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New Developments and Old Questions in Multifractal Cloud Modeling, Satellite Retrievals and Anomalous Absorption

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Introduction

In order to work out some fundamental problems occurring in radiative transfer in the atmosphere, one needs to have a physically based model of cloud fields and, more generally, of the atmosphere, which is able to yield realistic inhomogeneities over a wide range of scales, e.g., from at least 5000 km down to 50 cm—as argued in a companion paper (see Lovejoy et al. 1997, this volume).

As the physical foundations can rely neither on a reduced set of deterministic-like equations (heavily truncated in order to fit the very limited size of [super-] computers and with a subsequent large number of ad-hoc parameterizations) nor on ad-hoc stochastic models (e.g., with parameters tuned up in an ad hoc way), one needs to consider stochastic models respecting the symmetries of the fundamental equations, in particular, the scale symmetries.

In this respect, we will discuss refinements of the Fractionally Integrated Flux (FIF) multifractal model, which since Schertzer and Lovejoy (1997) has been used for modeling clouds as well as for modeling atmospheric dynamics in the framework of the "unified scaling model" of the atmosphere (Schertzer and Lovejoy 1985a, 1985b; Lovejoy and Schertzer 1985, 1986; Lovejoy et al. 1993; Lazarev et al. 1994).

These refinements were built on scaling symmetries by taking care of other symmetries (e.g., Galilean invariance), as well as some symmetry-breaking mechanisms (e.g., causality, which breaks the mirror symmetry along the time axis). These symmetries lead to dynamical models over a large range of scales. On the other hand, we discuss in a very inhomogeneous framework how to use mean photon paths to explore the fundamental issues in atmospheric radiative transfer of the anomalous atmospheric absorption and retrievals by remote sensing.

Fundamental Elements of Scaling Fields

To have a physically based model of turbulent fields, one needs to consider rather distinct elements as well as their interplay. Later, (see section on model limitations, we will discuss the indispensable need to do so in order to avoid some misleading confusion between them. In a rather general manner, we need to consider:

- the *fields* themselves such as the velocity ($v(\underline{x},t)$), temperature ($\theta(\underline{x},t)$), liquid water content ($\rho(\underline{x},t)$), radiance field ($I(\underline{x}, \underline{u}, t)$; where the unit vector $\underline{u}$ corresponds to the direction of the ray), etc.

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- their *increments* or *fluctuations*, e.g., for the velocity field

$$\Delta\underline{v}(\Delta\underline{x},\Delta t)=\underline{v}(\underline{x}+\Delta\underline{x},t+\Delta t)-\underline{v}(\underline{x},t) \quad (1)$$

which (with the fields) correspond to fundamental statistical observables of the fields. We did not explicitly indicate any dependence on the point $(\underline{x},t)$ since usually one assumes a statistical stationarity of the fields (and hence the increments) corresponding to statistical time and space translation invariance: their probability distribution does not depend on $(\underline{x},t)$. In this case, a consequence is that the structure functions of any order q, i.e., the statistical moment of the increment at this order:

$$S^{(q)}(\Delta\underline{x},\Delta t)=\langle|\Delta\underline{v}(\Delta\underline{x},\Delta t)|^{q}\rangle \quad (2)$$

depends only on the lag $(\Delta\,\underline{x},\,\Delta\,t)$.

- the densities of turbulent fluxes, such as the turbulent energy flux density ε, the density of the flux of the variance of the concentration field χ, etc. These quantities are generally hard to observe directly, contrary to the fluctuations, but since they correspond generally to some fundamental symmetry of the equations, they possess a further property of stationarity (than being translation invariant): they are "conservative" in the sense that they are rather independent of the scale of observation.

