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RADIATIVE TRANSFER IN MULTIFRACTAL ATMOSPHERES: FRACTIONAL INTEGRATION, MULTIFRACTAL PHASE TRANSITIONS AND INVERSION PROBLEMS

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Abstract. This paper is devoted to studying the inhomogeneity of the radiation field resulting from propagation through a multifractal cloud field by relating the orders of singularities and codimensions of both fields. This direct relationship is of fundamental importance for climate studies, whereas the inverse problem is fundamental for remote sensing. We point out similarities between smoothing by scattering and fractional integration, showing they are exactly analogous for certain cases: 1-D medium and plane parallel atmospheres (with a few extra hypotheses). We therefore deduce that there is a limited range of singularities susceptible to exhibiting identical multifractal characteristics before any inversion. The lower bound ($\gamma_{D'}$) is defined by a first order multifractal phase transition which occurs when the dimension D′ of the fractional integration is insufficient to smooth out the singularities of the cloud field, whereas the upper bound (γ_s) is defined by a second-order phase transition and corresponds to the limitations induced by the finite size of the samples. These two critical singularities drastically reduce the range of relevant singularities and justify some essentially ad hoc procedures used in multifractal estimation.

1. Introduction. In meteorology, climatology and remote sensing, fundamental uncertainties are related to cloud modelling and analysis. It is especially difficult to describe the complex interactions between radiative transfer, cloud microphysics and cloud dynamics: none of the three major aspects of clouds—their optical properties, their spatial and temporal variability—nor the associated precipitation are well understood. A particular aspect of interest here is that estimates of the total albedo depends largely on the variation of cloudiness in atmosphere. One of these problems, "the cloud absorptivity paradox" is still not resolved. The basic problem can be stated as follows: comparison of radiation budget estimates at the top and bottom of the atmosphere leads to the indirect inference that absorption considerably exceeds the largest values obtainable from theory, when the latter assumes homogeneity and no absorption. It is obvious that remote sensing needs a model of atmospheric variability to handle the observations, in order to deal with the inverse problem : how to deduce liquid water content variability from radiances variability.

The same holds for climate studies (see for review Somerville and Gauthier, 1994) and indeed recently, Cess et al. and Ramanathan et al. (1995) cite observations of an "anomalous absorption" of radiation in cloudy skies in comparison with the values predicted by usual models (homogeneous

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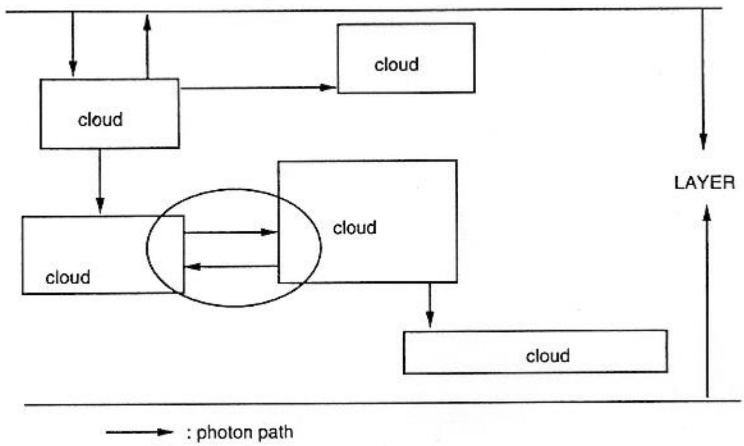

FIG. 1. *Illustration of photon trapping between high inhomogeneities of the cloud field : optical path as well as effective absorption increase considerably.*

atmosphere). If true, this would lead to large uncertainties in assessing climate change. Byrne et al. (1995), in order to explain this anomalous absorption, propose a simple model of Broken Clouds and measure an increase in the photon mean free path in comparison with the value calculated for a homogeneous atmosphere. The modelled media is a layer filled with a mixture of clouds and portions of clear sky. They argue that due to photons travelling horizontally (see Figure 1), columns of clear sky absorb more light than in the corresponding completely clear sky. These "holes" significantly increase the mean free path of photons and thus increase the absorption of the layer, i.e. photons are trapped due to cloud inhomogeneities. They show that there should be a new description of clouds and suggest the multifractal model. Our purpose is not to explicitly solve this problem of Anomalous Absorption but instead to study the effect of liquid water concentration variability on the light scattering, in the ideal case of perfect scattering. The object of this paper is to argue that homogeneous models are simply not relevant in relating the highly variable properties of clouds and radiation fields: however smoothed, the intensity of clouds' multiply scattered radiation fields reflects this extreme variability. Unfortunately, classical methods—of both radiative transfer and dynamical modelling—are limited to studying such relationships on an arbitrary scale (often considered as "characteristic"), although the interactions between clouds and radiation occur on a wide range of scale. We have argued for many years that the extreme variability of the radiation fields can be best understood in a multifractal framework (Gabriel et al. 1988, Lovejoy and Schertzer 1990, 1991, Schertzer and Lovejoy 1988, 1991). Indeed, the (scalar) multifractal model of cloud fields, as discussed below, is capable of respecting the clouds' texture, clustering, bands and intermittency, and the non-linear nature of the true dynamical processes at all scales. The simplest relevant dynamical model corresponds to a stochastic model of passive clouds, passively advected by a turbulent velocity field, using coupled cascade processes, non-linearly conserving the fluxes of energy and concentration variance (Schertzer, Lovejoy 1987, Wilson et al. 1991, Pecknold et al. 1993, 1995).

In order to propagate light in such an atmosphere, different analytical and numerical methods have been used. There have been some attempts for (mostly) 2D- cloud fields to capture some of the radiative effects of clouds (Gabriel et al. 1986 (for 3D), Lovejoy et al. 1989, 1990, Gabriel et al. 1990, Davis et al. 1990, 1991, 1992) : by evaluating the global radiative responses such as transmittance (T) or reflectance (R) for conservative scattering, using discrete angle phase functions in the (continuous angle) Radiative Transfer Equation (Chandrasekhar 1950), i.e. D.A. (Discrete Angle) Radiative Transfer, on respectively homogeneous, monofractal and some multifractal fields. More recently, for 1-D multifractal clouds, transport has been described by the diffusion equation (Silas et al. 1993, Lovejoy et al. 1995) and by radiative transfer studying the scattering statistics of a photon, dependent on the heterogeneous optical depth of the medium (Lovejoy et al. 1995). Work on related models includes Evans 1993, Barker 1992 and Cahalan 1989.

Our goal is to ascertain the consequences of the radiative transfer equation on multifractal statistics in a simplified 2D multifractal field, first by simulation and second by analytical calculation. We give some general considerations on multifractal fields and simulations in section 2, and on the radiative transfer equation in section 3. In section 4, we present the main argument that relates the radiative transfer equation to a fractional integration and then test it on a plane parallel multifractal atmosphere in section 5. Section 6 is mainly concerned with 2D-cuts of a multifractal atmosphere (although most of the results also apply for 3D) and in order to generalize the relation between the radiative transfer equation and fractional integration, we present the consequences of a D′-integration on a D-cut of a multifractal field.

2. General description of the multifractal model.

2.1. Physical basis. First we present the physical basis of the multifractal model. As advocated by Schertzer and Lovejoy (1987), it is important to first consider the fundamental and rather well defined case of passive scalar clouds. Although real clouds are not truly passive, their statistics may in fact be quite similar (Bromsalen 1994), and in any case, it is the simplest relevant nonlinear model. Such clouds result from the passive scalar advection of water concentration (ρ) by a velocity field ($\underline{v}$) in the limit of vanishing viscosity and diffusivity. The dynamical equations are the incompressible Navier Stokes equations and the equation of passive advection. They both conserve the fluxes of energy (density ε) and scalar variance (density χ) while effecting a transfer to smaller scales by a cascade process (introduced by Richardson 1922) down to the inner viscous scale:

$$\varepsilon = -\frac{\partial \underline{v}^2}{\partial t} = const \tag{2.1}$$

$$\chi = -\frac{\partial \rho^2}{\partial t} = const \tag{2.2}$$

The energy flux is mainly transferred from one scale to a neighboring scale. Considering the real space fluctuations (increments) at scale ℓ in the inertial range (viscosity scale $= \eta < \ell < L =$ outer scale) of the fields $\underline{v}$ and ρ, and considering the fluxes as rather homogeneous, we have scaling laws according to on the one hand *Kolmogorov* 1941, and on the other *Obukhov* 1949 and *Corrsin* 1951:

$$\Delta v(\ell) = \varepsilon^{1/3}\ell^{1/3} \tag{2.3}$$

$$\Delta \rho(\ell) = \varphi^{1/3}\ell^{1/3} \tag{2.4}$$

where

$$\varphi = \chi^{3/2}\varepsilon^{-1/2} \tag{2.5}$$

is the flux resulting from the non linear interaction between the velocity and water concentration.

