

# ON THE DETERMINATION OF UNIVERSAL MULTIFRACTAL PARAMETERS IN TURBULENCE

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**ABSTRACT.** The scaling behavior observed in turbulent flows has lately been the object of growing interest. Scaling exponents are fundamental since they describe the statistical properties over wide ranges of length scales. Multifractals are characterized by their scaling exponents and when generated by canonical cascade processes they generally will belong to specific universal classes. In this case the scaling exponents are specified by two parameters, the Lévy index  $\alpha$  and the codimension of the mean singularity  $C_1$ . In particular, these parameters are believed to characterize the probability distribution of the singularities of the Navier-Stokes equations. The first data analysis technique specifically designed to directly estimate these parameters, the "Double Trace Moment" is described. The methods are then used to analyse the scaling behaviour of turbulent velocity data, providing estimates of their universal multifractal parameters:  $\alpha \approx 1.3 \pm 0.1$  and  $C_1 \approx 0.25 \pm 0.05$ .

## 1. Introduction

In fully developed turbulence in three dimensions, scale invariance is a symmetry well known since at least Kolmogorov (1941). High variability or intermittency of velocity fields is characteristic of the signal measured in turbulent flows (Fig. 1). This behavior is observed over a wide range of length scales.<sup>1</sup> The question of how to characterize and relate these two fundamental properties has mainly been considered using the cascade phenomenon. The latter provides a mechanism for the transfer of energy

<sup>1</sup> In the atmosphere, the range of scales goes from  $\approx 10,000$  km to  $\approx 1$  mm.

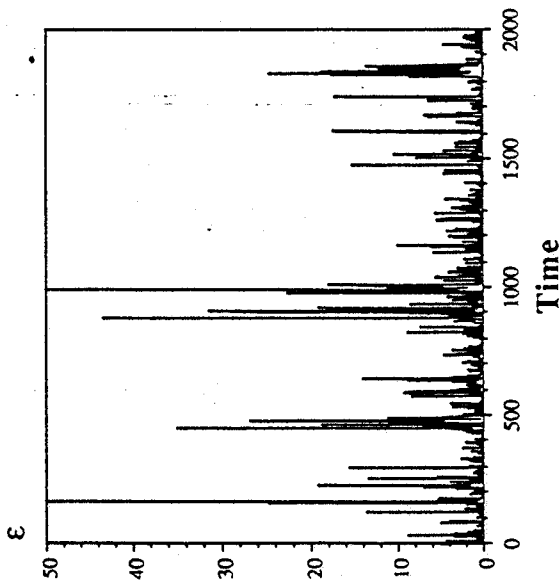


Fig. 1. Illustration of the measured signal of the velocity in turbulent flow.

flux from large to small structures. It also appears to be the generic process of scale invariant fields, characterized by a scaling exponent function: its (multi) fractal dimensions (Grassberger (1983), Hentschel & Proccacia (1983), Schertzer & Lovejoy (1983, 1985a, b), Benzi *et al.* (1984) and Parisi & Frisch (1985)). Thus, initial attempts to characterize turbulence by one parameter—a unique fractal dimension (Mandelbrot (1974) and Frisch *et al.* (1978)) have been followed by several empirical studies<sup>2</sup> of the infinite hierarchy of dimensions (Schertzer & Lovejoy<sup>3</sup> (1985b, 1987a), Meneveau & Sreenivasan (1987a, b, 1989) and Chhabra & Jensen (1989)).

However (canonical) multiplicative cascade processes for which stable attractive multifractal generators exist have the functional behavior of their scaling exponents completely governed by two universal parameters,  $\alpha$  and  $C_1$  (Schertzer & Lovejoy (1987a, b, 1989, 1991), Schertzer *et al.* (1988, 1991), Brax & Peshanski (1991) and Kida (1991)). The Lévy index  $\alpha$  characterizes the degree of multifractality ( $\alpha = 0$  corresponds to monofractal and  $\alpha = 2$ —the maximum—to “lognormal” multifractals<sup>4</sup>) and  $C_1$  is the codimension which characterizes the sparseness/inhomogeneity of the mean of the process

<sup>2</sup> Using the strange attractor formalism of multifractal.