- a (generalized) notion of time-space scale. The precise definition of the space-time scale $(s=\|(\Delta\underline{x},\Delta t)\|)$ will be discussed below in the section on FIF models; however, let us point out that it should be in a general way physically defined by the (scaling) behavior of the flux densities

$$\langle\varepsilon_{\|(\Delta\underline{x},\Delta t)\|}{}^{q}\rangle\propto\|(\Delta\underline{x},\Delta t)\|^{-K_\varepsilon(q)} \quad (3)$$

in particular, their canonical conservation

$$\langle\varepsilon_{\|(\Delta\underline{x},\Delta t)\|}\rangle=\text{Const.} \quad (4)$$

i.e., $K_\varepsilon(1)=0$. More generally $\max\{K_\varepsilon(q)-d(q-1),0\}$ measures how singular the flux is, i.e., how its moment of order q diverges at smaller and smaller scales. Similarly, we will have some scaling behavior of the structure functions, e.g.,

$$S^{(q)}(\Delta\underline{x},\Delta t)\propto|\|(\Delta\underline{x},\Delta t)\||^{-K_v(q)} \quad (5)$$

Since the fluctuations are equal to a gradient times a scale, we have in general $K_v(1)\neq 0$, and $\max\{q+K_v(q),0\}$ measures how singular the gradient is, i.e., how its moment of order q diverges at smaller and smaller scales.

The common property shared by the scaling moment functions $K_v(q)$ and $K_\varepsilon(q)$ is that they are linear in the case of homogeneity $(K_v(q)=-q;\,K_\varepsilon(q)=0)$ or of fractal inhomogeneity (the activity of the fields is concentrated on a fractal set), whereas they are nonlinear in the case of multifractal inhomogeneity (the different levels of activity define embedded fractals sets of distinct fractal dimension). However, beyond this common property, there are some important differences, and even opposition. We have already mentioned a few of them. Forgetting to distinguish them may lead to confusion (see later discussion of model limitations). One may note that products of conservative fluxes are in general no longer conservative.

Fractionally Integrated Flux (FIF) Multifractal Models

In the framework of FIF models, these previously reviewed elements are clearly defined, as is the interplay between them. The fluxes are directly obtained from multiplicative cascade models. In general, for reasons discussed above, this cascade is conservative; whereas the fields are obtained by (space-time) fractional integrations over various powers of fluxes or products of fluxes. The order of this fractional integration is directly related to the expected behavior of the fluctuations of the corresponding fields. However, the fluctuations so obtained are rather more involved than usually expected on the basis of (mono-) fractal ideas (see below for discussion of behavior of the increments).

Before examining these different questions in some detail, let us highlight some recent improvements of this general scheme. The first was to consider an isotropy in space and time, within the general framework of Generalized Scale Invariance (GSI): this corresponds to introducing differentiation operators with distinct (fractional) orders for temporal and spatial differentiation. However, in addition, the "arrow of time" requires the breaking of the mirror symmetry along the temporal axis, i.e., because of causal antecedence. Overall, we obtain a new and dynamical meaning of the fractional integration of fluxes: the scaling function involved in the corresponding convolution is no longer isotropic, mirror symmetric (i.e., acausal), and static, but is rather a Green's function (or propagator) of a dynamical solution of a time-space (fractional) differential equation.

Let us now consider some details in order to better perceive the fundamental questions.

Time-Space Framework and Generalized Scale Invariance

In relativity, the space-time scale notion is usually related to invariance of a characteristic velocity c therefore to an invariance of the (infinitesimal) metric:

$$ds^2 = dx_i dx^i - c^2 dt^2 \qquad (6)$$

On the contrary, within a turbulent medium, we have to conserve energy flux density, which, merely on dimensional grounds, rather leads us to consider quite a different notion of scale (which is no longer a metric):

$$ds^2 = dx_i dx^i - \varepsilon dt^3 \qquad (7)$$

The general framework of continuous cascades leads us to define a flux at any ratio of scale λ (with respect to some outer space-time scale $|(\underline{L},T)|$) by its generator, i.e.:

$$\varepsilon_\lambda = e^{\Gamma_\lambda} \qquad (8)$$

the scale notion being defined in the GSI (Schertzer and Lovejoy 1985a, 1985b; Lovejoy and Schertzer 1985, 1986) by

$$\| T_\lambda(\underline{x},t) \| = \lambda^{-1} \| (\underline{x},t) \| \qquad (9)$$

where T_λ is a generalized contraction operator, which as the usual (isotropic) contraction ($\underline{T}_\lambda = \frac{1}{\lambda}$) forms a (multiplicative) one parameter group ($T_\lambda \circ T_{\lambda'} = T_{\lambda\lambda'}$). It therefore admits an infinitesimal generator G, which in the linear case is a matrix ($\underline{T}_\lambda = e^{-Log\lambda \underline{G}}$) and is the identity ($\underline{G} = \underline{1}$) for the usual (isotropic) contraction.

