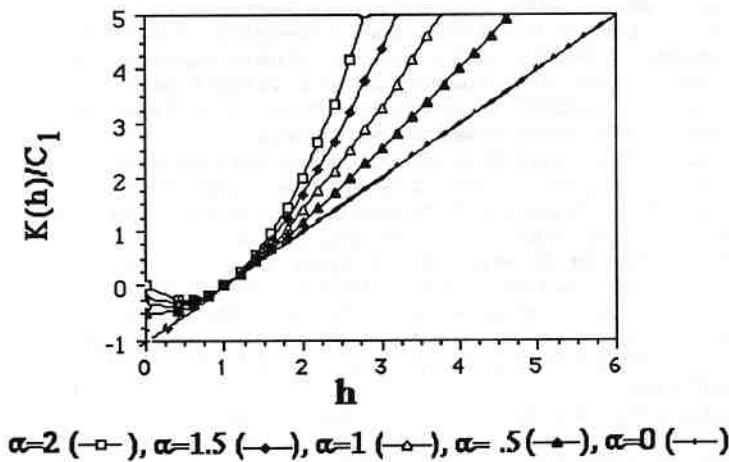
$\alpha = 2$ (—□—), $\alpha = 1.5$ (—•—), $\alpha = 1$ (—◇—), $\alpha = .5$ (—▲—), $\alpha = 0$ (———)

Fig. 6b: universal (bare) second characteristic function $K(h)/C_1$ ($\equiv h.F(h)/C_1$, $F(h)$ being the "free energy"), corresponding to the five classes; here $\alpha = 2, 1.5, 1, .5, 0$.

Let us point out briefly some consequences:
- the Lévy cases fill the gap between the two more or less classical cases the so-called lognormal ($\alpha = 2$) and the β-model, which now represent just two extremes of the whole spectrum of universal generators. The corresponding bare processes are log-Lévy, but not the dressed ones. The Lévy index α indicates how far we are from monofractality (the β-model) in the precise sense that it measures the convexity of $K(h)$ for $h \approx 1$ ($K''(1) = C_1 \alpha$).

- the very symmetric Gaussian case is the exception which assures the existence of negative order moments, on the contrary the asymmetric "extremal" character of the Lévy cases corresponds to the fact that we are "digging" wild regularities ("holes") with the algebraic extremes of the Lévy generator which preventing convergence of any negative order moment[1] (see the relevant fig. 7a, b).

- the links between regularity of the generator and the resulting flux are at first glance somewhat paradoxical since as α decreases the generator is more and more wild but the resulting fluxes more easily have finite positive order moments since the upper bound C_∞ of $C(h)$ ($\equiv\infty$ for $\alpha\geq1$; $\equiv\gamma_0<\infty$ for $\alpha<1$) decreases with α, leading to the finiteness of all moments for set A of dimension $D>C_\infty$. However, it is merely due to the fact that the wild behaviour of the generator is restricted to regularities (hence the particular problem of negative order moments since they interchange regularities with singularities and conversely). The β-model yields the extreme regularity since: $C_\infty = C_1 = C(h)$, any set A where the process is not degenerate, will have regular flux at all positive orders (but still none at negative orders!).

One may note that exact mathematical results have been obtained on the case $\alpha=2$ (Kahane, 1985, 1987) and $\alpha<1$ (Fan, 1989).

Fig. 7a : a Gaussian white-noise ($\alpha=2$), fluctuations are symmetric.
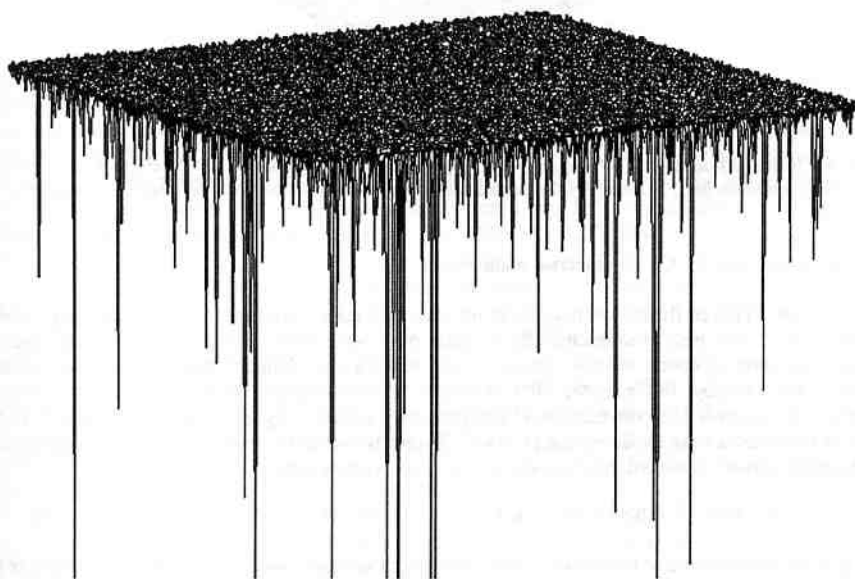

Fig. 7b: an extremal Lévy white-noise ($\alpha=1.5$), fluctuations are extremely asymmetric: only *negative* hyperbolic jumps are allowed for the generator, "digging" wild regularities ("holes")

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[1] Note that this may explain the many difficulties discussed in the physics literature connected with the estimation of moments of negative order of multifractal fields... since in general these moments don't exist!
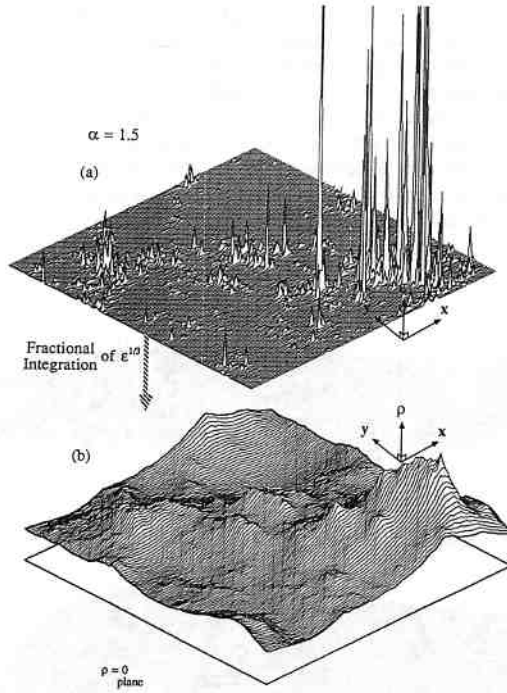
Fig. 8 : It shows from top to bottom, over $2^8 \times 2^8$ pixel grid, (a) the density ε of a conserved flux, obtained with a Lévy generator ($\alpha=1.5$), then (b) the associated concentration field obtained by fractional integration (of order 1/3).

5.3 Nonconservative (H, C_1, α) universal multifractals

From the conservatives fluxes, we may build up others by taking products of them or raising them to different powers. We may even fractionally integrate over them, which is especially desirable when for example, we want to obtain the concentration field itself, rather than the flux of the scalar variance (Schertzer and Lovejoy, 1987a, 1988, 1989; Wilson et al., this volume). However, by doing so, we will fundamentally add only one extra parameter[1] (the order of fractional integration) to our two basic C_1 and α. Indeed, a fractional integration/derivation of order[2] (b) on a power (a) of a conserved flux corresponds to an affine transformation on the orders of singularity (leaving $c(\gamma)$ invariant):

$$\gamma' = a\gamma - b \; ; \; h' = h/a \; ; \; K'(h') = K(h) - bh' \; ; \; c'(\gamma') = c(\gamma) \qquad (38)$$

staying in the same α-type of universality, but a and b both introduce deviations from conservation of the flux, hence the new parameter H. We can restrict our attention to transformations with a=1, since the

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[1] At least considering bare properties. Things are more involved when considering dressed properties: apparently equivalent processes may create quite different wild singularities.
[2] Integration when b is positive, derivation when negative. This explains the change of sign we made in comparaison with Schertzer and Lovejoy (1987a, 1990).

deviation $(K(a))$ resulting from raising to the power a the density (ε) of a conservative flux, corresponds at least formally[1] , to changing the order b of fractional integration to $H = b-K(a)$ on the conserved flux (obtained by fractional integration/derivation of order $-K(a)$ on the density ε^a).

Fig. 8 shows the two main steps needed to obtain a concentration field over a $2^8 \times 2^8$ pixel grid simulated ($a=b=1/3$) on a personal computer, fig. 8a shows the corresponding conserved flux (identified to ε), then the resulting concentration field after fractional integration (fig. 8b), see also Wilson et al. (this volume) for more discussion.