A crucial point of criticism concerning this first approach to turbulence has been that it was assumed that the energy transfer itself is not a fluctuating quantity. In a more refined scaling theory, Kolmogorov (1962) and Obukhov (1962) also considered high inhomogeneity in the energy transfer rate ε. In order to study this question of inhomogeneity of ε and χ, we will use cascade processes which, by iterating a scale invariant step, systematically reduce the scale of homogeneity to zero. For convenience we introduce a new variable, the scale ratio $\lambda = L/\ell$ $(1 < \lambda < \Lambda = L/\eta)$. At a given resolution λ, the corresponding intermediate quantities χ_λ and ε_λ are highly variable (intermittent) but scale invariant. The flux can be rewritten:

$$\varphi_\lambda \approx \chi_\lambda^{3/2}\varepsilon_\lambda^{-1/2} \tag{2.6}$$

and equation (2.4):

$$\Delta \rho_\lambda \approx \varphi_\lambda^{1/3}\lambda^{-1/3} \tag{2.7}$$

Statistical moments of χ_λ and ε_λ exhibit multiple scaling:

$$\begin{aligned} \langle \varepsilon_\lambda^q \rangle &\approx \lambda^{K(q)} \\ \langle \chi_\lambda^q \rangle &\approx \lambda^{K(q)} \end{aligned} \tag{2.8}$$

$K(q)$ is a convex function and these relations do not affect the validity of (eq. 2.3) and (eq. 2.4), i.e. intermittency does not change the relationships between $\underline{v}$ and ε, and ρ and χ.

The nonlinear dependence on q through $K(q)$ corresponds to the multiple scaling and expresses the fact that generally, as discussed below, the most intense and weakest regions will scale differently.

2.2. General considerations on multifractal fields. A multifractal field is associated with an infinite hierarchy of singularities γ ($\varepsilon_\lambda \approx \lambda^\gamma$) (Schertzer and Lovejoy, 1987). When it is stochastic, their frequency of occurrence can be described by codimensions $c(\gamma)$, i.e. at a scale ratio λ, the probability (Pr) of the fluctuations of the field diverging faster than λ^γ scales as $\lambda^{-c(\gamma)}$ (Schertzer and Lovejoy, 1987):

$$Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)} \tag{2.9}$$

In order to model a multiple scaling (multifractal) field, we seek a field ε_λ with resolution scale λ, satisfying equation (2.8), i.e

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)} \tag{2.10}$$

Therefore $K(q) \ln \lambda$ is the second Laplacian characteristic function (or cumulant generating function) of $\Gamma_\lambda = \ln \varepsilon_\lambda$, which is the generator of the process.

The functions $c(\gamma)$ and $K(q)$ are related to each other, using a Legendre transform (Parisi and Frish 1985):

$$K(q) = \max_\gamma (q\gamma - c(\gamma)) \tag{2.11}$$

$$c(\gamma) = \max_q (\gamma q - K(q)) \tag{2.12}$$

By mixing different processes of the same type, if we seek the limit when $\lambda \to \infty$, we converge by iterations to a universality class. In order to obtain universality we require generators that are both stable and attractive under addition. Extremal and stable Levy variables (Levy 1925), characterized by the Lévy index α ($0 \leq \alpha \leq 2$), respect these properties. Then $K(q)$ and $c(\gamma)$ have the following formulae (Schertzer, Lovejoy 1987):

$$K(q) + Hq = \frac{C_1}{\alpha - 1}(q^\alpha - q) \tag{2.13}$$

$$c(\gamma - H) = C_1\left(\frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha}\right)^{\alpha'} \tag{2.14}$$

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \text{ for } \alpha \neq 1 \tag{2.15}$$

In the particular case where $\alpha = 1$ we have:

$$K(q) + Hq = C_1 q \log(q) \tag{2.16}$$

$$c(\gamma - H) = C_1 \exp(\frac{\gamma}{C_1} - 1) \tag{2.17}$$

Parameters designated H, C_1 and α are of fundamental significance (Schertzer and Lovejoy 1987) and define the local multifractal hierarchy around the mean field ($q = 1$). They have the following signification:

- H describes the *deviation from conservation* of the flux:

$$\langle \Delta\rho_\lambda \rangle \approx \lambda^{-H} \tag{2.18}$$

  where $H = 0$ for conservative fields.
- C_1 describes the *mean inhomogeneity* as it is the codimension of the mean singularity: $C_1 = c(C_1 - H)$. In the case of conservative fluxes it is also the order of the mean singularity (and simultaneously the fixed point of $c(\gamma)$).
- α represents the degree of multifractality, given by the convexity of $c(C_1)$ around the mean singularity $(C_1 - H)$, measured by the radius of curvature: $R_c(\gamma = C_1 - H) = 2^{3/2}\alpha C_1$ which increases with the range of singularities (starting from zero with the monofractal β-model).

2.3. Simulation of multifractal clouds. In order to create a multifractal cloud respecting the symmetries of passive scalar advection, both discrete or continuous cascades can be used. The more realistic continuous cascade processes will be used here.

The quantity of interest here is the passive scalar $\Delta\rho_\lambda$ which is related directly to φ_λ through equation (2.7). The field that we produce is the flux φ_λ, and then $\Delta\rho_\lambda$ can be simulated by introducing the extra scaling $\lambda^{1/3}$ to the field $\varphi_\lambda^{1/3}$ by fractional integration (power law filtering).

In order to obtain multiscaling, we require the generator to be a noise with a possible weighting function, having the following properties (Schertzer and Lovejoy, 1987):

1) The spectrum of the field must scale as k^{-1}, in order to obtain the scaling behavior: this is a $\log\lambda$ divergence of $K(q)$.
2) The generator must be band-limited to wave-number within $[1, \lambda]$; this is to ensure that for scales smaller than λ^{-1}, the field will be smooth; λ^{-1} will therefore be the resolution of the field.
3) The probability distribution of the generator must fall off faster than exponentially for positive fluctuations. This is to ensure convergence of $K(q)$ for $q > 0$.
4) It must be normalized so that $K(1) = 0$. This is the condition of conservation of the mean of the field at varying scales; $< \varepsilon_\lambda >= 1$.

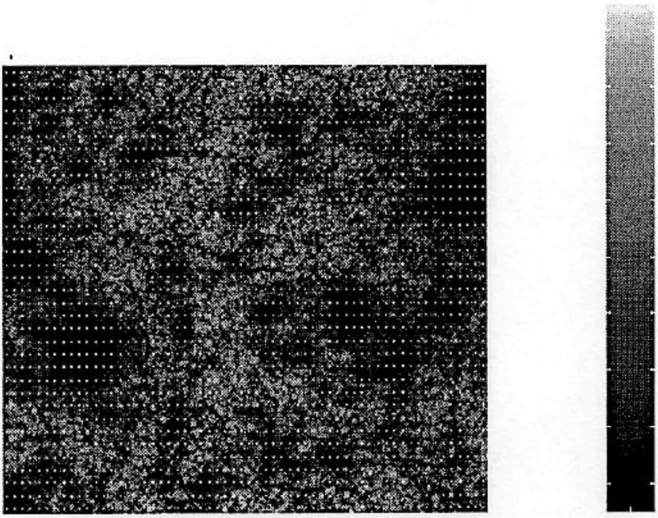

FIG. 2. *Simulation of a 2D multifractal cloud density field with* $\alpha = 1.35, C_1 = 0.75$ *and* $H = 1/3$ *(*C_1 *is chosen arbitrarily but* $\alpha = 1.35$ *and* $H = 1/3$ *have been estimated by Tessier et al. 1993). To increase the contrast we show the logarithm of the original field.*

In order to perform simulations, we start with a gaussian white noise or an extremal asymmetric Lévy distribution with large negative fluctuations in a finite bandwidth $[1, \lambda]$: the subgenerator $\gamma_\lambda(x)$. In Fourier space, we filter it by $\hat{f}(\underline{k})$ $(= |\underline{k}|^{-d/\alpha})$ in order to make a "1/f noise" and to obtain the generator $\hat{\Gamma}_\lambda(\underline{k}) = \hat{f}(\underline{k})\hat{\gamma}_\lambda(\underline{k})$ (by convention, the Fourier transform of a given quantity h is denoted $\hat{h}$). The conserved quantity $\varphi_\lambda(\underline{x})$ is then the exponentiation of $\Gamma_\lambda(\underline{x})$ and the multifractal density $\Delta\rho_\lambda$ is then obtained after having taken the 1/3 power of φ_λ and having filtered it (in Fourier space) by $|\underline{k}|^{-1/3}$.