<sup>3</sup> The multiple scaling and multiple fractal dimensions discussed in these papers was for the highly turbulent (and well measured) radar reflectivity field of rain in 1, 2, 3, 4 and 1.5 dimensions (a fractal network).

<sup>4</sup> As pointed out in Schertzer & Lovejoy (1987a, b), a more correct expression is mul-

( $C_1 = 0$ ) is space filling). These two parameters along with the Kolmogorov scaling exponents  $H$  give a complete description of the scaling properties of turbulence. Two independent analyses of turbulent flows, Kida (1991) and Schertzer *et al.* (1991), Schmitt *et al.* (1991a, b) have provided the first estimates for the universal velocity parameters<sup>5</sup> (with respectively  $\alpha = 1.65$  and 1.3). After summarizing the basic properties of multiplicative cascade processes (section 2), we describe the Double Trace Moment (DTM) technique for estimating the fundamental parameters  $\alpha$  and  $C_1$  in section 3 as well as its application to turbulence data (section 4).

## 2. Cascade Processes and the Turbulent Multifractal Formalism

Since the work of Richardson and Kolmogorov, the cascade phenomenology<sup>6</sup> of turbulence has become a basis for the investigation and simulation of intermittency and scale invariance. Cascade processes were invoked to explain how the energy is transferred from large to small structures in turbulence. An illustration of a (continuous) multiplicative cascade processes is given in Fig. 2. In such processes a multiplicative increment modulates the transfer of energy flux from parent eddy to each subeddy. After several iterations the energy flux is concentrated over smaller and smaller volumes, a typical manifestation of intermittency. The multiple scaling properties of the result (best described as mathematical measures<sup>7</sup>) is given by the following relations;

$$\epsilon_\lambda \approx \lambda^\gamma, \quad (2.1)$$

$$Pr(\epsilon'_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)}, \quad (2.2)$$

where  $\epsilon_\lambda$  is a field intensity threshold at resolution  $\lambda$ ,  $\epsilon'_\lambda$  a random field intensity and  $\lambda$  the ratio of the largest scale of interest  $L$  (e.g., the integration scale) to the smallest scale of homogeneity  $\ell$  (e.g., the dissipation scale):  $\lambda = L/\ell$ . When  $\lambda \rightarrow \infty$  (or  $\ell \rightarrow 0$ ),  $\gamma > 0$  is the order of the singularity and  $\gamma < 0$  is the “order of the regularity”. The probability distribution  $Pr(\epsilon'_\lambda \geq \lambda^\gamma)$  of the field intensities is also multiple scaling. Furthermore, examining low dimensional cut of the probability space (dimension  $D$ ),  $c(\gamma)$  has the geometric interpretation of a codimension as long as  $c(\gamma) \leq D$ . The corresponding fractal dimensions are obtained by subtracting:  $D - c(\gamma)$ .

<sup>5</sup> Gabriel *et al.* (1988) test the universality of the turbulent cloud radiance fields and (Lovejoy & Schertzer (1990)) estimate the parameters at infrared and visible wavelengths.

<sup>6</sup> This idea has initiated various cascade models of turbulence: the “pulse in pulse” in Novikov & Stewart (1964); the log-normal model in Yaglom (1966); the “ $\beta$ -model” in Mandelbrot (1974) and Frisch *et al.* (1978); the “ $\alpha$ -model” in Schertzer & Lovejoy (1983); the “ $\beta$ -model” in Meneveau & Sreenivasan (1987a,b); the continuous cascade process in Schertzer & Lovejoy (1978a,b) (see also Fig. 2).

<sup>7</sup> A classification of multifractals according to their extreme singularities is discussed in Schertzer & Lovejoy (1991) and Schertzer *et al.* (1990).