Dynamical Generation of Fluxes

The generator of a continuous cascade satisfies a (dynamical) equation:

$$g^{-1} * \Gamma_\lambda = \gamma_\lambda \qquad (10)$$

where γ_λ the sub-generator is a white noise (a Levy white noise for universal multifractals), g is a space-time scaling propagator

$$g(\underline{x},t) \propto |(\underline{x},t)|^{D_H} \qquad (11)$$

and precisely for Levy generator (of Lévy index α on a D dimensional space) in order to satisfy the scaling (Equation [3]), it turns out that (Schertzer and Lovejoy 1997, Schertzer et al. 1997)

$$D_H = \frac{D}{\alpha} \qquad (12)$$

The choice of the appropriate fluxes for passive clouds is discussed in Schmitt et al. (1996, 1997) and Schertzer et al. (1997).

Dynamical Generation of Fields by Fractional Integration of Fluxes

The concentration field itself satisfies a similar equation, but with a different (fractional) differential operator G^{-1}

$$G^{-1} * v_\lambda = \varepsilon_\lambda \qquad (13)$$

distinguish v, ρ

whereas the usual isotropic and static propagator corresponds to

$$G^{-1} = (-\Delta)^{\frac{1}{2}} \qquad (14)$$

the breaking of temporal symmetry corresponds to the fact that

$$\frac{\partial}{\partial t} \neq \left(-\frac{\partial^2}{\partial t^2}\right)^{\frac{1}{2}} \qquad (15)$$

More details are discussed by Marsan et al. (1996, 1997). One may furthermore note that there are structural relationships between FIF models and dynamical models directly derived from Navier-Stokes equations (Chigirinskaya et al. 1996, 1997), which strengthen the physical basis of FIF.

Behavior of the Increments

It is interesting to note that the increments of a fractionally integrated flux have the rather distinct behavior of a monofractal field, although they yield the expected trivial scaling. Indeed, they have rather distinct behaviors at quite larger or smaller scales with respect to the spatial lag. Due to the linearity of the convolution, the increment is the convolution of the same flux, but with the increment of the corresponding Green's function, i.e.

$$\Delta v_\lambda = \Delta G * \varepsilon_\lambda \qquad (16)$$

At larger scales due to a dipole effect (Schertzer et al. 1989), they correspond to a fractional integration, but with the gradient of the propagator

$$\|(\underline{x},t)\| \gg \|(\Delta\underline{x},\Delta t)\|: \quad \Delta G \approx (\Delta\underline{x},\Delta t)\nabla G = (\Delta\underline{x},\Delta t)\nabla\|(\underline{x},t)\|\frac{dG}{d\|(\underline{x},t)\|} \tag{17}$$

Hence, the order of integration for the increment is decreased by one compared with the field itself, while at small scales, we have a unipole effect, i.e., only the nearest point of the lag contributes

$$\begin{aligned} \|(\Delta\underline{x}+\underline{x},\Delta t+t)\| \ll \|(\Delta\underline{x},\Delta t)\|: \ \Delta G \approx G(\Delta\underline{x}+\underline{x},\Delta t+t) \\ \|(\underline{x},t)\| \ll \|(\Delta\underline{x},\Delta t)\|: \ \Delta G \approx G(\underline{x},t) \end{aligned} \tag{18}$$

This small-scale behavior being similar to the field itself justifies the choice of calculating the order of fractional integration of the flux on the basis of the expected scaling behavior of the increments (as expected e.g., with the help of some dimensional reasoning).