Wilson et al. (1991) and Pecknold et al. (1993) developed efficient algorithms yielding such multifractal clouds. We used them to simulate a 2D-cut of a multifractal cloud density field, and it is represented on Figure 2 with $\alpha = 1.35, C_1 = 0.75$ and $H = 1/3$ with respect to turbulence. It simulates a cloud with axes along vertical and horizontal.

3. General considerations on radiative transfer.

3.1. Theory. The radiative field is characterized by a monochromatic intensity (radiance) $I(\underline{x}, \underline{s})$ at a point $\underline{x}$ and in a direction $\underline{s}$, in a medium with an absorption coefficient κ and a density field $\rho(\underline{x})$. The Radiative Transfer Equation is then (Chandrasekhar 1950):

$$\underline{s} \cdot \underline{\nabla} I(\underline{x}, \underline{s}) = -\kappa\rho(\underline{x})[I(\underline{x}, \underline{s}) - j(\underline{x}, \underline{s})] \tag{3.1}$$

$j(\underline{x}, \underline{s})$ is the source function and for a scattering[1] atmosphere, it can be written in the form:

$$j(\underline{x}, \underline{s}) = \frac{1}{4\pi} \iint \sigma(\underline{s}, \underline{s}') I(\underline{x}, \underline{s}') d\omega_{\underline{s}'} \tag{3.2}$$

where $\underline{s}$, $|\underline{s}| = 1$, is a unit vector specifying some direction through a point $\underline{x}$, $d\omega_{\underline{s}'}$ the solid angle, and $\sigma(\underline{s}, \underline{s}')$ the scattering coefficient corresponding to the fraction of intensity scattered from one direction $\underline{s}$ to another $\underline{s}'$ with the following condition of normalization:

$$\int \sigma(\cos\theta) \frac{d\omega}{4\pi} \leq 1 \; with \; \cos\theta = \underline{s} \cdot \underline{s}' \tag{3.3}$$

In the case of perfect scattering, the scattering coefficient is normalized to unity and the absorbed radiation reappears totally as scattered radiation. We also have to introduce some relevant quantities depending on the radiance. First the total intensity:

$$J(\underline{x}) = \frac{1}{4\pi} \int I(\underline{x}, \underline{s}) d\omega_{\underline{s}} \tag{3.4}$$

also the net flux:

$$\underline{F}(\underline{x}, \underline{s}) = \frac{1}{\pi} \int I(\underline{x}, \underline{s}) \underline{s} d\omega_{\underline{s}} \tag{3.5}$$

and the K-tensor[2]:

$$\underline{\underline{K}}(\underline{x}) = \frac{1}{4\pi} \int I(\underline{x}, \underline{s}) \underline{s} \otimes \underline{s} d\omega_{\underline{s}} \tag{3.6}$$

Using these quantities we can also express the Radiative Transfer Equation in its integrated form, considering an isotropic (all directions are identically distributed) and perfect scattering (absorbed light is totally reemitted) case. Integrating eq. (3.1) over s, and using definition (3.5) yields equation (3.7), and taking the tensor product of eq. (3.1) and $\underline{s}$, then integrating over s, with the aid of eq. (3.6), yields equation (3.8):

$$div\underline{F} = 0 \tag{3.7}$$

$$div\underline{\underline{K}} = -\frac{\kappa\rho}{4} \underline{F} \tag{3.8}$$

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[1] The contributions to the source function are only due to scattering.

[2] which generalizes the K-integral of Chandrasekhar (1950) ($\otimes$ is the tensorial product).

3.2. The radiative transport calculations. The two major problems in order to simulate the R.T.E. are the differentiation due to the gradient of intensity and the inherent continuity of the scattering function. The gradient differentiation can be discretized using finite difference approximations, however these approximations are only valid in a medium without large fluctuations, i.e. $\delta\tau \ll 1$, where:

$$\delta\tau = \kappa\rho\delta z \tag{3.9}$$

is the elementary optical depth (or thickness) of the cloud between positions z and $z + \delta z$. Since multifractal fields have enormous dynamic ranges (diverging as $\lambda \to \infty$) it is therefore important to use numerical schemes which are stable for occasionally large values of $\delta\tau$.

A classical way to discretize the scattering function was developed by Chandrasekhar (1950), inspired by Schuster (1905) and Schwartzschild (1906), who divided the radiative field in 2n streams along 2n directions and obtained 2n linear equations. Whereas Chandrasekhar (1950) developed a discretization of directions in order to get a quadrature, i.e. the best approximation in a given sense for any order n, cruder approximations were considered in toy models. For instance, D.A. radiative transfer (Lovejoy et al. 1990) corresponds to the case n=1 or n=2, the phase function describing scattering only through 90° for n=2.

More elaborated discretizations of directions in the radiative transfer equation are obtained by Legendre polynomials expansion (Chandrasekhar 1950, in the case of plane parallel atmospheres) or more generally, with spherical harmonics (Appendix C). The solution is a sum of Legendre polynomials of increasing degree which give angular dependency, weighted by coefficients dependent on the position in the cloud. The net advantage of this method is a drastic simplification of the scattering term in the equation. Another approach is to expand the Green's function of the radiative transfer equation with respect to the (rather trivial) Green's function of its linearization, i.e. without the scattering term. Despite appealing features, this approach, discussed in Appendix B, still faces some theoretical difficulties.

We now consider the problem of finite differences. The spatially explicit discretization for the intensity at a position $\underline{x}+\delta\underline{x}$ is (the discrete scattering coefficient is written $\sigma_{s,s'}$, s and s' being different directions):

$$I(\underline{x} + \delta\underline{x}, s) = I(\underline{x}, s) - \delta\tau I(\underline{x}, s) + \sum_{s'} \delta\tau\sigma_{s,s'} I(\underline{x}, s') \tag{3.10}$$

This scheme will lead generally to inconsistencies. Indeed, it is rather easy to understand when considering a D.A radiative transfer on a 2D cut. We have only 4 possible directions: $s, -s, \pm s_\perp$. The elementary (for a given $\delta\tau$) transmission T, reflexion R and diffusion S then have the corresponding form:

$$\mathrm{T} = 1 - \delta\tau(1 - \sigma_{s,s}) \tag{3.11}$$

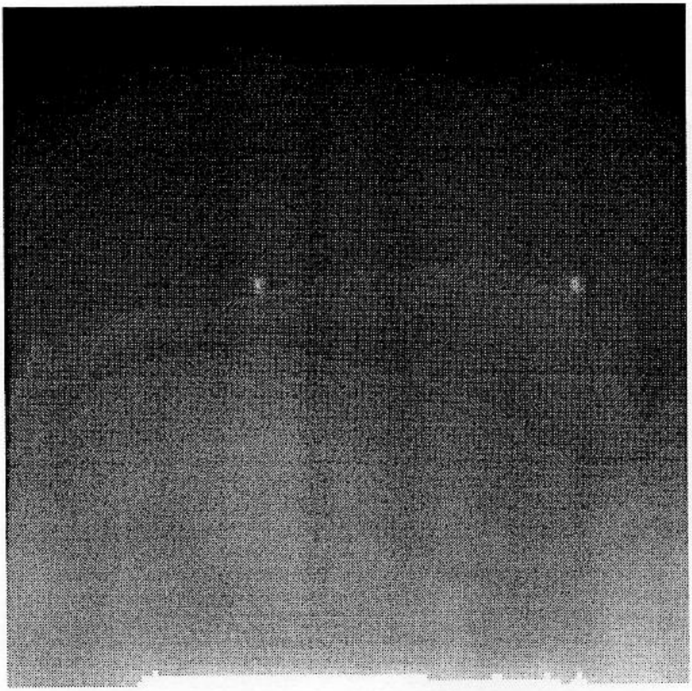

FIG. 3. *Transmission field for a cloud density field with* $\alpha = 1.35, C_1 = 0.75$ *and* $H = 1/3$ *(highest values are black and smallest white).*

$$\text{R} = \delta\tau\sigma_{s,-s} \tag{3.12}$$

$$\text{S} = \delta\tau\sigma_{s,s_\perp} \tag{3.13}$$

It is easy to remark that in case of large fluctuations, $\delta\tau \to \infty$, the transmittance will become negative, reflection and diffusion will diverge. In order to avoid divergences[3] induced by finite difference approximations, we use the semi-implicit scheme (Borde 1991, Borde et al. 1993):

$$\begin{aligned} I(\underline{x}, s) &= I(\underline{x} - s\delta\underline{x}, s) - \delta\tau(1 - \sigma_{s,s})I(\underline{x}, s) \\ &\quad + \sum_{s' \neq s} \delta\tau\sigma_{s,s'}I(\underline{x} - s'\delta\underline{x}, s') \end{aligned} \tag{3.14}$$

We represent in Figures 3 and 4, the radiative field transmitted and reflected for the multifractal cloud density field represented in Figure 2 with $\alpha = 1.35$, $C_1 = 0.75$ and $H = 1/3$. Initial conditions are a transmission at unity from above ($I_- = 1$) and the three other directions are zero ($I_+ = I_{\pm\perp} = 0$). The propagation is vertical.