6. GENERALIZED SCALE INVARIANCE

6.1 Scaling anisotropy

We have already pointed out that the time-space domain is usually strongly (scaling) and anisotropic. This remark can be rendered a bit more obvious when considering formal scaling transformation on the velocity field (as done in Schertzer and Lovejoy (1987a, Appendix D2)) since (on purely dimensional grounds):

$$x \to x/\lambda, \ v \to v/\lambda^H \ \Rightarrow t \to t/\lambda^{1-H} \tag{39}$$

thus as soon as $H \neq 1$, we have space/time anisotropy (in case of homogeneous turbulence we would have $H=1/3^2$).

However, strong anisotropy is already present in the space domain with oriented forces such as buoyancy (due to gravity) as well as the Coriolis force (due to the earth's rotation). These forces, which may introduce anisotropic differential operators, e.g., a fractional differential operator with the order of differentiation depending on the direction instead of (isotropic) gradients, are responsible for the (fractional) differential stratification and rotation of the atmosphere respectively. For instance, in order to avoid the classical but untenable 2D/3D dichotomy between large and small scale atmospheric dynamics, we have proposed an anisotropic scaling model of atmospheric dynamics (Schertzer and Lovejoy (1983, 1984, 1985a,b, 1987a,b); Lovejoy and Schertzer (1985); Levich and Tzvetkov[3] (1985)). In this model, the anisotropy introduced by gravity via the buoyancy force results in a differential stratification and a consequent modification of the effective dimension of space, involving a new "elliptical" dimension (d_{el}, see below), with resulting anisotropic shears. In isotropy, $d_{el}=3$, while in completely flat (stratified) flows, $d_{el}=2$. Empirical and theoretical evidence is given indicating that for the horizontal components of the velocity filed d_{el} is rather the intermediate value $d_{el} \approx 23/9=2.5555...$ (and $d_{el} \approx 2.22...$ for the rainfield (Lovejoy et al., 1987)).

Indeed, the requisite scale changes T_λ can be far more general than simple magnifications or reductions. It turns out that practically the only restrictions on T_λ are that it has group properties, viz: $T_\lambda = \lambda^{-G}$ where G is the generator of the group of scale changing operations, and that the balls $E_\lambda = T_\lambda(S_1)$ (S_1 being the unit sphere) decrease with λ. In the simplest case of "Generalized Scale Invariance" ("GSI"), G is a matrix -"linear GSI" (Schertzer and Lovejoy, 1983, 1984) E_λ are self-affine ellipsoids (see fig. 9) rather than the self-similar spheres of the isotropic case ($G=1=$ identity), it already allows a tremendous variety of behaviour, since the only constraint on G turns out to be that every (generalized) eigenvalue of G has a non-negative real part (Schertzer and Lovejoy, 1985b), we can speak more concisely of positive (generalized) spectrum ($\sigma(G)$):

$$\sigma(G) \geq 0 : \inf_\mu [\operatorname{Re} \sigma(G)] \geq 0 \ ; \ \sigma(G) = \{\mu \in C \mid G - \mu 1 \text{ non-invertible on } C \times \Re^d\}. \tag{40}$$

anisotropic ("elliptical") scale ϕ_{el} is then defined by the volume of the E_λ (hence is a measurable property, rather than a metric property)

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[1] Indeed there is a priori no equivalence between the different ways of maintaining conservation of fluxes.

[2] This is indeed consistent with the value empirically measured in rain according to preliminary results (Lovejoy and Schertzer, this volume), and work in progress.

[3] They also pointed out the possible breaking of mirror symmetry for atmospheric dynamics, hence the importance of the associated helicity.

$$\phi_{el}{}^{d_{el}} (E_\lambda) = \phi^d (E_\lambda) = \lambda^{d_{el}} \phi^d (S_1) = \lambda^{d_{el}} \phi_{el}{}^{d_{el}} (S_1) \qquad (41)$$

where the effective dimension of the space, the "elliptical" dimension d_{el} of the space, is simply[1] the trace of G: d_{el}=Tr(G). This anisotropic framework allows rather straightforward extensions of Hausdorff measures and dimensions, still respecting the divergence rule (eq. 2):

$$\int_A d^{D_{el}} x = \lim_{\substack{\delta \to 0 \\ \phi_{el}(E_i) < \delta}} \inf_{A \supset \cup E_i} \sum_{E_i} \phi_{el}{}^{D_{el}} (E_i) \qquad (42)$$
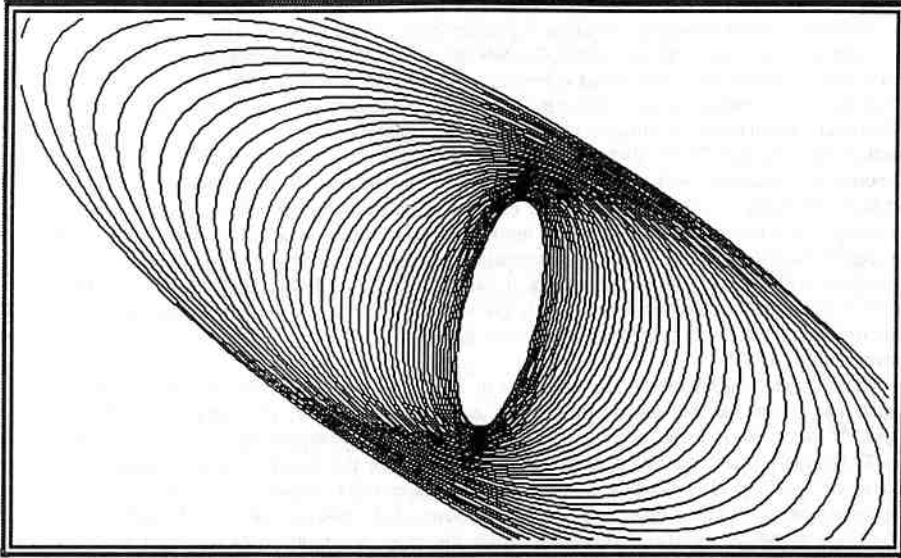
Fig. 9: Family of "ellipsoids" E_λ obtained by linear GSI. Due to linearity the E_λ remain convex.

In two dimensions it is rather convenient to use a representation of quaternions (Schertzer and Lovejoy, 1985b; Lovejoy and Schertzer, 1985) for the generator:

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[1] However on any Euclidean subspace, G needs to be appropriately normalized (as discussed by Schertzer and Lovejoy (1987b)).

$$1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \ I=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \ J=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \ K=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{43}$$

with the following commutation rules:

$$IJ = -JI = -K; \ JK = -KJ = I; \ KI = -IK = -J \ ; \ 1 = -I^2 = J^2 = K^2 \tag{44}$$

in this representation G is a linear combination of $1, I, J, K$:

$$G = d1 + cK + eI + fJ \ ; \ Tr(G) = 2d = d_{el} \tag{45}$$

I is a rotation generator, J and K are stratification generators (as seen by the commutation rules they corresponds to each other via a rotation I), and 1 remains of course the generator of isotropic contraction. Indeed, the effect of rotation dominates as soon as a is complex (negative a^2), otherwise stratification dominates (positive a^2):

$$a^2 = c^2 + f^2 - e^2 \ ; \ \ u = \log(\lambda) \ ; \ \det(G) = d^2 - a^2$$

$$T_\lambda = \lambda^{-G} = \lambda^{-d} \ [1 \cosh(au) - (G-d1) \sinh(au)/a] \tag{46}$$

in the stratification dominated case, the axis of the ellipsoids rotates only by $\tan^{-1}(e/a)$ when λ goes from 0 (infinite outer scale) to ∞ (zero inner scale), and interchange of minor and major axis occur at $\lambda=1$ (the sphero-scale, which corresponds to isotropy), yielding a total rotation of $\pi/2 + \tan^{-1}(e/a)$ for each of the axes). The condition of positivity for the spectrum is simply:

$$\sigma(G) \geq 0 \iff \det(G) \ \text{and} \ Tr(G) \geq 0 \iff d \geq 0 \ \text{and} \ d^2 \geq a^2 \tag{47}$$

thus always satisfied in the case of rotation dominance. Note we use the following fundamental identity for the commutators ($[X,Y] = XY - YX$) :

$$[\lambda^A, \lambda^B] = \lambda[A,B] \tag{48}$$

One may also note that reducing the scale by a factor λ via $T_\lambda = \lambda^{-G}$ in the physical space corresponds to magnification by the same scale ratio of wave-vector k in Fourier space by the transposed $\tilde{T}_{\lambda-1}$ of $T_{\lambda-1}$ since wave vector are conjugate via the scalar product (denoted $(\cdot,\cdot)$) of physical coordinates, indeed:

$$(\tilde{T}_{\lambda-1} \ k, T_\lambda \ x) = (k \ , T_{\lambda-1} T_\lambda x) = (k, x) \tag{49}$$

hence the generator of anisotropy in Fourier space is the transposed $\tilde{G}$ of G, in the quaternion representation (with obvious notation):

$$(\tilde{d}, \tilde{c}, \tilde{e}, \tilde{f}) = (d, c, -e, f) \tag{50}$$

6.2. Local scale transformations, nonlinear GSI

In the framework of linear GSI, the anisotropic scale transformations remains global, i.e. invariant for any time-space translation The same is true of multifractality, even if overly simplistic presentations of multifractals tend incoherently to speak of "local fractal dimension"[1]. However, we already argued for the indispensable necessity of using local scale transformations, as in the original (Weyl's) local gauge invariance. We then have to consider nonlinear GSI, with nonlinear scale operator T_λ. The main new feature is that the generator of the scale transformation becomes local and T_λ is only a semi-group. The general framework is that of differential manifolds, and on each tangent space we will return to consider a