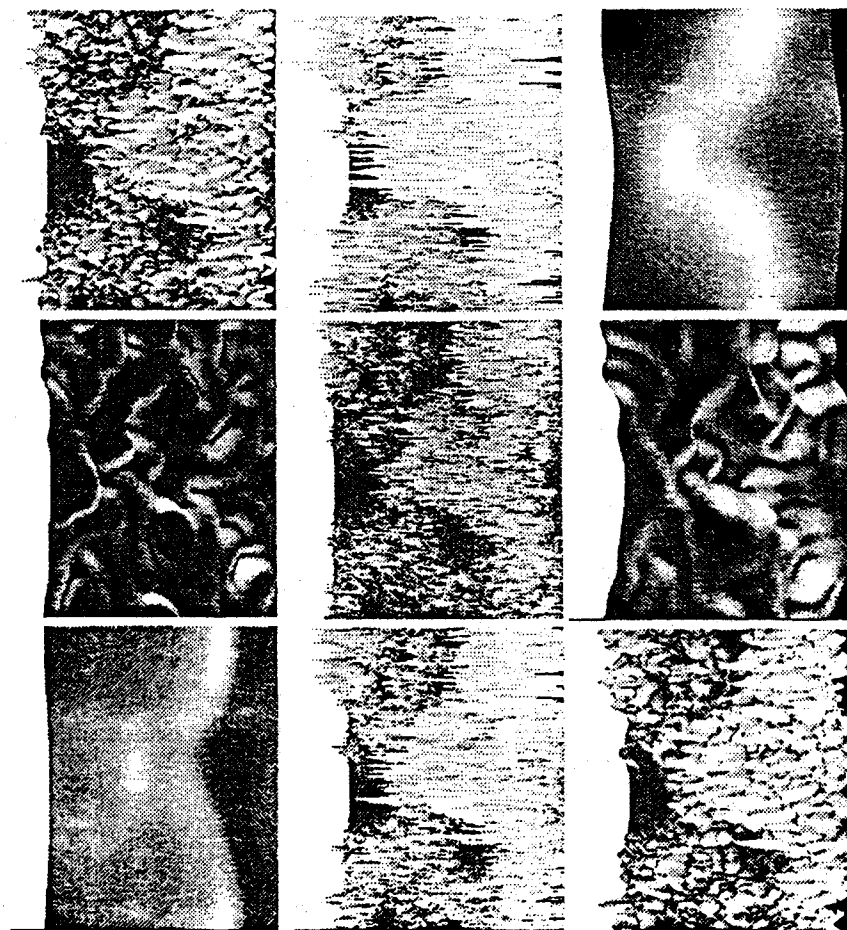


Fig. 2. The three top images and the two images on the second row correspond to five stages in the construction of a continuous multiplicative process. The resolutions are, respectively, one half of the image size,  $1/8^{\text{th}}$ ,  $1/32^{\text{th}}$ ,  $1/128^{\text{th}}$  and, finally,  $1/256^{\text{th}}$  for the center image. These correspond to the "bare" quantities. (Notice that the vertical axis for the central image has been compressed for visualization purposes.) The third image on the second row and the three images on the last row, correspond to the "dressed" quantities, obtained by averaging. Their resolutions are, respectively,  $1/128^{\text{th}}$  of the image size,  $1/32^{\text{th}}$ ,  $1/8^{\text{th}}$  and, finally, one half for the last image (bottom right-hand corner). The parameters used for these images,  $H = 1/2$ ,  $C_1 = 0.25$  and  $\alpha = 1.3$ , are those estimated for the turbulent velocity field (Schertzer *et al.* (1991), Schmitt *et al.* (1991a, b), see also below).

The function  $c(\gamma)$  is a codimension since the probability measures the fraction of the (infinite dimensional) probability space occupied by the singularities exceeding the order  $\gamma$ . Unlike the singularity spectrum function,  $f(\alpha) = D - c(\gamma)$  with  $\alpha = D - \gamma$ , (introduced in Halsey *et al.* (1986) for studying the multifractal probability measures associated with strange attractors), the codimension function is intrinsic to the process: it is independent of the observing space dimension  $D$ . The strange attractor notation can not be used in turbulence since the relevant cascades are stochastic processes, i.e.,  $D \rightarrow \infty$ , the corresponding multifractal measures ( $\epsilon$ ) are not probability measures. Equalities in (2.1) and (2.2) are valid up to slowly varying functions of  $\lambda$  (e.g., log's) and normalization constants.