[3] In order to cancel the negative transmittances, A.Davis (private communication) noted that while using this scheme for a multifractal cloud, he put these negative transmittances to zero.

FIG. 4. *Reflection field for simulated cloud density field with* $\alpha = 1.35, C_1 = 0.75$ *and* $H = 1/3$.

4. General argument. For largely homogeneous media, there are the classical optically thin and thick limits. In the case of extreme inhomogeneity, as in multifractal atmospheres, thin and thick "parts" of the medium are so entangled that the two limits might not exist independently and the two could exist at different scales. However, the distinction could be relevant by performing a similar analysis in a scale invariant way, i.e. directly on singularities. For instance, most of the transfer will occur in areas where the singularities will be rather low, whereas most of the scattering will occur when the singularities are rather high. Therefore one may expect the existence of a critical singularity separating 'transparent' (low order) singularities where the radiance field mostly flows through, keeping the trivial scaling of the source flux, and for 'opaque' (high order) singularities where scattering becomes more and more effective, yielding a more and more non trivial scaling for the radiance field. The field becomes more and more singular with respect to volume integration. By directly considering single scattering statistics on multifractal clouds, this has been quantified in Lovejoy et al. 1995.

It turns out that these rather general arguments are rather similar to what happens for a fractional integration, as detailed in section 6—yielding a first order multifractal phase transition which occurs when the dimension D′ of the fractional integration is no longer smoothing out the singularities of the integrated field, whereas the finite size of the sample induces a second order phase transition.

This is not surprising since in at least two cases, the radiance inhomogeneities are obtained by integration over cloud inhomogeneities (section 5).

5. Multifractal plane parallel and 1D atmospheres.

5.1. Multifractal 1D atmosphere. Going back to section 3, we consider the radiative transfer equation (eq. 3.1) with emphasis on its integrated form (eq. 3.7 and eq. 3.8). If we take a one dimensional field with radiation entering at the top of it, I_+ being the upgoing intensity and I_- being the downgoing one, these equations become:

$$\frac{dF(z)}{dz} = 0 \tag{5.1}$$

$$\frac{dK(z)}{dz} = -\kappa\rho(z)F(z) \tag{5.2}$$

with

$$F(z) = I_- - I_+ \tag{5.3}$$

and

$$K(z) = I_- + I_+ \tag{5.4}$$

Thus if we combine (5.3) and (5.4) with (5.1), we find that the K-function is proportional to the intensity, irrespective of direction. Integrating (5.2), we obtain the result that intensities are a 1D-integration of the density field.

5.2. Multifractal plane-parallel atmosphere. To illustrate the general argument, we derive the relationship between the intensity field and the cloud density field, and the corresponding conditions necessary for a plane parallel field (which is two dimensional and homogeneous along the horizontal and heterogeneous along the vertical).

Thus every quantity in the Radiative Transfer Equation only depends on the vertical coordinate z, and its integrated form becomes (v stands for vertical):

$$F_v = const \tag{5.5}$$

$$\frac{\partial K_v}{\partial z} = -\kappa\rho F_v \tag{5.6}$$

The flux being constant we can easily integrate eq. (5.6). The right hand side depends on:

$$\Delta\tau = \int_z^{z+\Delta z} \kappa\rho(z')dz' \tag{5.7}$$

and we obtain a simple relationship:

$$\Delta K_v \propto \Delta\tau \tag{5.8}$$

Using the Discrete Angle radiative transfer scheme we can express the relationships between the different quantities in a very simple manner:

$$\Delta F_v = 0 = \Delta I_- - \Delta I_+ \tag{5.9}$$

$$\Delta K_v = \Delta I_+ + \Delta I_- = 2\Delta I_\pm \tag{5.10}$$

The following relationship is then straightforward:

$$\Delta I_v \propto \Delta K_v \tag{5.11}$$

Another way to have a third equation completing the first two integrated R.T.E's is to consider an expansion of the scattering coefficient and intensities as a series of Legendre polynomials (see section 3.2 and Appendix C):

$$\sigma(\underline{s}, \underline{s}') = \sum_\ell \omega_\ell P_\ell(\cos\alpha) \tag{5.12}$$

$$I(z, \underline{s}) = \sum_n P_n(\cos\theta) I_n(z) \tag{5.13}$$

with θ the angle between $\underline{s}$ and the vertical and α the angle between $\underline{s}$ and $\underline{s}'$. The three first orders of functions $I_n(z)$ depend respectively on J, F and K (see equations 3.4, 3.5, 3.6). With this method, Chandrasekhar (1950) pointed out a solution of $I(z, s)$ as a function of $F(z, s)$ and of τ in conservative cases. It corresponds to the solution of least anisotropy:

$$I(\tau, \mu) = \frac{3}{4}\, F[(1 - \frac{1}{3}\omega_1)\tau + \mu] \tag{5.14}$$

with $\mu = \cos\theta = \underline{s} \cdot \underline{s}'$. These two methods, the first combining (5.8) and (5.11) and the second directly with (5.14) give us the same relationship between I_v and τ:

$$\Delta I_v \propto \Delta\tau \tag{5.15}$$

The vertical variation of the light intensity transferred through a plane-parallel cloud is proportional to the corresponding variation of the optical depth $\Delta\tau$ which is a 1D-integration of the density field ρ (eq. 5.8).

6. Full multifractal atmosphere.

6.1. General presentation. We study now a 2D-cut[4] for convenience in computation and presentation, with no loss of generality in 3D (this time we conserve the heterogeneity along the two directions) and we consider first the analytical aspect.

We might be interested in horizontal averages of the different quantities in order to restrict the dependence on the vertical coordinate only. Integrating the R.T.E along the horizontal (indicated by overbars) we obtain:

$$\frac{\partial \overline{F_v}}{\partial z} = 0 \tag{6.1}$$

$$\frac{\partial \overline{K_v}}{\partial z} = -\kappa \overline{\rho F_v} \tag{6.2}$$

Although equation (6.1) is similar to equation (5.5), equation (6.2) is much more complicated than equation (5.6) due to the non-linearity introduced by the correlation term $\overline{\rho F_v}$, and there is no simple analytical technique to determine any general solution.

Going back to the general argument, we hypothesize that for any density field these equations should have the same consequences as a fractional integration, i.e. a radiative field should have the same statistics as those of an integrated field. We have also to be sure that it is related to the full original multifractal statistics. Thus we study the influence on statistics of a (fractional) D$'$-integration on a D-cut of a multifractal field.

6.2. Non trivial consequences of a D$'$-integration on a D-cut of a multifractal field. Whereas the exponents $c(\gamma)$ and $K(q)$ are preserved for any D$'$-cut of a multifractal process observed in D-space D > D$'$, they are not preserved for a D$'$-integration. This can be understood with the notions of "bare" and "dressed" quantities, which are different due to the divergence of high order statistical moments for the latter (Schertzer, Lovejoy 1987). For the same resolution λ, the "bare" field is the result of a multiplicative cascade partially developed from large scales down to λ, whereas the "dressed" field is obtained by integration over λ of a completed cascade. In the limit $\lambda \to \infty$, the bare quantity ε_λ becomes singular and is implicitly given by the measure $\Pi_\lambda(A)$ (which converges):

$$\Pi_\lambda(A) = \int_A \varepsilon_\lambda d^D \underline{x} \quad (\lambda = \frac{L}{\ell}) \tag{6.3}$$

where A is a set of dimension D. Corresponding dressed quantities $\varepsilon_{\lambda(d)}$ (d for dressed) are expressed from the definition of integrated fluxes of the

[4] Physically, a vertical cross section of the atmosphere.