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[1] Indeed, they often falsely present fractal dimensions as pointwise and/or scale dependent notions.

linear scale transformation which is the tangent application of T_λ. Hence, we will once again find the condition of a positive (generalized) spectrum (eq. 47), but here in each tangent space at every point of the manifold. As preliminary examples, we may consider nonlinear GSI (still on the plane) with a representation of quaternions (Schertzer and Lovejoy, 1985b, 1989; Lovejoy and Schertzer, 1985), hence with spatially variable coefficients (d,c,e,f). In order to obtain the image point x_λ (belonging to E_λ, the "ellipsoid" of scale reduced by factor λ) of a point x_1 (belonging to the sphero-scale S_1) we now have to solve the nonlinear differential equation:

$$dx_\lambda/d\lambda \quad = -G(x_\lambda)/\lambda \tag{51}$$

this leads to the striking figures, note due to nonlinearity the E_λ are no longer convex sets (see fig. 10).
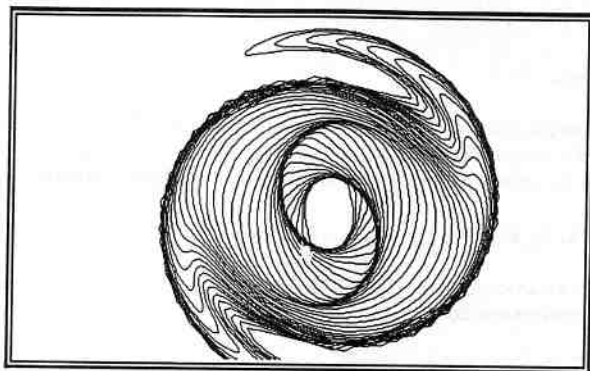

Fig. 10 : Family of "ellipsoids" E_λ obtained by nonlinear GSI . Due to nonlinearity the E_λ are no longer convex sets.

These preceeding considerations are not at all incidental, since the quaternions are nothing but one of the simplest examples of Lie algebras. Indeed, more generally we have to consider scale changing operators T_λ as forming a Lie group acting on a manifold, and their (local) generators in the associated Lie algebra. More specifically we have to consider the subset satisfying the condition of positive spectra, and the scale changing operators are then obtained by their exponentiation on the manifold, i.e. by integration of eq. 51. Doing so we indeed obtain a connection on this manifold (local properties get transportable on the global manifold, or at least on a part of it). The structure of the Lie algebra $\{G| G = \Sigma_i g^i G_i, g^i \in R \text{ or } C\}$ of the

generators depends primarily on its structure constants c_{ijk} given by the commutation rule (eq. 44 is a particular case):

$$[G_i, G_j] = c_{ijk} G_k \qquad (52)$$

The choice of the algebra depends on the symmetries other than scaling which we want to be respected (as a consequence of eq. 48, the generators S of these symmetries must commute with the generator G of the scale changing operator T_λ). In this respect, Lie groups lead to the important Casimir invariant which commutes with every generator of the group and in particular for compact groups they appear as the sum of the squares of the generators (e.g. the square of the angular momentum in orthogonal group O(3) in dimension 3), in the quaternion case, the corresponding invariant of this type is:

$$1^2 + I^2 + J^2 + K^2 = 2 \, 1 \qquad (53)$$

6.3 Stochastic GSI

We can now add a new and important ingredient: the random behaviour of the scale transformation T_λ, i.e. of its generator G which up to now were considered deterministic. This opens up a wide variety of possible behaviour. Indeed, we can understand it as being produced by the infinitesimal (and random) generator (γ, G), more specifically a conservative flux as resulting from the nonlinear integration on the manifold of (eq. 30), coupled with the nonlinear scale transformation (eq. 51):

$$\partial \varepsilon_\lambda(x_\lambda)/\partial \lambda = \gamma_\lambda(x_\lambda) \, \varepsilon_\lambda(x_\lambda) \; ; \; dx_\lambda/d\lambda = -G(x_\lambda)/\lambda \qquad (54)$$

But now G is random as γ. Of course the statistical interrelations between both are of prime importance. Note that the *linear stochastic* case is rather simple since $G(x_\lambda)$ corresponds simply to the action of a random matrix (still denoted G) on the vector x_λ. The constraints will be on the one hand those already discussed on the monotony of the ellipsoids E_λ (i.e., eq. 40) and on the other hand the statistical behaviour of G will be essentially the same[1] type as those of the singularities γ, except we have to consider a matrix pink noise. Once again, the decomposition into elementary components (such as quaternions) can be quite helpful (e.g. we have only to consider the random behaviour of (d, c, e, f) subject to $d^2 \geq a^2$ (eq. 47).

The extension to the nonlinear case is more or less straightforward still using decomposition of the Lie groups (for instance fig. 11 gives an example nonlinear stochastic GSI using quaternions). Note that the 1/f noise may also be obtained by the same kind of integration on the manifold of a white-noise, but of fractional order. In this respect, Fourier techniques[2] seem to be manageable with the use of the transposed generator $\tilde{G}$.

Understanding such a symmetry group raises highly important theoretical and empirical questions. On the one hand, it renders much more precise and abstract the question of the generation of nonlinear variability: we need no longer care much about more or less complex sets of coupled partial differential equations, but rather to find their generators (γ, G), the analog of the "action" to use the consecrated expression in physics. Furthermore, it may also be important to realize that such an approach is also very concrete, and may indeed extract valuable information through the large geophysical data sets which are more and more frequently available. Indeed, empirical determination of the generators may be sought in generalizing the elliptical dimension sampling, designed up to now to explore GSI (as done by Lovejoy et al. (1987)), since the theoretical arguments (Schertzer and Lovejoy, 1987a (Appendix B2)) leading to this method seems to have rather straightforward generalizations for nonlinear and/or stochastic GSI. Nevertheless, their concrete exploration will require in certain cases gargantuan amounts of data, particularly since geophysical multifractals are often hard (with wild singularities), not soft.

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[1] Deterministic GSI corresponds to a unique and homogeneous singularity for the generator of scale transformation (e.g. a unique matrice in the linear deterministic GSI.

[2] Although presumably under their localized version, i.e., the "wavelet" techniques.
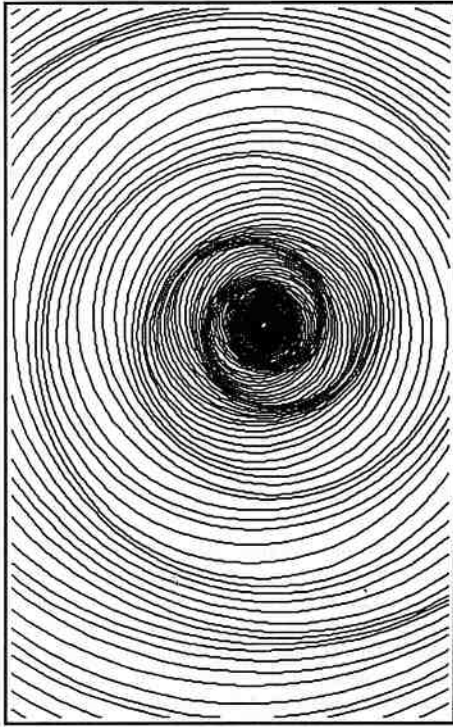
Fig. 11 : Family of "ellipsoids" E_λ obtained by stochastic nonlinear GSI . Due to nonlinearity the E_λ are no longer convex sets and their shapes randomly fluctuate scale by scale.

7. CONCLUSIONS

We sharpened the theoretical foundations of the singular statistics of multifractal fields. We discussed in a rather general manner the conditions of their appearance. We clarified the fundamental difference between "bare" and "dressed" properties at a given (non-zero) scale, i.e. the important differences between a process with small scale interactions cut-off and one with the full range of interactions. We thus emphasized the nontrivial behaviour of geophysical observables depending on the type of the process, as well as on the observation (both its scale and dimension). We also pointed out general properties of the generators of multifractal fields and their links with notions of classical statistical physics, emphasizing their particularities.