Galilean Invariance

Classical mechanics require that the construction of the FIF model be Galilean. The Galilean transform from one frame $\mathfrak{R}$ to another $\mathfrak{R}'$, moving with the speed $\underline{u}$ in respect to $\mathfrak{R}$ (and with parallel axis), corresponds to the following (linear) transform $\underline{\underline{A}}(\underline{u})$

$$\begin{aligned} (\underline{x}',t') = \underline{\underline{A}}(\underline{u}).(\underline{x},t): \\ \underline{x}' = \underline{x} - \underline{u}t; \quad t' = t \end{aligned} \tag{19}$$

The generator $\underline{\underline{U}}$ of this transform is

$$\underline{\underline{A}}(\underline{u}) = e^{-u\underline{\underline{U}}} = \underline{\underline{1}} - u\underline{\underline{U}}; \ U_{i,j} = \delta_{i,4}\,(i \le 3), U_{4,j} = 0 \tag{20}$$

This induces (Schertzer et al. 1997) a corresponding Galilean transform of the GSI generalized contraction operator $\underline{\underline{T}}_\lambda$

$$\bullet\underline{\underline{T}}_\lambda = \underline{\underline{A}}(\underline{u}) \bullet \underline{\underline{A}}^{-1}(\underline{u}) \tag{21}$$

since the generator $\underline{\underline{G}}$) of $\underline{\underline{T}}_\lambda$ in general does not commute with $\underline{\underline{U}}$ (with the notable exception of space-time isotropy $\underline{\underline{G}} = \underline{\underline{1}}$):

$$\underline{\underline{T}}'_\lambda \ne \underline{\underline{T}}_\lambda; \underline{\underline{G}}' \ne \underline{\underline{G}} \tag{22}$$

and, indeed, a non-diagonal $\underline{\underline{G}}'$ results from a diagonal $\underline{\underline{G}}$ (the latter corresponding to self-affine structure functions). However, this Galilean transform gives us a Galilean notion of ratio of scales with the help of Equation (9); therefore, space-time FIF models are actually Galilean invariant.

Some Limitations of Other Models

We emphasized the need to distinguish between different fundamental elements of the scaling model. Indeed, ignoring this may lead to some serious difficulties which we illustrate by a few examples. The first example concerns the overly general discussion of the question of the statistical stationarity of the cascade, specifically since we pointed out two rather different meanings of stationarity, related to two distinct observables, i.e., fluxes and fluctuations.

This distinction also brings into question the relevance of any model which tries to build up directly a field, without distinguishing fluctuations and fluxes: one is compelled to try to satisfy distinct and often contradictory constraints. For instance, in the case of the so-called "bounded cascade" models (Bell 1987; Cahalan 1994 and references therein), one tries to use a rather standard model of flux but for the concentration field itself, one is compelled to introduce an ad-hoc smoothing operation (e.g., to obtain a smooth enough spectrum). Furthermore, the scaling notions themselves even need to be revised so that the model may be claimed to be scaling or multifractal at all. Not only is the physics soon lost (we already underlined the basic role of fluxes), but so is the logical coherence of cascades. Furthermore, there is no need to proceed this way.

In contrast to the space-time FIF models, models involving the rather ad-hoc addition of a temporal dimension to a spatial model of turbulence also face severe limitations. This is the case for models based on a Markovianization of Lagrangian dynamics (Over and Gupta 1996, Lima and Vilela Mendes 1996). Indeed in this case, one hypothesizes a very special property within the so-called Lagrangian framework, which is in fact the Galilean framework moving with the average speed of the phenomena (e.g., rain field). We have already argued from the theoretical necessity to respect the Galilean invariance. Let us mention that the interplay between Lagrangian and Eulerian statistics can be empirically assessed (Seuront et al. 1997).

Radiative Transfer and FIF Models

The radiative transfer equation (κ being the (constant) extinction coefficient and $\sigma(\underline{u},\underline{u}')$ being the phase function between the directions $\underline{u}$ and $\underline{u}'$)

$$\text{div}\,[\underline{u}\,I(\underline{x},\underline{u},t)] \equiv \underline{u}\ \underline{\text{grad}}\ [I(\underline{x},\underline{u},t)] = -\kappa\rho(\underline{x},t)[I(\underline{x},\underline{u},t) - \int_{|\underline{u}'|=1} d^d\underline{u}'\sigma(\underline{u},\underline{u}')I(\underline{x},\underline{u}',t) \qquad (23)$$

generates an infinite hierarchy of directional moments of the radiance, i.e., averages over the radiance field $I(\underline{x},\underline{u},t)$ of higher and higher tensor powers $\underline{u}^{\otimes n} = \underline{u}\otimes\underline{u}\otimes....\otimes\underline{u})$ of the unit vector $\underline{u}$ defining the direction of the ray