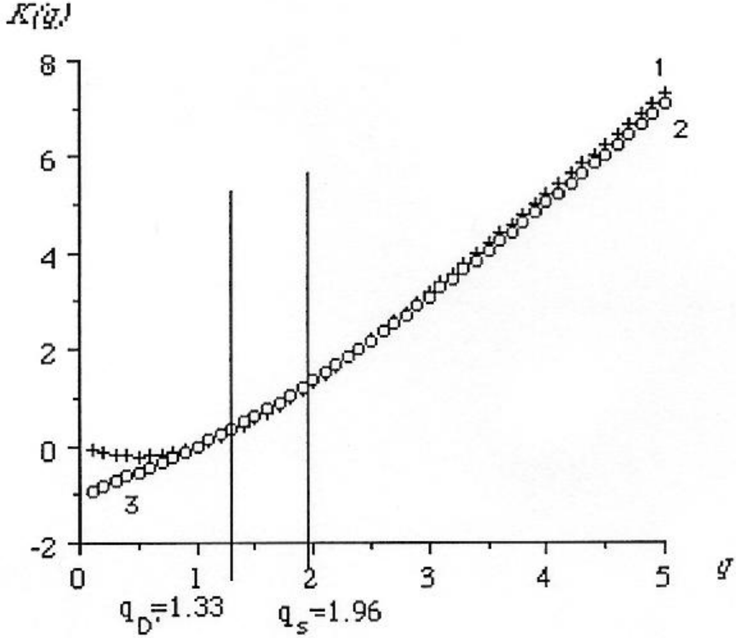

FIG. 5. *$K(q)$ for the D'-integrated field and the D-field with $\alpha = 2$ and $C_1 = 0.75$ $(H = 0)$; with D=2 and D'=1.*
Slopes:
1 $K(q) = -3,0112 + 2,0581q$
2 $K_{D'}(q) + q - 1 = -3,0220 + 2,0156q$
3 $K_{D'}(q) + q - 1 = .1.0263 + 1.0087q$ *for* $q < q_{D'}$
Captions: $^{+}$ *D-integrated field;* $^{\circ}$ *D'-field*

bare quantity ε_λ:

$$\varepsilon_{\lambda(d)} = \frac{\Pi_\infty(B_\lambda)}{vol(B_\lambda)} \tag{6.4}$$

where $\text{vol}(B_\lambda) = \lambda^{-D}$ is the D-dimensional volume of a ball of size λ^{-1} and by definition, for a set $A, \Pi_\infty(A) = \lim_{\lambda\to\infty} \Pi_\lambda(A)$. These dressed quantities will display a divergence of moments above a critical order q_D, defined by:

$$K(q_D) = D(q_D - 1) \tag{6.5}$$

This situation corresponds to a hyperbolic behaviour of the probability distribution tails (extreme events) and the dressed characteristic function

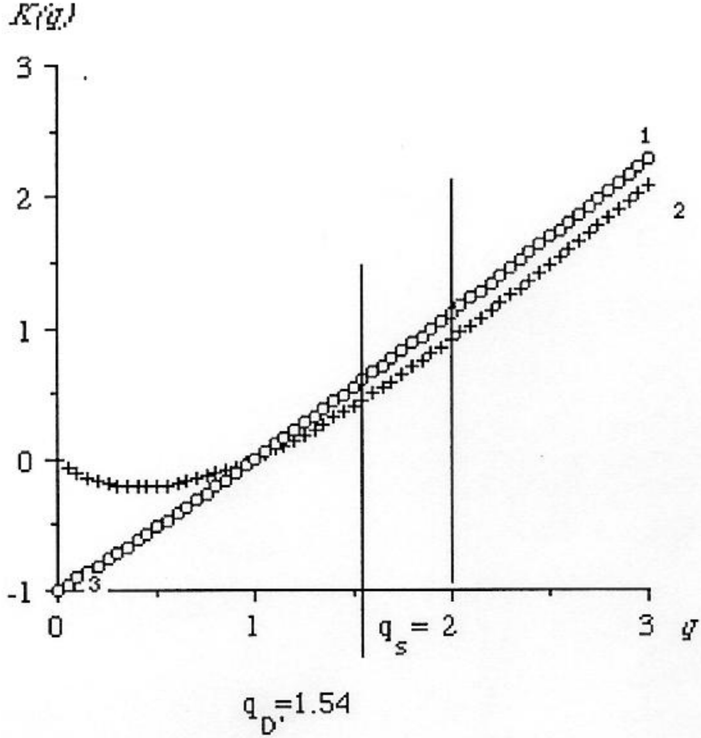

FIG. 6. *$K(q)$ for the D'-integrated and D-field with* $\alpha = 1.35$ *and* $C_1 = 0.75$ $(H = 0)$. *D=2 and D'=1. Slopes:*
1 $K_{D'}(q) + q - 1 = -1,2464 + 1,1758q$
2 $K(q) = -1,4711 + 1,1827q$
3 $K_{D'}(q) + q - 1 = -1,0103 + 0,99967q$ *for* $q < q_{D'}$
Captions: $^+$ *D-integrated field;* $^\circ$ *D'-field*

of moments $K_{(d)}(q)$ is of constant slope above q_D (Schmitt et al. 1994), whereas below q_D, it is identical to the bare characteristic function of moments. This critical order leads a discontinuity in the first derivative of $K_{(d)}(q)$ and corresponds to a first order multifractal phase transition[5].

In our case, instead of integrating the bare field ε_λ (on a set A of dimension D) over a given scale $\ell = \frac{L}{\lambda}$, we choose an arbitrary order of integration D$'$ which can be whether integer or fractional with D$'$ < D.

[5] This terminology is borrowed from statistical thermodynamics and corresponds to an analogy between the moment order q and the inverse temperature β, as well as between $K(q)$ and the Massieu potential $\sum(\beta)$. See Schertzer and Lovejoy 1995 and references therein, for discussions on first and second order of multifractal phase transitions, as well as their consequences.

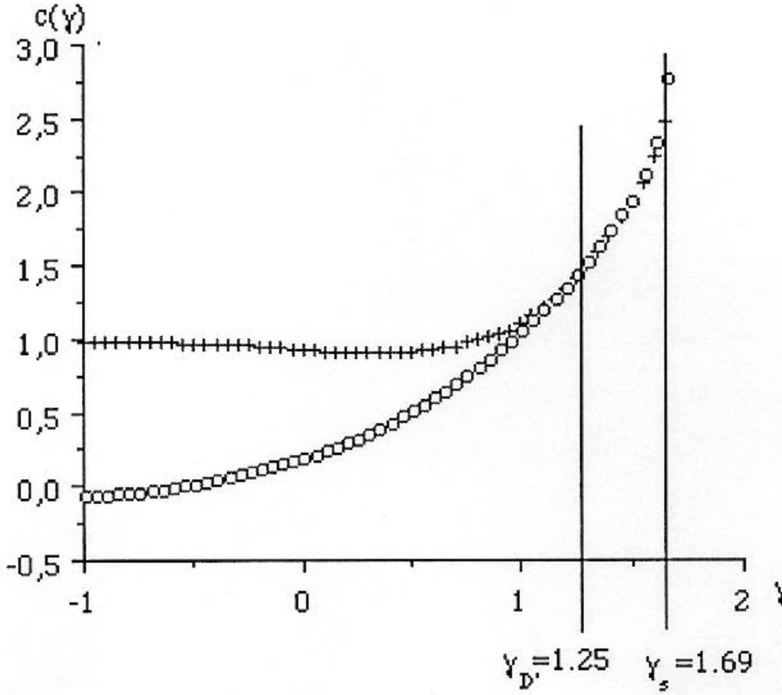

FIG. 7. *Codimensions of the D-field and the D'-integrated field for $\alpha = 2$ and $C_1 = 0.75$, $(H = 0)$, for D=2 and D'=1. Captions:*
+ D'-integrated field; ° D-field

Characterizing the original field ε_λ by $K(q)$ and $c(\gamma)$, we correspondingly characterize the field obtained after D′-integration by $K_{D'}(q)$ and $c_{D'}(\gamma)$. We consider the case where D′ is an integer and then we decompose A as follows: $A = A' \times B$, where A' is a set of dimension D′ and B is the set of dimension D-D′. The field of interest here can be obtained as:

$$\Pi_\lambda(A') = \sum_{A'} \varepsilon_\lambda \lambda^{-D'} \tag{6.6}$$

Recalling the general expression for trace-moments (Schertzer, Lovejoy, 1987):

$$Tr_A \varepsilon_\lambda^q = \int_A \langle \varepsilon_\lambda^q \rangle d^{qD}x \approx \lambda^{K(q)-(q-1)D} \tag{6.7}$$