We demonstrated the existence of three-parameter (H, C_1, α) universality classes of the generic multifractal processes. These three fundamental exponents characterize the degree of flux non conservation flux (H), the deviation of the mean field from homogeneity (C_1), and the deviation of the process from monofractality $(0 \leq \alpha \leq 2)$. These three exponents correspond to fundamental properties of the process, for instance: to the fractional order of integration over a conservative flux (H), the sparseness of average singularities measured by the codimension (C_1), and the type and regularity of the generator ($2 - \alpha$ indicates the deviation from Gaussianity). The five main sub-classes of these (H, C_1, α) universal generators are:

Gaussian generator ($\alpha=2$), unbounded Lévy generator ($2>\alpha>1$), bounded Lévy generator ($1>\alpha>0$), a very special in-between case Cauchy generator ($\alpha=1$) … as well as the once celebrated β-model ($\alpha=0$)!

We also investigated the anisotropic and/or local scaling symmetry. This leads us to generalize the idea of scale invariance far beyond the familiar self-similar (or even self-affine) notions. We pointed out some of the basic ingredients of the nonlinear Generalized Scale Invariance (or anisotropic/local/stochastic scale transformations).

8. ACKNOWLEDGMENTS

APPENDIX A: GENERALIZED CENTRAL THEOREM, EXTREMAL LEVY STABLE GENERATORS
(in collaboration with R. Viswanathan[1])

A.1. Fixed points for sums of independent and identical distributed (i.i.d.) random variables and central limit theorems

In this sub-section we review briefly the classical features of Lévy stable variables, stressing that these variables emerge as generalizations of Gaussian variables, which then are seen to be a very particular case of Lévy stable variable. Indeed, we are interested in the universal stable and attractive fixed points of renormalized sum of independent and identical distributed (i.i.d.) variables, consider first the stable fixed points of renormalized sum ($\overset{d}{=}$ means equality in probability[2]):

$$X_i \overset{d}{=} X_1 \quad i=1,n \text{ are stable points under renormalized sum iff} \tag{A1}$$
for any (integer) n (≥ 2), there exists a (positive) b_n and a (real) a_n
$$\sum_{i=1,n} X_i \overset{d}{=} b_n X_1 + a_n$$

The well-known Gaussian case corresponds to:

$$<X_1^2> \leq \infty \Rightarrow b_n = n^{1/2}, a_n = (n-1)<X_i> \tag{A2}$$

hence the assumption of finite variance which has been considered as so "natural" that it has become a kind of dogma. The usual central limit theorem corresponds simply to the limit n→∞ in eq. A1:

$$X = \lim_{n \to \infty} [(\sum_{i=1,n} X_i) - a_n]/b_n \tag{A3}$$

even though the X_i on the r.h.s. are not necessarily assumed to be Gaussian, the X will be, hence the Gaussian law is attractive :

$$<X^2> = <X_i^2> \leq \infty \Rightarrow b_n = n^{1/2}, a_n = n <X_i> - <X> \tag{A4}$$

one may note that the average $<X>$ can be arbitrarily set to 0 (as usual) due to the expression of the a_n.

Lévy (1925,1954) generalized the Gaussian case by relaxing the hypothesis of finite variance for the X_i (which implies finiteness of every statistical moment for the limit itself) introducing on the contrary an

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[1] Now at Mitsubishi, London, England.
[2] Note in order to be consistent, the use of this symbol requires that the corresponding variables should be mutually independent.

order of divergence (α, $0<\alpha<2$) for the moments of the X_i (α is often called the Lévy index) which satisfies either A1 or A3:

$$h<\alpha \Rightarrow <|X_i|^h> < \infty \text{ and } h'\geq\alpha \Rightarrow <|X_i|^{h'}> = \infty \quad \text{i. e. : } \Pr(|X_i|\geq s) \sim s^{-\alpha} \tag{A5}$$
$$\text{A1 or A3} \Leftrightarrow b_n = n^{1/\alpha} \text{ (and, if } \alpha>1; a_n= n<X_i> - <X>)$$

the variables X_i are very often termed "hyperbolic variables" (or even "hyperbolics") due the algebraic fall-off of their probability distribution tails, which are themselves sometimes termed "fat tail" due to their (unusually) important contribution. Hence, the Lévy stable variables are the stable and fixed points of (renormalized) sums of i. i. d. hyperbolic variables. Note that for $\alpha\leq1$, as the mathematical expectations of the X_i and their sums are divergent, the required recentring is a bit more involved than that indicated for $1<\alpha\leq2$ (subtracting out the averages) and will be only discussed later. The very special Gaussian case appears as the (extreme) regular case $\alpha=2$, after a highly critical transition since for any $\alpha=2-\varepsilon$ (ε arbitrarily small) we have divergence of all orders greater than α whereas all divergences are suppressed for $\alpha=2$. One may note that the stable variables were introduced in a slightly different form (Lévy 1925, 1954) who addressed the stability under any linear combination:

$$X_i \stackrel{d}{=} X_2 \text{ are said stable under linear combination iff} \tag{A6}$$
$$\text{for any (positive) } b_1 \text{ and } b_2, \text{ there exists (real) a and (positive) b, such that:}$$
$$b_1X_1+ b_2 X_2 \stackrel{d}{=} b X_1+a$$

It is rather easy to check (by induction) that eq. A1 and eq. A6 are equivalent. One may furthermore note that "any n" in A1 can be equivalently reduced to "n=2,3" due essentially to the density of numbers 2^j 3^k among positive numbers, j and k being relative integers (Zolotarev, 1986).

Note that there exists a sub-class of stable variables which do not require recentring (i.e. a=0, -it is rather obvious in the cases $1<\alpha\leq2$). These special cases (to which, Lévy (1925) restricted his study) are frequently called "strictly stable" (Feller 1971, Zolotarev 1986), more rarely the complementary cases (i.e., a≠0) are called (Lévy, 1954) "quasi-stable".

A.2. Characteristic functions of Lévy laws

With a few notable exceptions ($\alpha=2$, 1, 1/2) the probability distributions of Lévy stable variables are not expressible in a closed form. However, the second (Fourier or Laplace) characteristic function is easily expressible due to the basic properties of stability. K(h) is the logarithm of the first characteristic function Z(h), i.e. the (Fourier or Laplace) transform of the probability distribution dP(x)) and the argument h is purely imaginary in case of Fourier (h=ih'), real in case of Laplace (we will discuss later the restrictive conditions under which such a transform is possible) and a complex number (h+ih') in the case of Fourier-Laplace (or two-sided Laplace) transform:

$$e^{K(h)} = Z(h) = <e^{hX}> = \int e^{hx} \, dP(x) \tag{A7}$$

the fundamental property of the fixed point (eq. A1) or the equivalent form (eq. A6) are easily transposed for the characteristic functions:

$$X_i \stackrel{d}{=} X_1 \text{ (i=1,n) of second characteristic function K(h)} \tag{A1'}$$

are stable points under renormalized sum iff. for any (integer) n (≥2), there exists a (positive) b_n and a (real) such that:

$$a_n n K(h) = K(b_n h) + a_n h;$$

and:

$$X_i \overset{d}{=} X_2 \text{ of second characteristic function } K(h) \qquad (A6')$$

are said stable under linear combination iff. for any (positive) b_1 and b_2 there exists a (real) a and a (positive) b such that:

$$K(b_1 h) + K(b_2 h) = K(bh) + ah$$

the limit theorems correspond to (K_i second characteristic function of the X_i, K second characteristic function of X):

$$K(h) = \lim_{n \to \infty} n[K_i (h/b_n) - h a_n/n b_n] \qquad (A3')$$

We may infer, especially from A6', that these characteristic functions, up to the recentring term, should be of power law form whose exponent α is bounded above by 2 (the extreme regular Gaussian case) and must of course be positive (to avoid divergence at h=0). It is obvious that the case $\alpha=1$ is very special since this hyperbolic exponent becomes equal to the (linear) recentring term exponent, we may guess that conflict and compensation between the two terms will introduce logarithm corrections. For other values of α the (linear) recentring term has no importance and we can restrict our attention to strictly stable cases ($a_n=0$). As h^α (or h Log(h) for $\alpha= 1$) is not analytic (except once again in the Gaussian case $\alpha=2$) in the complex plane, we clearly expect on the one hand divergence of moments of order greater or equal to α, on the other hand that the inferred "power law form" may be rendered more precise in order to obtain a second characteristic function (e.g., Z(h) must be positive definite in case of Fourier transform (Bochner's theorem) or absolutely monotone in case of Laplace transform). Indeed, considering the symmetric (or symmetrized) probability distributions lead to the following law (partially) known ... since Cauchy 1853, but essentially obtained by Lévy (1925) of Fourier characteristic functions, since K(h) must be also symmetric:

$$K(ih') = -\lambda_\alpha |h'|^\alpha \qquad (A8)$$

with the obvious Gaussian case when $\alpha=2$, and Cauchy case when $\alpha=1$. The λ_α characterizes the width of the probability distribution as in the Gaussian case ($\lambda_2 = \sigma^2/2$) but doesn't correspond to the evaluation of an α-moment, since it diverges, but rather the rate of divergence of this moment.