$$K^{(n)}(\underline{x},t) = \int \underline{u}^{\otimes n}\, I(\underline{x},\underline{u},t)d^d\underline{u} \qquad (24)$$

which satisfies the following hierarchy of equations

$$\text{div}\,[K^{(n+1)}(\underline{x},t)] = \kappa\rho(\underline{x},t)[K^{(n)}(\underline{x},t) - \int_{|\underline{u}'|=1} d^d\underline{u}'S^{(n)}(\underline{u}')I(\underline{x},\underline{u}',t)] \qquad (25)$$

$$S^{(n)}(\underline{u}') = \int_{|\underline{u}|=1} d^d\underline{u}\ \ \underline{u}^{n\otimes}\,\sigma(\underline{u},\underline{u}')$$

we will see that $S^{(n)}(\underline{u}')$ has rather simple expression for $n = 0,1$ and the corresponding directional moments (of order $n+1$) are of fundamental importance. Indeed, the classical flux of radiance is the first order directional moment

$$\underline{F}(\underline{x},t) = K^{(1)}(\underline{x},t) = \int \underline{u}\,I(\underline{x},\underline{u},t)d^d\underline{u} \qquad (26)$$

and it corresponds, loosely speaking, to the mean path of photons: the larger its amplitude, the more focused is the beam along this vector. On the contrary, the symmetric (second order) tensor $\underline{\underline{K}}$

$$\underline{\underline{K}}(\underline{x},t) = K^{(2)}(\underline{x},t) = \int \underline{u}\otimes\underline{u}\,I(\underline{x},\underline{u},t)d^d\underline{u} \qquad (27)$$

rather measures its dispersion, in particular by its trace

$$J(\underline{x},t) = \text{Tr}(\underline{\underline{K}}(\underline{x},t)) \equiv (\underline{e}_i\otimes\underline{e}^i):\ \underline{\underline{K}}(\underline{x},t) \qquad (28)$$

which corresponds to the (classical) total radiance

$$J(\underline{x},t) = K^{(0)}(\underline{x},t) = \int d^d\underline{u}\ I(\underline{x},\underline{u},t) \qquad (29)$$

The higher order directional moments are obtained by simple contraction or double contraction of the first two directional moments, as soon as one considers an "n-fluxes approximation" (here $n=2d$ with d= the dimension of space; which corresponds to "Discrete Angle" phase functions (Lovejoy et al. 1990), i.e., considering only rays along $2d$ orthogonal directions ($\underline{u} = \pm\underline{e}_i;\ \ i = 1,d$). In this (simple) case, $K_{ij} = 0$ for $i \neq j$, and the relation of the present notation to that of Lovejoy et al. (1990) is $F_i = I_i-$, $K_{ii} = I_i+$ for the ith component. Indeed, the orthogonality condition, implies (by [over-] simplifying the integrations over directions) that

$$K^{(2p)}(\underline{x},t) = \frac{1}{2}\int d^d\underline{u}(\underline{u}^{\otimes 2(p+1)}):\ \underline{\underline{K}}(\underline{x},t);\ K^{(2p+1)}(\underline{x},t) = \frac{1}{2}\int d^d\underline{u}(\underline{u}^{\otimes 2(p+1)}).\underline{F}(\underline{x},t) \qquad (30)$$

and we have the following (orthogonal) decomposition of the radiance

$$I(\underline{x},\underline{u},t) = \frac{1}{2}[\underline{u}.\underline{F}(\underline{x},t) + (\underline{u}\otimes\underline{u}):\ \underline{\underline{K}}(\underline{x},t)] \qquad (31)$$

whereas, in general cases, one must consider a rather involved decomposition in spherical harmonics. In the following, we will often consider the limited expansion of the radiance field corresponding to Equation (31), without requiring a discretization of angles. However, we will not discuss the realizability conditions, i.e., conditions ensuring that the corresponding radiance remains positive.