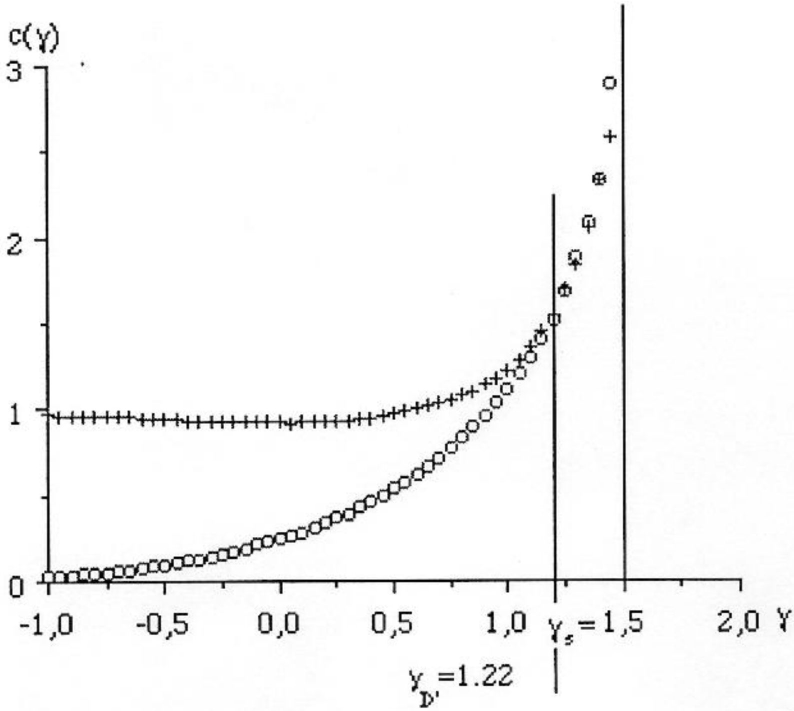

FIG. 8. *Codimensions of the D-field and the D'-integrated field with* $\alpha = 1.35$ *and* $C_1 = 0.75$ $(H = 0)$. *D=2 and D'=1. Captions:*
+ *D'-integrated field;* ° *D-field*

Here, the relevant trace-moment should be :

$$Tr_{A'}(\varepsilon_\lambda^q) \approx \lambda^{K(q)-D'(q-1)} \tag{6.8}$$

The behaviour of the moment of interest $\langle \Pi_\lambda^q(A') \rangle$ is given by this trace-moment on the one hand, and on the other can define a trace-moment as in equation (6.7):

$$\begin{aligned} Tr_B(\Pi_\lambda(A'))^q &\approx \sum_B \lambda^{-q(D-D')} \langle (\Pi_\lambda(A'))^q \rangle \\ &\approx \lambda^{-(D-D'(q-1)} \langle (\Pi_\lambda(A'))^q \rangle \end{aligned} \tag{6.9}$$

By analogy with the general theory (eq. 6.5), we should have a critical moment $q_{D'}$ given by the divergence of the trace-moment of equation (6.8) in the limit $\lambda \to \infty$:

$$K(q_{D'}) = D'(q_{D'} - 1) \tag{6.10}$$

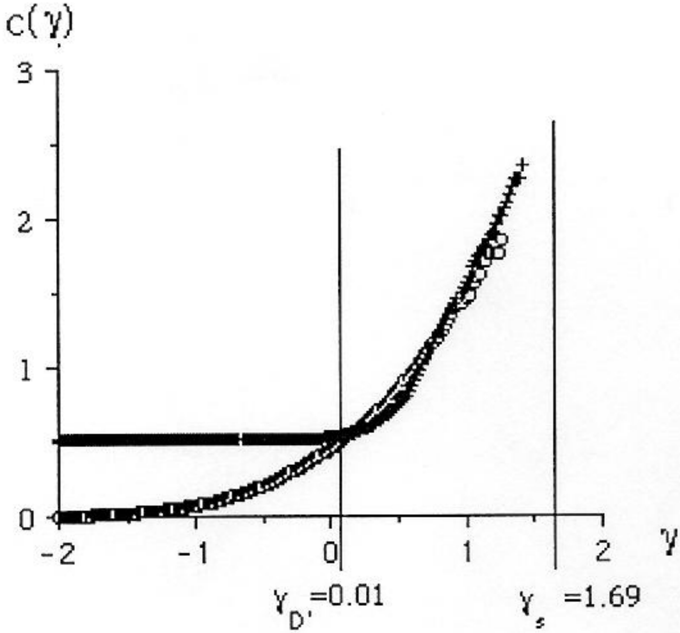

FIG. 9. *Codimensions of the D-field and the fractional D'-integrated field with* $\alpha = 2$ *and* $C_1 = 0.75$ $(H = 0)$. *for D=2 and D'=0.5. Captions:*
+ *D'-integrated field;* ° *D-field*

Above this exponent, the trace-moment diverges.

By definition, we should have :

$$\langle (\Pi_\lambda(A'))^q \rangle \approx \lambda^{K_{D'}(q)} \tag{6.11}$$

Now if we suppose that above the moment of order $q_{D'}$, the moment $\langle \Pi_\lambda^q(A') \rangle$ is almost equivalent to the trace-moment defined in equation (6.8), and also that the trace on A is equivalent to taking first a trace on A' and then on B, we should obtain:

$$Tr_B(\Pi_\lambda(A')^q) \approx Tr_A(\varepsilon_\lambda^q) \tag{6.12}$$

and finally that:

$$K_{D'}(q) - (D - D')(q - 1) = K(q) - D(q - 1) \tag{6.13}$$

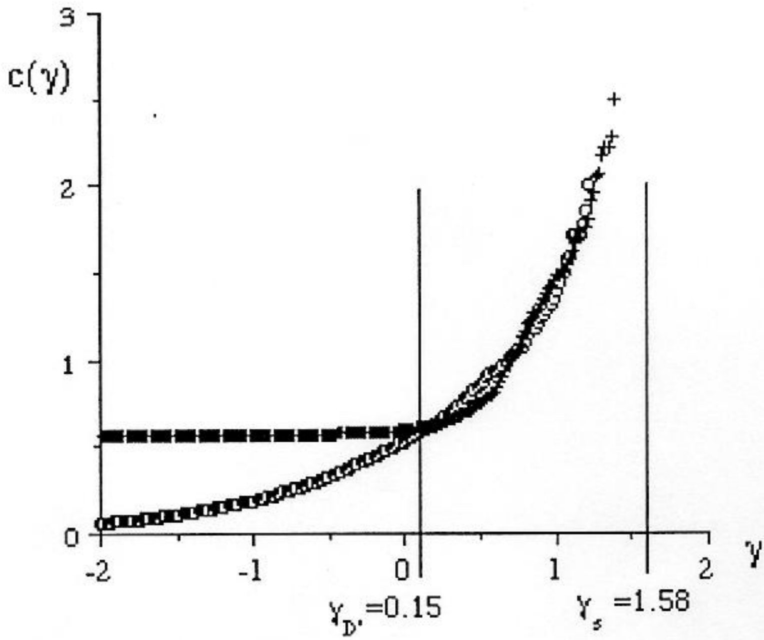

FIG. 10. *Codimensions of the D-field and the D'-integrated field with* $\alpha = 1.35$ *and* $C_1 = 0.75$ $(H = 0)$. *D=2 and D'=0.5. Captions:*
+ *D'-integrated field;* ° *D-field*

Then

$$K_{D'}(q) = K(q) - D'(q-1) \quad \text{for all}\, q > q_{D'} \tag{6.14}$$

For $q < q_{D'}$, the traces converge which implies that the moments converge as well and should lose their dependency on λ. Thus equation (6.11) implies $K_{D'}(q) = 0$. The main consequence is that the moments calculated for the D$'$-integrated field are smoothed out below the $q_{D'}$-moment.

The D-cut ε_λ also exhibits a critical singularity γ_s (Schertzer et al. 1991), due to finite sample size limitations. This singularity is the maximum one attainable and is given by :

$$c(\gamma_s) = D + \frac{\log(N_s)}{\log(\lambda)} \tag{6.15}$$

where Ns is the total number of samples. This expression gives by Legendre transform (eq. 2.11 and 2.12) a critical order of moments q_s. The order q_s induces a discontinuity in the second derivative of the characteristic function of moments and represents a second order multifractal phase transition.

As a conclusion we can say that: below a critical order of moment $q_{D'}$, dependent only on the order of fractional integration and representing a first order multifractal phase transition, $K_{D'}(q)$ is null; above, $K_{D'}(q)$ is simply related to $K(q)$, however, above another critical order of moment q_s, dependent on the sample size and representing a second order multifractal phase transition, $K(q)$ and $K_{D'}(q)$ become linear.

Finally, these two critical orders form the range of order of moments that give information about the statistics of the D-cut from the statistics of the D$'$-integrated field.