A.3. Particular properties of extremal Lévy laws

The symmetric case corresponds to limit of sums of symmetric hyperbolics or mixing with equal probability (p=q=1/2) positive (with probability p) and negative (with probability q) one-sided hyperbolic distributions (concretely: just multiplying by a random sign positive one-sided hyperbolics). Asymmetric cases correspond to p≠q. It is time to stress that if we want to have a Laplace transform. we can only consider extremal (asymmetric) hyperbolics, simply because algebraic fall-off could not tame an exponential divergence, hence we restrict here our attention to negative hyperbolics (p=0, q=1). However, note that the corresponding limits, the extremal Lévy stables, are not always one sided (in the sense of having only positive or negative values) -precisely one sided probability distributions only occur for $0<\alpha<1$.
In order to assess different statements, it is interesting to consider the characteristic functions under their "canonical form" i. e:

$$K(h) = \int (e^{hx} -1 +hx) \, dF(x) \approx Z(h) -1+a'h \qquad (A9)$$

dF is called the Lévy canonical measure or the spectral measure, which needs not be a probability measure. Indeed $\int dF(x)$ is not bounded, with the exception of the Gaussian case, and we introduced a recentring term a' just to cancel the second term of the exponential development (near h=0) when needed (for $1<\alpha<2$). This form corresponds on the one hand to a first order term development of $\log[1+(Z(h)-1)]$, which is the only term kept in the limit theorem besides recentring and normalization of K(h) (i.e. K(0)=0 corresponding to $\int dP(x)=1$). On the other hand, it corresponds to the limit of (Poisson) random (renormalized) sums of i. i.

d. variables, instead of uniform sums as discussed up to now. Indeed considering the characteristic function of the limit $(n \to \infty)$ of the Poisson compound probability distribution generated by renormalized sum of n i. d. d. hyperbolic variables (as given by A1) lead us to a new version of the limit theorem (earlier stated under the forms A3 and A3') keeping in mind that the second characteristic function $K(h)$ of the Poisson compound probability distribution is $c(Z(h)-1)$, where c is the parameter of the Poisson process, $Z(h)$ the first characteristic function of the generating probability distribution):

$$K(h) = \lim_{n \to \infty} K_n(h)$$

$$K_n(h) = n[Z_n(h) - 1]$$

$$Z_n(h) = Z_i (h/b_n) \exp(- h a_n/n b_n) \tag{A3''}$$

the "canonical form" (eq. A9) is obtained by slightly recasting this equation to take directly into account arbitrary centering directly on K (no longer on Z_n or Z).

Let us consider the negative hyperbolic generation of extremal Lévy stable by negative hyperbolic, it suffices to put $dF(x) \propto 1_{x<0} x^{-\alpha} dx/x$ ($1_{x<0}$ being the indicator function of the negative x) and with repeated use of the identity:

$$\Gamma(\beta) = z^{-\beta} \int_0^\infty e^{-zt} t^\beta dt/t \ ; \ Re(z) \geq 0 \tag{A10}$$

and integrations by parts, we obtain easily for $dF(x) = 1_{x<0} C(2-\alpha) x^{-\alpha} dx/x$:

$$\alpha \neq 1: \ K(h) = C h^\alpha \Gamma(3-\alpha)/\alpha(\alpha-1); \ \alpha \neq 1$$
$$\alpha = 1: \ K(h) = C h \log(h) \tag{A11}$$

one may note that the expressions for the corresponding Fourier transforms are a bit more complex (i.e. Fourier transforms, so convenient for symmetric laws, are inconvenient for extremal (and more generally for asymmetric laws), on the contrary Laplace is only fitted for the extremal, useless for the others):

$$\alpha \neq 1:$$
$$K(h) = |h|^\alpha \ C \ [\Gamma(3-\alpha)/\alpha(\alpha-1)] \ [\cos(\pi\alpha/2) + -i(\text{sgn}(h) (p-q) \sin(\pi\alpha/2)]$$
$$\alpha = 1:$$
$$K(h) = -|h| \ C \ [\pi/2 + i \ (\text{sgn}(h)(p-q)\log|h|] \tag{A12}$$

As a last general remark, one may note (from eq. A11 or eq. A12) it is only in the case of extremal stable distribution ($p-q=\pm1$) that an analytic extension on the whole complex plane of K is possible (but with a cut along the ray $\arg(h) = -3\pi/4$, as it is for $\alpha=2$, or that the double-sided Laplace transform applies only to extremal stable variables.

APPENDIX B: HAUSDORFF MEASURES, FRACTIONAL INTEGRALS AND DERIVATIVES

B.1. Hausdorff measures

We first recall the geometric definition of Hausdorff measures and dimensions for a compact set A. The Hausdorff measure relative to a convex function g (m_g) of A is defined as:

$$m_g(A) = \lim_{\lambda \to \infty} m_{\lambda, g}(A)$$

$$m_{\lambda, g}(A) = \inf_{\substack{A \supset \cup B_i. \\ \text{diam}(B_i) < \delta = l_0/\lambda}} \sum_{\{B_i\}} g(\text{diam}(B_i)) \qquad (B1)$$

where $m_{\lambda, g}(A)$ can be understood as the Hausdorff measure (relative to the function g) with resolution λ, and corresponds to the infimum over all possible coverings by balls B_i such that the diameter (diam(B_i)) is smaller than the resolution scale $\delta = l_0/\lambda$.

The D-dimensional Hausdorff measure of A, $m_D(A) = \int_A d^D x$ is obtained in the particular case $g(t) = t^D$ and the Hausdorff dimension (D) of the set A is obtained by the divergence rule:

$$\int_A d^{D'} x = \infty, \text{ for } D' < D \; ; \qquad \int_A d^{D'} x = 0, \text{ for } D' > D \qquad (B2)$$

One may note that the D-measure of A is not necessarily finite and non-zero. In order to obtain a finite and non-zero value of the D-like measure of A, one may have to change slightly of the basic function g of the Hausdorff measure introducing on iterates of the logarithm:

$$g \rightarrow g(t) = t^{D(A)} (\text{Log}_1 t)^{\Delta 1} (\text{Log}_2)^{\Delta 2} \ldots \Rightarrow 0 < m_g(A) < \infty$$
$$\text{Log}_i = \text{Log}(\text{Log}_{i-1}); \; \text{Log}_1 = \text{Log} \qquad (B3)$$

the logarithmic correction exponents Δ_i on the i-th iterate of the logarithm, are called sub-dimensions and correpond to the fact that the volume of an elementary ball (l^D) is now 'corrected' by factors of the type $[\text{Log}_i (l/l_0]^{\Delta i}$. For instance, Mauldin and William (1986) have shown that such log-corrections arise in evaluating the dimension of the support of the β-model with the help of a simple box-counting algorithm, and Lovejoy and Schertzer (this volume) give indications of the presence of log corrections in experimental analysis using the same algorithm.

On the other hand, it is worthwhile to note that in fact that the D-dimensional Hausdorff measure of A can be defined direcrtly in a measure sense (for balls of topological dimension d), i.e.:

$$\phi^d(B_i) = \int_{B_i} d^d x$$

$$\int_A d^D x = \lim_{\lambda \rightarrow \infty} \inf_{\substack{A \supset \cup B_i. \\ \phi(B_i) < \delta = l_0/\lambda}} \sum_{\{B_i\}} \phi^D(B_i) \qquad (B4)$$

This measure definition allows us to deal with more complex integrands (such as h powers of the flux "density"[1] in the trace moments, Schertzer and Lovejoy (1987a)) or anisotropic scaling (sect. 6.1) by replacing balls B_i by their anisotropic analogues and henceforth introducing "elliptical Hausdorff dimensions" instead of the isotropic ones.

B.2. Fractional integrals and derivatives

These correspond to extensions to non-integer orders (H) of integrations (I^H) or differentiations (D^H). These extensions are rather straightforward in Fourier space for 1-dimensional (scalar) functions, since integrations -up to a constant of integration discussed below- or differentiations of integer order n, correspond respectively to division or multiplication by $(ik)^n$, where k is the wave number (Fourier transforms of physical space quantities will be denoted by a circonflex ($\hat{\ }$)):

$$\hat{I}^{-H}(f) = \hat{D}^H(f) = (ik)^H \hat{f}(k)) \qquad (B6)$$

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[1] which is no longer a function in the limit of zero homogeneity-scale length.