Fluctuations of the Radiance Field and Photon Mean Path Integration

Let us consider the fluctuations of the radiance field $\Delta I(\Delta\underline{x},\underline{u},t)$, with merely the assumption of perfect scattering, i.e., without any absorption

$$S^{(0)}(\underline{u}) = 1 \qquad (32)$$

Without any other assumption than the usual one corresponding to considering that the phase function $\sigma(\underline{u},\underline{u}')$ depends only on the angle of the two directions $\underline{u},\underline{u}'$, and more precisely on $\mu = \underline{u}.\underline{u}' = \cos(\underline{u},\underline{u}')$, one obtains

$$S^{(1)}(\underline{u}') = S^{(1)}\underline{u}';\ S^{(1)} = \int_{|\underline{u}'|=1} d^d\underline{u}(\underline{u}.\underline{u}'')\,\sigma(\underline{u}.\underline{u}'') \Rightarrow |S^{(1)}| < S^{(0)} \qquad (33)$$

and may note that the simple case of isotropic scattering yields $S^{(1)} = 0$. Therefore Equation (25) for n=1,2 corresponds to

$$\text{div}\,(\underline{F}(\underline{x},t)) = 0 \qquad (34)$$

$$\text{div}\,(\underline{\underline{K}}(\underline{x},t)) = -(1 - S^{(1)})\kappa\rho(\underline{x},t)\underline{F}(\underline{x},t) \qquad (35)$$

Due to the divergence free condition on the flux, these equations can be integrated along elementary flux tubes (see Figure 1 for illustration) with elementary section $\underline{u}dA(\underline{x},t)$ and due to Equation (34)

$$\underline{F}.\underline{u}dA = \text{Const.} \tag{36}$$

As discussed earlier, this rather corresponds to integration along mean photon paths: the flux tube becoming larger as soon as the scattering increases. More precisely, we are considering a volume of the atmosphere delimited by a top and bottom surface that are both orthogonal to the flux $\underline{F}$, with a given top insolation intensity I_0 along the downward vertical (direction $\underline{e}_1$) and a bottom albedo $\omega_0 \leq 1$, then the variation of the radiance along any flux tube with elementary section $\underline{u}dA(\underline{x},t)$ is (due to Equations [31] and [36]) given by

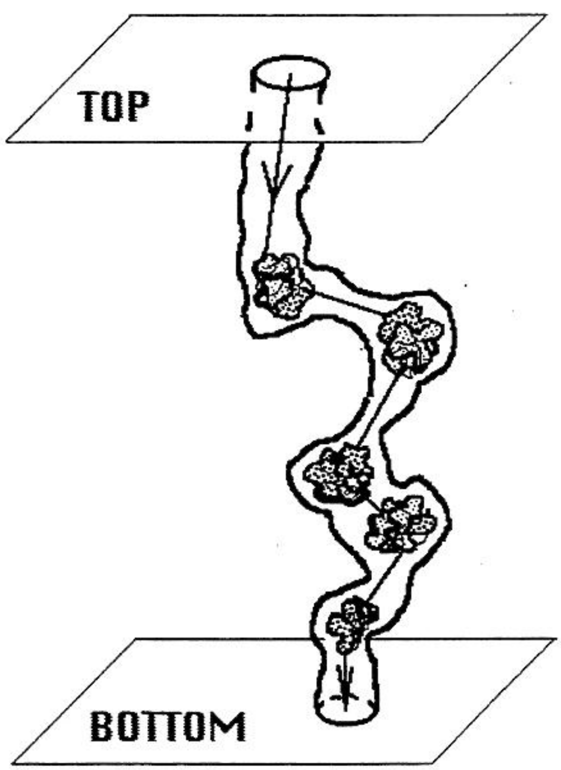

Figure 1. It illustrates a 'mean photon path' in a perfectly scattering atmosphere. It corresponds to an elementary radiance flux tube between the top and the bottom of the atmosphere. The cross-section of the tube increases as soon as the concentration of the scatters increases and the intensity of the flux vector decreases.

$$\Delta\{I(\underline{x},\underline{u},t)dA(\underline{x},t)\} = \frac{1}{2}\,\Delta\{(\underline{u}\otimes\underline{u}):\underline{\underline{K}}(\underline{x},t)dA(\underline{x},t)\} \tag{37}$$