In order to illustrate this, for D=2 and D$'$=1, we create a D-cut of a multifractal field (100 realizations), in a conservative case ($H = 0$) for convenience and no loss of generality, and $C_1 = 0.75$. We integrate it, then calculate the moments of both fields and their respective $K(q)$. We represent $K(q)$ and $K_{D'}(q) + D'(q-1)$ (as defined previously) versus q in Figures 5 for $\alpha = 2$, and 6 for $\alpha = 1.35$. For both α, we have a perfect correspondence with theory and we can notice that the range of q's (Δq) between q_D and q_s is very small (for $\alpha = 2, \Delta q = 0.3$ and for $\alpha = 1.35, \Delta q = 0.46$). The usual technique to estimate α and C_1 (the DTM or Double Trace Moment) (Lavallée et al. 1991) calculates and analyses moments of the field after having raised it to a power η at each scale. The range of η chosen should be large enough to preserve accuracy but is also limited by divergence of moments. Equivalently, the limited range of relevant q's reduces the range of relevant η's. In this case, the DTM is not accurate enough as $\Delta\eta$ is of order of Δq. With the Legendre transform, we get the correspondingly critical singularities $\gamma_{D'}$ and γ_s. We can also represent this relationship with the codimensions $c(\gamma)$ and $c_{D'}(\gamma)$:

$$\gamma < \gamma_{D'} \Longrightarrow c_{D'}(\gamma) = 0 \tag{6.16}$$

$$\gamma > \gamma_{D'} \Longrightarrow c_{D'}(\gamma) = c(\gamma + D') - D'$$

as shown on Figures 7 (for $\alpha = 2$) and 8 ($\alpha = 1.35$). Once again we calculate the codimensions $c(\gamma)$ and $c_{D'}(\gamma)$ and compare them. One can notice that the behavior for small γ's makes the codimension representation more striking than that of $K(q)$.

The calculations for fractional integration (D$'$ is no longer an integer) are more tricky because we are compelled to perform a summation on all the embedded space, i.e. some renormalisation procedures have to be set properly. However it appears from performing simulations that the behaviour of $c_{D'}(\gamma)$ and $K_{D'}(q)$ are rather similar. As codimensions exhibit in a clearer manner the expected properties, we represent in Figures 9 and 10 the comparison of codimensions of a field with D=2 and the codimensions of this field D$'$-integrated with D$' = 0.5$. Figure 9 shows a field with

$\alpha = 2$ and Figure 10 with $\alpha = 1.35$.

We have yet to compare the codimensions of simulated radiative fields and fractionally integrated fields, which would require extensive calculations on a supercomputer, we would also have to determine the order of the closest fractional integration.

The bad news is that direct application of the DTM on an integrated field gives wrong values of α and C_1. However, the good news is that the advantage of fractional integration is that it allows a reverse transformation (differentiation, see Appendix A and Lavallée et al. 1993 for application on data) which restores the original statistics of the field. Using this transformation on any integrated field returns its validity to the DTM technique.

7. Conclusion. In order to preserve its heterogeneity, its scale invariance and its physical properties, we describe the atmosphere within a theoretical framework: the universal multifractal model. Due to its universality property, it depends only on three fundamental parameters C_1, α and H, already experimentally measured for many fields in the atmosphere (Tessier et al. 1993, Schmitt et al. 1992, 1993, 1994). It seems to be a pertinent way to deal with all the problems involving clouds that persist in climatology and for remote sensing, in the sense of dealing directly with singularities of the heterogeneous water concentration field.

We focused on the case of perfect scattering, as this phenomenon should be strongly related to the problem of anomalous absorption and inversion methods for radiances. In order to propagate light in this medium, we analytically set up different expressions of the solution of the radiative transfer equation; first a discrete angle scheme, which has also been used for simulation, and also a formulation of the radiance with Legendre polynomials, spherical harmonics and Green's functions. The general argument is that the radiance field is a fractional integration of the cloud density field. It is shown as being verified for 1D and plane parallel atmospheres and we would like to generalize it for 2D and 3D cases.

We showed that the idea that (fractional) D$'$-integration on a D-cut of a multifractal field : conserves the codimension of the mean C_1, the Lévy index α, and increments the conservation parameter H, is too crude.

D$'$-integration has important consequences when applied on a multifractal field. It smoothes the low singularities and reduces the range of singularities necessary to preserve the validity of our usual estimation techniques for C_1, α and H. This phenomenon occurs until a critical singularity corresponding to a first order phase transition. Another singularity due to sample size limitations, representing a second order multifractal phase transition, is the upper limit of the range of singularities following the desired statistics.

Physically it means that if the effect of the radiative transfer equation is similar to a fractional integration, one can only observe directly the large concentrations of water in clouds, whatever the scale of observation.

However, considering fractional integration, this effect can be eliminated by fractional differentiation which returns the initial statistics, a method that could explain the fact that one can analyze the statistics of the gradients of radiances instead of the radiances themselves. More generally, the determination of the critical singularities should help to overcome many dead-locks encountered until now in inversion methods for radiances.

In order to refine our study, especially if we want to compare fractional integration and the radiative transfer equation, we would have to make large simulations on supercomputers. Analytically, a more rigorous foundation could be found by using Green's functions. Finally an important part of further study will be dedicated to anomalous absorption.

A. Appendix: Fractional integration and differentiation. Fractional integration and differentiation correspond to extensions to non- integer orders (H) of integrations (I^H) or differentiations (D^H). We denote D^H by $\partial^H/\partial t^H$, where t is the argument of the function. These extensions are rather straightforward in Fourier space or one-dimensional functions, since integrations—up to a constant of integration discussed below—or differentiations of integer order n, correspond respectively to division or multiplication by $(ik)^n$ where k is the wave number (Fourier transforms of physical space quantities will be denoted by a tilde ($\sim$):

$$\tilde{\mathrm{I}}^{-H}(f) = \tilde{\mathrm{D}}^{H}(f) = (ik)^H \tilde{f}(k) \tag{A.1}$$

In the usual physical space, for a non-integer H we obtain an ordinary (i.e. for integer order n) differentiation (D^n, positive n) or integration (I^{-n}, negative n) of a convolution:

$$\mathrm{I}^{-H} f = \mathrm{D}^H f = \frac{1}{\Gamma(n-H)} \mathrm{D}^n (f * x^{n-H-1}) \tag{A.2}$$

Here Γ is the Euler Gamma function:

$$\Gamma(t) = \int_0^\infty e^{-u} u^{t-1} du \; t > 0 \tag{A.3}$$

(A.2) is more general than (A.1), since it is written directly in physical space, but introduces ambiguities in the definition on non-integer integration or differentiation because they will clearly depend on the domain of definition of convolutions.

The same techniques can be extended to functions defined on R^d, however the analysis becomes more complex because various combinations of partial derivatives are now possible. Nevertheless one can consider the following strongly isotropic extension:

$$\mathrm{I}_d^{-H} f = \mathrm{D}_d^H f = \frac{1}{\Gamma(n-H)} \mathrm{D}_d^n (f * |x|^{n-H-d}) \tag{A.4}$$

$$\tilde{\mathrm{I}}_d^{-H}(f) = \tilde{\mathrm{D}}_d^{H} f = |k|^H \tilde{f}(k) \tag{A.5}$$

which in fact corresponds to fractional powers of the Laplacian (or of the Poisson solver):

$$\mathrm{I}_d^{-H} f = \mathrm{D}_d^{H} f = (-\Delta)^{H/2} \tag{A.6}$$

B. Appendix: Green's function approach. The strongly nonlinear dependence on directions introduced by multiple scattering can be perceived by first introducing the Green's function G_0 of the partial differential part of the transfer equation (on a spatial domain D and of boundary ∂D):

$$\underline{u}.\nabla G_0(\underline{x}, \underline{x}'; \underline{u}, \underline{u}') + \kappa\rho G_0(\underline{x}, \underline{x}'; \underline{u}, \underline{u}') = \delta_{\underline{x}-\underline{x}'}\delta_{\underline{u}-\underline{u}'} \tag{B.1}$$

which allows us to write the transfer equation in a purely integral form:

$$I(\underline{x}, \underline{u}) = \int_{\partial D\times S_d} G_0(\underline{x}, \underline{x}'; \underline{u}, \underline{u}')\kappa\rho(\underline{x}')j_0(\underline{x}', \underline{u}')d^{d-1}\underline{x}'d^{d-1}\underline{u}' + \int_{D\times S_d} G_0(\underline{x}, \underline{x}'; \underline{u}, \underline{u}')\kappa\rho(\underline{x}')j(\underline{x}', \underline{u}')d^{d-1}\underline{x}'d^{d-1}\underline{u}' \tag{B.2}$$