In the usual physical space for non-integer H, we obtain an ordinary (i.e. of integer order n) derivation (D^n, positive n) or integration (I^{-n}, negative n) of a convolution:

$$I^{-H}f = D^H f = 1/\Gamma(n-H)) \, D^n[f * x^{n-H-1}] \tag{B7}$$

here Γ is the Euler gamma function and intervenes as in Appendix A where we already encounter integration of the same type. Eq. B7 is more general than B6, since directly written in physical space, but introduces ambiguities in the definition on non-integer integration or differentiation[1] because they will clearly depend on the domain of defintion of the convolutions (cf. e.g. Ross (1975)). The same techniques can be extended to functions defined on R^d, however the analysis becomes more complex because various combinations of partial derivatives are now possible. Nevertheless, one can consider the following strongly isotropic extension:

$$I_d^{-H}f = D_d^H f = 1/\Gamma(n-H)) \, D_d^n[f * |x|^{n-H-d}];$$
$$\hat{I}_d^{-H}(f) = \hat{D}_d^H(f) = |k|^H \, \hat{f}(k)) \tag{B8}$$

which in fact, corresponds to fractional powers of the Laplacian (or of the Poisson solver):

$$I_d^{-H}f = D_d^H f = (-\Delta)^{H/2} \tag{B9}$$

Extensions for $R^{d'}$-valued (vectorial or even tensorial) fields can be also considered, but the variety of possible differential operators still increases, although this variety can be reduced by considering certain symmetries as previously.

As a final and important remark for applications, one must take into account the modification of the average of the integrand by the fractional integration, i.e. consider closely the role of the constant of integration. When working in the Fourier space (on a periodic box of size L), it corresponds to the modification of the Fourier component at wave number k=0. Indeed, splitting f in its average ($\bar{f}$) and fluctuating parts (f') we obtain:

$$f = \bar{f} + f':$$
$$Im(f) = Im(\bar{f}) + Im(f'); \qquad Im(\bar{f}) = \bar{f} \, L^m/m \tag{B10}$$

These considerations are especially important when we must preserve the sign of a field after fractional integration, e.g. the sign of the extreme fluctuations of an extremal "white" Lévy noise, in order to obtain extremal "pink" Lévy noise by fractional integration (as needed for a Lévy generator of a multiplicative cascade process, see also Wilson et al., this volume).

APPENDIX C: CHARACTERISTIC FUNCTIONALS AND UNIVERSAL GENERATORS

C.1. Characteristic functionals of generators

We consider the finite exponential increments Γ_λ as noises concentrated in the wave number band $[1/l_0, \lambda/l_0]$ (filtered out or strongly damped for other wave numbers) obtained by filtering their limit Γ:

$$\Gamma_\lambda = \Gamma * F_\lambda \tag{C1}$$

where F_λ is the filter and * denotes the convolution product corresponding to a product in Fourier space. This is an explicit definition of the scale of homogeneity $l = l_0/\lambda$, although at a more sophisticated

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[1] It is important to note that fractional derivations are obtained with the help of integrations, thus are depending in fact on "constants of integration" (contrary ot their integer counter parts)!

mathematical level Γ needs to be defined as a limit (Eq. 27) and not the Γ_λ as restrictions of Γ. Fourier transforms of physical space quantities (e.g. $\Gamma_\lambda(x)$) will be denoted by a circonflex (e.g., $\hat{\Gamma}_\lambda(k)$), k indicating a wave vector and k (=$|k|$) the corresponding wave vector, and we will take for sake of notational simplicity, $l_0=1$. Hence, we have in the Fourier space:

$$\hat{\Gamma}_\lambda(k) = \hat{\Gamma}(k)\,\hat{F}_\lambda(k) \tag{C1'}$$

If we strictly filter out any wave number ouside the range [1,λ], $\hat{F}_\lambda$ is simply the indicator function $\hat{1}_{S_\lambda}$ of the spherical (hyper) volume S_λ delimited by the spheres of radius 1 and λ, both centered at the origin of Fourier space ($\hat{1}_{S_\lambda}(k) = 1$ if $1 \leq k \leq \lambda$, 0, otherwise). However, its Fourier transform F_λ corresponds to Bessel functions. The second characteristic functional (or cumulant generating functional) K_λ of the generator Γ_λ is defined by the scalar product with any "test function f", as:

$$K_\lambda(f) = \log\left(\langle \exp \int \Gamma_\lambda(x)\, f(x)\, d^d x \rangle\right) \tag{C2}$$

We have not only a similar definition (with the hermitian product) in (complex) Fourier space for the characteristic functional of $\hat{\Gamma}_\lambda(k)$, denoting by $\hat{f}^*$ the conjugate of any complex function $\hat{f}$ (and the same for noises, e.g. $\hat{\Gamma}_\lambda^*$):

$$\hat{K}_\lambda(f) = \log\left(\langle \exp \int \hat{\Gamma}_\lambda(k)\, \hat{f}^*(k)\, d^d k \rangle\right) = \log\left(\langle \exp \int \hat{\Gamma}_\lambda^*(k)\, \hat{f}(k)\, d^d k \rangle\right) \tag{C3}$$

but, as the scalar product (on L^2 space, and more generaly the duality product[1] between the dual spaces L^α and $L^{\alpha'}$, $1/\alpha + 1/\alpha' = 1$) is conserved by Fourier transforms, we have also the equality between these two characteristic functionals:

$$K_\lambda(f) = \hat{K}_\lambda(\hat{f}) \tag{C4}$$

In order to have multiple scaling, the characteristic functionals K_λ and $\hat{K}_\lambda$ of Γ_λ and $\hat{\Gamma}_\lambda$, respectively, must be logarithmically divergent, namely:

$$K_\lambda(f) = \hat{K}_\lambda(\hat{f}) = (\text{Log } \lambda)\, K(f) \tag{C5}$$

at least for the test functions corresponding to n-points statistics, i.e. :

$$f(x) = \Sigma_{i=1,n}\, h_i \delta_{x_i}; \quad \hat{f}(k) = \Sigma_{i=1,n}\, h_i e^{i(k.x_i)}; \tag{C6}$$

C.2. Universal generators

In order to satisfy eq. C5 , it is rather obvious that Γ_λ should be a coloured noise. Indeed, as we have shown that universal generators should be either Gaussian ($\alpha=2$) or extremal Lévy-stable ($0<\alpha<2$), let $\Gamma_{\alpha\lambda}$ be defined as

$$\Gamma_{\alpha\lambda} = s^* \gamma_\alpha * F_\lambda; \quad \hat{\Gamma}_{\alpha\lambda}(k) = \hat{s}(k)\, \hat{\gamma}_\alpha(k)\, \hat{F}_\lambda(k) \tag{C7}$$

γ_α being a (unit) white noise, either Gaussian ($\alpha=2$) or extremal Lévy-stable of index α ($0<\alpha<2$), $\hat{\gamma}_\alpha$ its Fourier transform, and s a non-random weighting function determined below, corresponding in fact to fractional integration (see Appendix B), hence we consider fractional Lévy noises. Loosely speaking, the (unit) white noise noises may be understood as $\gamma_\alpha(x)$ (=$\int \gamma_\alpha \delta_x dx$) for the different x, are independently

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[1] This is in fact a convenient way to define Fourier transforms of distributions or "generalized functions". In the L^2 case, this property is known as Parseval's theorem.

identically distributed according to either (symmetric) Gaussian law ($\alpha=2$) or (extremal) Lévy-stable law (of index α). Hence, their characteristic functionals generalizes what we obtain for the extremal Lévy characteristic funtion (eq. A11), in the sense that for any function f(x):

$$K_{\gamma\alpha}(f) = Log(<\exp(\int f(x)\gamma_\alpha d^d x)>) = \int f(x)^\alpha d^d x \qquad (C8)$$

As the same relation holds for the convolution product, the desired log divergence (eq. C5) of $\Gamma_{\alpha\lambda}$ is obtained for fractional intergration of order d/α' (Appendix B, with as usual $1/\alpha+1/\alpha'=1$)), i.e.:

$$s(x) \propto |x|^{-d/\alpha}; \quad \hat{s}(k) \propto |k|^{-d/\alpha'} \qquad (C9)$$