Noting that the flux tube is also an envelope tube for $\underline{u}.\underline{\underline{K}}(\underline{x},t)$, we then obtain that

$$\Delta\{I(\underline{x},\underline{u},t)dA(\underline{x},t)\} = -\frac{1}{2}\,(1 - S^{(1)})(\underline{F}.\underline{u}dA)\Delta\tau(\Im(t)) \tag{38}$$

where $\Delta\tau(\Im(t))$ is the increment of the optical depth along the flux tube axis $\Im(t)$, i.e., the following curvilinear integral

$$\Delta\tau(\Im(t)) = \kappa \int_{\Im(t)} ds\,\rho(s,t) \tag{39}$$

With the help of the top and bottom boundary conditions, it yields the elementary flux along the tube

$$\underline{F}(\underline{x},t).\underline{u}(\underline{x},t)dA(\underline{x},t) = \frac{(1-\omega_0)I_0}{1 + \frac{1-S^{(1)}}{2}\,\Delta\tau(\Im(t))}\,dA(\underline{0},t) \tag{40}$$

which at the top of the atmosphere ($\underline{x} = \underline{0}$) is a quantity of fundamental interest for satellite remote sensing, since it corresponds to the difference between the top insolation and the measured radiance.

At first glance, the results obtained are rather the straightforward extensions of those established for the (academic) one-dimensional or plane parallel atmospheres (Naud et al. 1996a, 1996b) and therefore rather support the corresponding general claim of similarity between (fractional) integration and radiative transfer in a multifractal media. However, the new element of complexity is that instead of being straight as in the academic examples, the mean photon paths ($\Im(t)$) should be rather convoluted, with important fluctuation of their flux tube cross-sections due to inhomogeneities of the scatters. One has therefore to consider monodimensional integration along multifractal paths. Nevertheless, it does not change the general phenomenology, i.e., the lowest singularities are smoothed out, whereas the highest are missing due to insufficient sampling. The two corresponding critical singularities bound a "window of direct inversion," within which the singularities of the multifractal field can be directly retrieved from the radiance field fluctuations. However, the precise definition of these critical singularities will depend on the multifractal properties of the mean photon paths.

On the other hand, the relative simplicity of the general results may explain the apparent robustness of the "independent pixel approximation" (IPA) in many lengthy numerical simulations on generally optically thin clouds. Indeed, most of these simulations suffered from important numerical limitations and tended to lead to flux tubes rather close to vertical columns. However, note that as expected for thick clouds, the IPA does indeed lead to an anomalous scaling exponent for the albedo and transmittance with mean optical thickness; numerically, Davis et al. (1992) already found important errors in clouds with optical thickness $\tau = 100$.

Multifractal Singular Perturbations and the Anomalous Absorption

We now consider the problem of anomalous absorption in an apparently nearly perfect scattering case. The general idea to explain its appearance is that the length of the photon path increases so much that the result is an *effective* absorption coefficient by water vapor, which is much larger than the *bare* one obtained without taking into account the scattering by water liquid. In fact, we find that the case of perfect scattering (see earlier discussion of fluctutuations of the radiance field) corresponds to the rather naive *external* solution of a singular perturbation problem, whereas the quite different *internal* solution give us quite more insights in estimating the effect of the thin but primarily important *boundary layers* where most of the increase of the effective absorption occurs.

In weakly absorbing but strongly scattering atmospheres, the small parameter in the problem is

$$a^2 = \kappa_v \rho_v / \kappa \rho \qquad (41)$$

where the v subscript refers to the vapor phase, which is considered to be purely absorbing, whereas the liquid phase is considered to be purely scattering. Therefore the equations corresponding to Equation (23) are

$$\mathrm{div}(\underline{F}) = -\kappa_v \rho_v \mathrm{Tr}(\underline{\underline{K}}) \qquad (42)$$

$$\mathrm{div}(\underline{\underline{K}}) = -(\kappa\rho + \kappa_v \rho_v)(1 - S^{(1)})\underline{F} \qquad (43)$$

The external solution is obtained by fully ignoring the vapor phase, i.e., the solution discussed earlier, by considering $a^2 = 0$. The internal solution, denoted by a tilde, is obtained by considering $a^2 \to 0$ as a singular limit in boundary layers, where within their thin thickness δ, important variations of the divergence occur (see Figure 2 for an illustration). Indeed, by nondimensionalizing the distances by the thickness δ of the boundary layer, and by correctly