$$I(\underline{x}, \underline{u}) = G_0 * (\kappa\rho j_0) + G_0 * (\kappa\rho j) \tag{B.3}$$

where $j_0(\underline{x}, \underline{u})$ denotes the boundary sources, $j(\underline{x}, \underline{u})$ the scattering sources (eq. 3.2) and $*$ denotes (generalized) convolutions over positions and/or directions. By iteration of equations (3.1) and (B.3) we are lead to the following Von Neumann series:

$$I(\underline{x}, \underline{u}) = G_0 * (\kappa\rho j_0) + G_0 * (\kappa\rho\sigma) * G_0 * (\kappa\rho j_0) + \cdots .. + G_0 * [(\kappa\rho\sigma) * G_0]^{*n} * (\kappa\rho j_0) + \cdots \tag{B.4}$$

which displays the complexity of the multiple scattering. This approach is rather formal, since on the one hand the convergence of the series may be questioned and on the other a part of the solution depends on boundary conditions. We are studying radiative properties of a cloud field occupying a domain D of large horizontal extension, with boundary conditions corresponding to a null horizontal flux:

$$\underline{F}_h(\underline{x}) = 0 \; x \in \partial D \tag{B.5}$$

or at least (e.g. by imposing cyclic conditions):

$$\int_{\partial D} \underline{F}(\underline{x}')d^{d-1}\underline{x}' = 0 \tag{B.6}$$

On the contrary on a certain sub-domain $\partial D'(\partial D \supset \partial D')$:

$$\underline{F}_v(\underline{x}) \neq 0 \; x \in \partial D' \tag{B.7}$$

whose unknown values are part of the solution.

C. Appendix: Legendre polynomials and spherical harmonics. Chandrasekhar (1950), in case of plane-parallel atmosphere, proposed to express the scattering function and the radiance in terms of Legendre polynomials. We consider:

$$\sigma(\underline{s}, \underline{s'}) = \sum_{\ell} \omega_\ell P_\ell(\cos\alpha) \; with \; \cos\alpha = \underline{s}.\underline{s'} \tag{C.1}$$

$P_\ell(\cos\alpha)$ is a Legendre polynomial of order ℓ for $\cos\alpha$. Using the addition theorem for spherical harmonics we get:

$$\sigma(\cos\alpha) = \sum_{\ell} \omega_\ell \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \mathrm{Y}_\ell^m(\theta,\varphi) \mathrm{Y}_\ell^{m^*}(\theta',\varphi') \tag{C.2}$$

with unit vectors $\underline{s}$ and $\underline{s'}$ expressed in spherical coordinates, respectively (θ,φ) and (θ',φ') (their modulus is 1), where $\mathrm{Y}_\ell^m(\theta,\varphi)$ represents a spherical harmonic depending on two angles (θ,φ) and of order ℓ, and $\mathrm{Y}_\ell^{m^*}(\theta',\varphi')$ represents the complex conjugate of a spherical harmonic, of order ℓ and angles (θ',φ'). We choose to express the intensity in terms of spherical harmonics:

$$I(\underline{x},\theta,\varphi) = \sum_{\ell} \sum_{m} I_{\ell,m}(\underline{x}) Y_\ell^m(\theta,\varphi) \tag{C.3}$$

where the position dependence is represented by $\underline{x}$ and the direction by (θ,φ). Thus we have a coefficient $I_{\ell,m}(\underline{x})$ that depends only on position and which we can express as:

$$I_{\ell,m}(\underline{x}) = \iint d(\cos\theta) d\varphi I(\underline{x},\theta,\varphi) \mathrm{Y}_e^{m^*}(\theta,\varphi) \tag{C.4}$$

Then replacing these new expressions of I and σ we get the following expression for the scattering term A of equation (3.1):

$$\frac{1}{4\pi} \iint d(\cos\theta') d\varphi' I(\underline{x},\theta',\varphi') \sigma(\theta,\varphi;\theta'\varphi') = \tag{C.5}$$

$$\sum_{\ell} \frac{\omega_\ell}{2\ell+1} \sum_{m=-\ell}^{\ell} \mathrm{Y}_\ell^m(\theta,\varphi) \iint d(\cos\theta') d\varphi' I(\underline{x},\theta',\varphi') \mathrm{Y}_\ell^{m^*}(\theta',\varphi')$$

Thus from (C.3) we find:

$$\mathrm{A} = \sum_{\ell} \frac{\omega_\ell}{2\ell+1} \sum_{m=-\ell}^{\ell} \mathrm{Y}_\ell^m(\theta,\varphi) I_{\ell,m}(\underline{x}) \tag{C.6}$$

Now we consider the radiative transfer equation (3.1), multiplying each term by $Y_{\ell'}^{m'^*}(\theta,\varphi)$ and integrating over $\cos\theta$ and φ :

$$\text{(C.7)} \qquad \iint d\cos\theta d\varphi(\underline{s}\cdot \underline{gradI}(\underline{x},\theta,\varphi))Y_{\ell'}^{m'^*}(\theta,\varphi) = -\kappa\rho(\underline{x})(1-\frac{\omega_{\ell'}}{2\ell'+1})I_{\ell',m}(\underline{x})$$

We can see that this new expression noticeably simplifies the right hand side term, however the left hand side part becomes rather complicated, due to the angular dependence in the vector $\underline{s}$. It is possible to express this term as a function of the coefficients $I_{\ell,m}(\underline{x})$ by virtue of the spherical harmonics recurrence formulae. Defining the following coefficients:

$$\text{(C.8)} \qquad a_{\ell,m} = \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}}$$

$$\text{(C.9)} \qquad b_{\ell,m} = \sqrt{\frac{(\ell+m)(\ell+m-1)}{(2\ell+1)(2\ell-1)}}$$

We can express:

$$\text{(C.10)} \qquad \cos\theta Y_\ell^{m^*}(\theta,\varphi) = a_{\ell+1,m}Y_{\ell+1}^{m^*}(\theta,\varphi) + a_{\ell,m}Y_{\ell-1}^{m^*}(\theta,\varphi)$$

$$\text{(C.11)} \qquad \sin\theta\cos\theta Y_\ell^{m^*}(\theta,\varphi) = \frac{1}{2}\Big(-b_{\ell,m}Y_{\ell-1}^{m-1'}(\theta,\varphi) + b_{\ell,-m}Y_{\ell-1}^{m+1^*}(\theta,\varphi) - b_{\ell+1,m+1}Y_{\ell+1}^{m+1^*}(\theta,\varphi) + b_{\ell+1,-m+1}Y_{\ell+1}^{m-1^*}(\theta,\varphi)\Big)$$

$$\text{(C.12)} \qquad \sin\theta\cos\varphi Y_\ell^{m^*}(\theta,\varphi) = \frac{1}{2}\Big(-b_{\ell,m}Y_{\ell-1}^{m-1^*}(\theta,\varphi) + b_{\ell,-m}Y_{\ell-1}^{m+1^*}(\theta,\varphi) - b_{\ell+1,m+1}Y_{\ell+1}^{m+1^*}(\theta,\varphi) + b_{\ell+1,m+1}Y_{\ell+1}^{m-1^*}(\theta,\varphi)\Big)$$

Thus the final expression of the radiative transfer equation is:

$$\text{(C.13)} \qquad \frac{1}{2}\frac{\partial}{\partial x}\Big[-b_{\ell,m}I_{\ell-1,m-1}(\underline{x}) + b_{\ell,-m}I_{\ell-1,m+1}(\underline{x}) - b_{\ell+1,m+1}I_{\ell+1,m+1}(\underline{x}) + b_{\ell+1,-m+1}I_{\ell+1,m-1}(\underline{x})\Big]$$

$$+\frac{1}{2i}\frac{\partial}{\partial y}\Big[-b_{\ell,m}I_{\ell-1,m-1}(\underline{x})-b_{\ell,-m}I_{\ell-1,m+1}(\underline{x})$$

$$+b_{\ell+1,m+1}I_{\ell+1,m+1}(\underline{x})+b_{\ell+1,-m+1}I_{\ell+1,m-1}(\underline{x})\Big]$$

$$+\frac{\partial}{\partial z}\Big[a_{\ell+1,m}I_{\ell+1,m}(\underline{x})+a_{\ell,m}I_{\ell-1,m-1}(\underline{x})\Big]$$

$$=-\kappa\rho(\underline{x})(1-\frac{\omega_\ell}{2\ell+1})I_{\ell,m}(\underline{x})$$

This expression is still rather complicated and in order to perform simulations with this formula, we need to find other recurrence formulae.

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