One may note that the fact that the exponents are not the same are not the same in the physical space (d/α) and the Fourier space (d/α'), with the notable exception of the Gaussian generator ($\alpha=\alpha'=2$), corresponds to the introduction of interrelations between the componenents of the noise by the Fourier transform, i.e. the Fourier transform of a white Lévy noise is a coloured Lévy noise (the Fourier transform of a white Gaussian noise remains white). Nevertheless, the Fourier transform of a white Lévy noise can be "whitened" down by dividing its components by $|k|^{-d}$ in order to obtain a flat (generalised) spectrum. We recall that such a generalised spectrum can be defined (Schertzer and Lovejoy 1987a; see also Fan, 1989) for fractional Lévy noises (as defined by eq. C7) as:

$$E_{f\lambda}(k) = \int_{\partial S\lambda} |\hat{s}(k)|^\alpha \, d^{d-1} k \qquad (C10)$$

∂S_λ being the surface of the sphere (S_λ) of radius λ), corresponding to the usual definition of the spectrum in the Gaussian case and to its natural extension for Lévy-stable noises. In particular, we still have:

$$\hat{K}_{f\lambda}(k) = \int_1^\lambda E_{f\lambda}(k) d k \qquad (C11)$$

thus log divergence of $\hat{K}$ requires a k^{-1} spectrum, obtained with eq. C9. More precisely, taking:

$$\hat{s}(k) = \left[\frac{|k|^{-d} C_1}{\alpha - 1} \right] 1/\alpha' \qquad (C12)$$

leads to the (non normalized) universal characteristic function ($\alpha\neq1$):

$$K(h) = \frac{C_1 \alpha'}{\alpha} h^\alpha \qquad (C13)$$

hence the corresponding (normalized) universal characteristic function and singularity codimension function (eq. 34-35).

REFERENCES

Anselmet, F., Y. Gagne, E.J. Hopfinger, and R. A. Antonia, 1984: High order velocity structure functions in turbulent shear flows, J. Fluid Mech., 140, 63-75.

Arneodo, A., G. Grasseau, M. Holschneider, 1988: Wavelet transform of multifractals. Phys. Rev. Lett., 61, 2281-2284.

Benzi, R., G. Paladin, G. Parisi, A. Vulpiani, 1984: J. Phys. A, 17, 3521.

Bialas, A., R. Peschanski, 1986: Moments of rapidity distributions as a measure of short-range fluctuations in high-energy collisions, Nucl. Phys. B, B 273, 703-718.

Brax, Ph., R. Peschanski, 1990: Multifractal analysis of intermittency in random cascading models. Nuclear Physics B, (submitted).

Cahalan, R. F., 1990: Landsat Observations of Fractal Cloud Structure, this volume.

Cho, H. R., 1990: Energy spectrum and intermittency of semi-geostrophic flow, this volume.

Corrsin, S., 1951: On the Spectrum of Isotropic Temperature Fluctuations in an isotropic Turbulence, J. Appl. Phys., 22, 469-473 .

Davis, A., S. Lovejoy and D. Schertzer, 1990: Radiative transfer in multifractal clouds, this volume.

Derrida, B., E. Gardner, 1986: Magnetic properties and the function q(x) of the generalised random-energy model. J. Phys. C: Solid State phys., 19, 5783-5798.

Essex, C., 1990: Correlation dimension and data sample size , this volume.

Fan, A H., 1989a: Chaos additif et multiplicatif de Lévy. C. R. Acad. Sci. Paris, I, 308,151-154.

Fan, A H., 1989b: Décomposition des mesures et recouvrements aléatoires. Ph.D. Thesis, Paris-Sud U., Orsay.

Farge, M., G. Rabreau, 1988: Transformée en ondelettes pour détecter et analyser les structures cohérentes dans les écoulements turbulents bidimensionnels. C. R. Acad. Sci. Paris, II, 1479 1486.

Feller, W. 1971: An introduction to probability theory and its applications, vol.2. Wiley, New-York.

Frisch, U., P. L. Sulem, and M. Nelkin,.1978: A simple dynamical model of intermittency in fully developed turbulence. J. Fluid Mech., 87, 719-724.

Gabriel, P, S. Lovejoy, D. Schertzer, 1988a: Multifractal analysis of satellite resolution dependence. J.Geophys. Res. Lett., 1373-1376.

Gardner, E, B. Derrida, 1989: Magnetic properties and the function q(x) of the generalised random-energy model. J. Phys. A: Math Gen., 22, 1975-1981

Gnedenko, B. and A. N. Kolmogorov, 1954: Limit distribution for sums of independent random variables. Addison-Wesley, Cambridge, Massachusetts.

Gnedenko, B. , 1969: The theory of probability. MIR, Moscow

Grassberger, P., 1983: Generalized dimensions of strange attractors. Phys. Lett., A 97, 227.

Grassberger, P., 1986: Are there really climate attractors? Nature, 322, 609.

Grassberger, P., I. Procaccia, 1983: On the characterization of strange attractor. Phys. Rev. Lett., 50, 346.

Grossman, A., J. Morlet, 1987: Decomposition of functions into wavelets of constant shape and related transforms. Mathematics and physics lectures on recent results, ed. L. Streit, World Scientific, Singapore.

Gupta, V.K, E. Waymire, 1990: On scaling and log-normality in rainfall?, this volume.

Gurvitch E., M. Yaglom, 1967: Breakdown of eddies and probabilty distributions of small-scale turbulence, Phys. Fluids, 16, Suppl. S, S59-S65.

Halsey, T.C., M.H. Jensen, L.P. Kadanoff, I. Procaccia, B. Shraiman, 1986: Fractal measures and their singularities: the characterization of strange sets. Phys. Rev. A, 33, 1141-1151.

Hentschel, H.G.E., I. Proccacia, 1983: The infinite number of generalized dimensions of fractals and strange attractors, Physica, 8D, 435-444.

Herring, J.R., D. Schertzer, M. Lesieur, G.R. Newman, J.P. Chollet, and M. Larchevêque, 1982: A comparative assessment of sopectral closures as applied to passive scalar diffusion. J. Fluid Mech., 124, 411-420.

Hubert P., J. P. Carbonnel, 1988: Caractérisation fractale de la variabilité et de l'anisotropie des précipitations tropicales. C. R. Acad. Sci. Paris, 2, 307, 909-914.

Hubert P., J. P. Carbonnel, 1990: Fractal caracterisation of intertropical precipitations variability and anisotropy, this volume.

Kahane, J. P., 1985: Sur le Chaos Multiplicatif. Ann. Sci. Math. Que., 9, 435-444.

Kahane, J. P., 1987a: Martingales and Random Measures, Chinese Ann. Math., 8B1, 551-554.

Kahane, J. P., 1987b: Multiplications aléatoires et dimensions de Hausdorff. Ann. Inst. Henri Poincaré, 23, 289-296.

Kahane, J. P., 1988: Désintégration des mesures selon la dimension. C. R. Acad. Sci. Paris. I, 306,107-110.

Kolmogorov, A. N., 1949: Local structure of turbulence in an incompressible liquid for very large Reynolds numbers. Proc. Acad. Sci. USSR., Geochem. Sect., 30, 299-303

Kolmogorov, A. N., 1962: A refinement of previous hypothesis concerning the local structure of turbulence in viscous incompressible fluid at high Reynolds number. J. Fluid Mech., 13, 82-85

Kraichnan, R. H., 1971: An Almost-Markovian Galilean-invariant turbulence model. J. Fluid. Mech., 83, 349-367.

Kraichnan, R. H., 1980: Realizability inequalities and closed moment equations. Nonlinear Dynamics (Ann. N. Y. Acad. Sci., 357), ed. R. H. G. Helleman., New. York. Academy of Sciences.

Ladoy, P., D. Schertzer and S. Lovejoy, 1986: Une étude d'invariance locale-regionale des temperatures, La Météorologie, 7, 23-34..

Ladoy, P., S. Lovejoy and D. Schertzer, 1990: Extreme variability of climatological data: scaling and intermittency, this volume.

Landau, L. D., and E. M. Lifshitz, 1963: Fluid Mechanics, Pergamon, New York.

Lavallée, D,. D. Schertzer and S. Lovejoy, 1990: On the determination of the codimension function, this volume.

Leray, J., 1934: Sur le mouvement d'un liquide visqueux emplisent l'espace. Acta Math., 63, 193-248.

Levich E., and E. Tzvetkov, 1985: Helical inverse cascade in three-dimensional turbulence as a fundamental dominant mechanism in meso-scale atmospheric phenomena, Phys. Rep., 128, 1-37.

Levich, E., I Shtilman, 1990: Helicity fluctuations and coherence in developed turbulence, this volume.

Lévy, P., 1924: Théorie des erreurs, la loi de Gauss et les lois exceptionnelles. Bull. Soc. Math.. 52, 49-85.

Lévy, P., 1925: Calcul des Probabilités, Gauthier Villars, Paris

Lévy, P., 1954: Théorie de l'addition des variables aleatoires, Gauthier Villars, Paris

Lilly, D. K., 1983: Meso-scale variability of the atmosphere in Mesoscale Meteorology Theories. Observations and Models. edited by D. K. Lilly and T. Gal'Chen, pp.13-24, D. Reidel, Hingham, Mass.