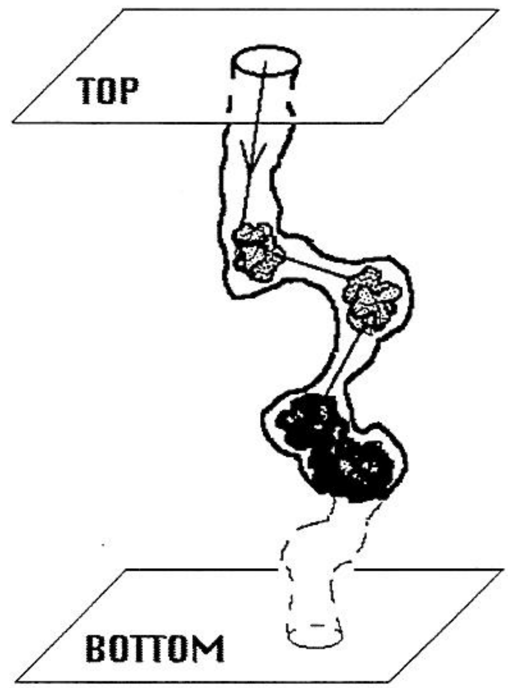

Figure 2. Illustration of a typical elementary radiance flux tube in a slightly absorbing, but strongly scattering atmosphere. Before reaching the bottom of the atmosphere, it reaches a thin boundary layer, where the dissipation becomes strong, thanks to extreme scattering. The radiance is no longer conservative, but decays exponentially fast within this layer.

nondimensionalizing the flux with a scale of intensity a times smaller than the intensity scale for $\underline{K}$:

$$\underline{x} = \delta \underline{\tilde{x}} \; ; \underline{F}(\underline{x},t) = a I_0 \underline{\tilde{F}}(\underline{\tilde{x}},t); \; \underline{\underline{K}}(\underline{x},t) = I_0 \underline{\underline{\tilde{K}}}(\underline{\tilde{x}},t) \qquad (44)$$

when the effective optical depth $\tilde{\tau} = \kappa\rho\delta a$ is of order one (which requires $\kappa\rho\delta \gg 1$ as expected and which defines the order of the boundary layer thickness), we obtain the following system:

$$\tilde{\mathrm{d}}\mathrm{iv}(\underline{\tilde{F}}) = -\tilde{\mathrm{T}}\mathrm{r}(\underline{\underline{\tilde{K}}}) \qquad (45)$$

$$\tilde{\mathrm{d}}\mathrm{iv}(\underline{\underline{\tilde{K}}}) = -(1 - S^{(1)})\underline{\tilde{F}} \qquad (46)$$

As the divergence of the flux $\underline{\tilde{F}}$ is now of order one, as is that of the tensor $\underline{\tilde{K}}$, this leads to the appearance of an effective absorption as important as the scattering! Both decay exponentially fast inside of the boundary layer:

loosely speaking, the photon paths end at the bottom of the boundary layer. In any case, the external bottom boundary condition (as with albedo) is no longer relevant, but rather the internal boundary layer condition is. Independently of the details of the latter, the asymptotic matching between the internal and external solutions preserves the fact that the amplitude of $\underline{F}$ is a times smaller than $\underline{\underline{K}}$. This corresponds to an apparent albedo ω_0^{eff}

$$\omega_0^{\text{eff}} \approx \frac{1-a}{1+a} \approx 1-2a \tag{47}$$

Applying Equation (40) with this albedo to the section of photon path between the top of the atmosphere and the top of the boundary layer ($\Im_B(t)$) yields the following estimate of the downward flux:

$$\underline{F}(\underline{x},t) \cdot \underline{u}(\underline{x},t)\, dA(\underline{x},t) = \frac{2a \quad I_0}{1+\frac{1-S^{(1)}}{2}\,\Delta\tau(\Im_B(t))}\, dA(\underline{0},t) \tag{48}$$

which shows two competing effects of the boundary layer: the shortening of the effective mean photon path and the increase of the effective albedo.

Conclusions

After having clarified some fundamental issues of multifractal cloud modeling, we derived some new theoretical results on perfect scattering as well as anomalous absorption, which are valid in any dimension and are not qualitative only, as in previous work.

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