Lorenz, E. N., 1963: Deterministic nonperiodic flows. J. Atmos. Sci., 20, 130.

Lovejoy, S., D. Schertzer, 1985: Generalized scale invariance in the atmosphere and fractal models of rain. Wat. Resour. Res, 21, 1233-1250.

Lovejoy, S., D. Schertzer, 1986: Scale invariance, symmetries fractals and stochastic simulation of atmospheric phenomena. Bull AMS 67, 21-32.

Lovejoy, S., D., Schertzer, 1989: Comment on "Are Rain Rate Processes Self-Similar?" by B.Kedem and L. S. Chiu. Wat. Resour. Res, 25, 3,577-579.

Lovejoy, S., D. Schertzer,1990a: Multifractals, universality classes and satellite and radar measurements of cloud and rain fields, J. Geophy. Res., 95, 2021-2034.

Lovejoy, S., D., Schertzer, 1990b: Multifractal analysis techniques and the rain and cloud fields from 10^{-3} to 10^6m, this volume.

Lovejoy, S., D. Schertzer, A.A. Tsonis, 1987: Functional box-counting and multiple elliptical dimensions in rain. Science, 235, 1036-1038.

Mandelbrot, B., 1974: Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier. J. Fluid Mech., 62, 331-350.

Mandelbrot, B, 1982: The Fractal Geometry of Nature, Freeman, 465pp.

Mandelbrot, B, 1984: fractals in Physics: squig clusters, diffusions, fractal meaures and the unicity of fractal dimensionality. J. Stat. Phys., 34, 895-930.

Mandelbrot, B., 1986: Self-affine fractal sets, I, the basic fractal dimensions, Fractals in Physics. edited by L. Pietronero and E. Tosatti, pp. 3-15, Elsevier North-Holland, New York.

Mandelbrot, B., 1989: An introduction to multifractal distribution functions. Fluctuations and Pattern Formation, Eds. H. E. Stanley and N. Ostrowsky, Kluwer, Dordrecht-Boston

Mauldin, R.D., S.C. Williams, 1986: Random recursive constructions: asymptotic geometric and topological properties. Trans. Am. Math. Soc., 295, 325-346.

Meneveau, C. ,K. R. Sreenivasan, 1987: Simple multifractal cascade model for fully developed turbulence. Phys. Rev.Lett., 59, 13, 1424-1427.

Meyer, Y., 1987: Wavelets viewed by a mathematician. Proceedings Ondellettes, méthodes temps-freequences et espaces de phases, CIRM, Luminy.

Moisseev, S. S., P. B. Rtkevitch, A. V. Tur and V. V. Yanovskii, 1988: Vortex dynamo in a convective medium with helical turbulence. Sov. Phys. JETP, 67, 2, 294-299

Monin, A.S., A. M. Yaglom, 1975: Statistical Fluid Mechanics, vol. 2. MIT Press, Boston.

Nerenberg, M. A. H., T. Lookman and C. Essex, 1990: On the existence of low dimensionsal climatic attractors, this volume.

Nicolis, C., G. Nicolis, 1984: Is there a climate attractor. Nature, 311, 529.

Novikov, E. A., R. Stewart, Intermittency of turbulence and spectrum of fluctuations in energy-dissipation, Izv. Akad. Nauk. SSSR. Ser. Geofiz., 3, 408-412, 1964.

Obukhov, A., 1949: Structure of the Temperature Field in a Turbulent Flow, Izv. Akad. Nauk. SSSR Ser. Geogr. I Jeofiz., 13, 55-69.

Obukhov, A., 1962: Some specific features of atmospheric turbulence. J. Geophys. Res, 67, 3011-3014

Parisi, G., U. Frisch, 1985: A multifractal model of intermittency, in Turbulence and predictability in geophysical fluid dynamics and climate dynamics, 84-88, Eds. Ghil, Benzi, Parisi, North-Holland.

Perrin, J., 1913 (and 1948): Les Atomes, NRF-Gallimard, Paris.

Pietronero, L., and A.P. Siebesma 1986: Self-similarity of fluctuations in random multiplicative processes. Phys. Rev. Lett., 57, 1098-1101.

Rényi, A., 1966: Calcul des Probabilités. Dunod, Paris, 620 pp.

Richardson, L. F., 1926: Atmospheric diffusion shown on a distance neighbor graph. Proc. R. Soc., London. Sec. A, 110, 709-722.

Richardson, L. F., 1922, (Republished by Dover, New York,1965.): Weather prediction by numerical process, Cambridge U. Press.

Ross, B.: Fractional Calculus and its applications. Lecture Notes in Mathematics 457, Springer-Verlag, Berlin.

Sarma, G., 1989: Analyses et simulations multifractales des champs de nuages, applications à la télédétection. Engineer's Thesis, Ecole Nationale Supérieure des Techniques Avancées, Paris.

Schertzer, D., S. Lovejoy, 1983: On the dimension of atmospheric motions. Preprint Vol., IUTAM Symp on Turbulence and Chaotic Phenomena in Fluids, 141-144.

Schertzer, D., S. Lovejoy, On the dimension of atmospheric motions, in Turbulence and chaotic phenomena in fluids, edited by T. Tatsumi, pp.505-508, Elsevier North-Holland, New York, 1984.

Schertzer, D., S. Lovejoy, 1985a: The dimension and intermittency of atmospheric dynamics. Turbulent Shear Flow 4, 7-33, B. Launder ed., Springer, NY.

Schertzer, D., S. Lovejoy, 1985b: Generalized scale invariance in turbulent phenomena. P.C.H. Journal, 6, 623-635.

Schertzer, D., S. Lovejoy, 1987a: Physically based rain and cloud modeling by anisotropic, multiplicative turbulent cascades. J. Geophys. Res., 92, 9693-9714.

Schertzer, D., S. Lovejoy, 1987b: Singularités anisotropes, et divergence de moments en cascades multiplicatifs. Annales Math. du Qué., 11, 139-181.

Schertzer, D., S. Lovejoy 1989: Generalized Scale invariance and multiplicative processes in the atmosphere. Pageoph, 130, 57-81.

Schertzer, D., S. Lovejoy, 1990: Nonlinear variability in geophysics: Multifractal simulations and analysis, in Fractals: Physical Origin and Consequences, Ed. L. Pietronero, Plenum, New York, 49-79.

Schertzer, D., S. Lovejoy, R. Visvanathan, D. Lavallée, and J. Wilson, 1988: Universal Multifractals in Turbulence, in Fractal Aspects of Materials: Disordered Systems, Edited by D.A. Weitz, L.M. Sander, B.B. Mandelbrot, 267-269, Materials Research Society, Pittsburg, Pa.

Seed, A., D. Lavallée, S. Lovejoy, D. Schertzer, G.L. Austin, 1990: Multifractal analysis of radar rain measurements. (in preparation).

Smith, L A , 1988: Intrinsic limits on dimension calculations. Phys. Lett. A, 133, 6, 283-288

Smith, L A , J. D. Fournier, E. A. Spiegel , 1986: Lacunarity and intermittency in fluid turbulence. Phys. Lett. A, 8,9.

Stanley, H. E. P. , P. Meakin, 1988: Minireview on multifractals. Nature, 6, 116.

Todoeschuck, J. P. , O.G. Jensen, 1990: !/f geology and seismic deconvolution, this volume.

Tessier, Y., S. Lovejoy, D. Schertzer, 1989: Multifractal analysis of global rainfall from 1 day to 1 year. XXIV th. European Geophysical Society Assembly, Barcelonna.

Viswanathan, R., C. Weber and P. Gibart, 1990: Stochastic coherence and the dynamics of global climate models and data, this volume.

Von Neumann, J., 1963: Recent theories in turbulence, in Collect. Works, 6, 437-450, Pergamon, New York.

Voss, R., 1983: Fourier Synthesis of Gaussian fractals: 1/f noises, landscapes and flakes. Proceedings, Siggraph conf., Detroit, p1-21.

Waymire, E., and V.K. Gupta, 1981: The mathematical structure of rainfall representations, parts 1-3, Water Resour. Res., 17, 1261-1294.

Wilson, J., D. Schertzer, S. Lovejoy, 1990: Physically based cloud modelling by scaling multiplicative cascade processes, this volume.

Yaglom, A. M. 1966: The influence of the fluctuation in energy dissipation on the shape of turbulent characteristics in the inertial interval, Sov. Phys. Dokl., 2, 26-30.

Zolotarev, V. M, 1986: One-dimensional stable distributions. Translations of Mathematical Monographs, Americ. Math. Soc., Providence, Rhode Island.