Using the definition of statistical moments, the corresponding scaling behavior of the  $q^{\text{th}}$  order statistical moments is characterized by a scaling exponent  $K(q)$ :

$$\langle \epsilon_\lambda^q \rangle = \lambda^{K(q)} \propto \int \lambda^{q\gamma} \lambda^{-c(\gamma)} d\gamma. \quad (2.3)$$

The brackets  $\langle \dots \rangle$  in (2.3) indicates the statistical ensemble average, and the integral is equivalent to the Laplace transform of the probability distribution. In the limit  $\lambda \rightarrow \infty$ , the scaling exponents<sup>8</sup>  $K(q)$  and  $c(\gamma)$  are related by a Legendre transformation (Parisi & Frisch (1985) and Halsey *et al.* (1986)). The monotonically increasing function  $C(q) \equiv K(q)/(q-1)$  is the (dual) codimension function associated with the statistical moments of the field intensities. The codimensions  $c(\gamma)$  and  $C(q)$ , are not only scale invariant but also independent of the dimension  $D$  of the space in which the field is embedded. They are the fundamental functions characterizing the scaling properties of the processes under study (Schertzer *et al.* (1991)). When the mechanism of redistribution of the energy flux in the cascade process is not homogeneous, the process leads to multifractal measures described by the equation given above. Without any *a priori*, the functions  $c(\gamma)$  or  $C(q)$  could be any (respectively convex or increasing) functions. However, when the conservation principle that governs the redistribution of the energy flux is canonical<sup>9</sup> and either the turbulence is "mixed" or the cascade is "densified" by the excitation of a continuum of intermediate scales, the process

<sup>8</sup> The scaling exponent  $\tau(q) = (q-1)D - K(q)$  is also used in the literature (Halsey *et al.* (1986)).

<sup>9</sup> As in statistical physics, the canonical conservation indicates that the statistical ensemble averages of the processes ( $\epsilon_\lambda$ ) is constant (taken for convenience as 1). On the other hand, microcanonical cascade processes correspond to an exact conservation of the energy flux from the eddy to its sub-eddies (see Mandelbrot (1974) and Meneveau & Sreenivasan (1987)). Despite their apparent simplicity the microcanonical cascade constraint is quite complex and restrictive since it applies at all scales and prohibits large fluctuations. It is still not known how to generally make continuous microcanonical cascades nor are there

leads to universal multifractals (Schertzer & Lovejoy (1987a, b), Schertzer *et al.* (1991)). In this case the scaling exponents have their functional behavior specified by two parameters  $\alpha$  and  $C_1$ :

$$c(\gamma) = \begin{cases} C_1 \left( \frac{\gamma}{C_1 \alpha} + \frac{1}{\alpha} \right)^{\alpha'} & , \text{ with } \frac{1}{\alpha} + \frac{1}{\alpha'} = 1, (\alpha \neq 1, 0 \leq \alpha \leq 2) \\ C_1 \exp \left( \frac{\gamma}{C_1} - 1 \right) & , \text{ if } \alpha = 1; \end{cases} \quad (2.4)$$

$$K(q) = \begin{cases} \frac{C_1}{\alpha-1} (q^\alpha - q) & , \text{ if } \alpha \neq 1; \\ C_1 q \ln(q) & , \text{ for } \alpha = 1. \end{cases} \quad \text{with } q \geq 0 \text{ for } \alpha < 2 \quad (2.5)$$

The derivation of these equations relies on the application of the (generalized) central limit theorem for the addition of random variables to the generator of the cascade (Schertzer & Lovejoy (1987a, b, 1989)). The most basic parameter,  $\alpha$ , takes values between 0 and 2 and indicates the class to which the probability distribution belongs. When  $\alpha = 2$ , the process has a Gaussian generator whereas  $0 < \alpha < 2$  corresponds to the less well known infinite variance cases of the central limit theorem: the (stable) "Lévy" distributions with index  $\alpha$ . Decreasing the value of  $\alpha$  allows us to identify three qualitatively different cases; first a regime with unbounded singularities ( $1 < \alpha < 2$ ), followed by the (asymmetric) Cauchy generator multifractals ( $\alpha = 1$ ) and finally the third situation with  $0 < \alpha < 1$  corresponding to Lévy generators<sup>10</sup> with bounded singularities.<sup>11</sup> The case  $\alpha = 0$  coincides with the popular monofractal "β-model" (Mandelbrot (1974) and Frisch *et al.* (1978)). The second parameter,  $C_1$ , is the fractal codimension of the singularities contributing to the average values of the field. It is related to the sparsity of the average level of intensity.

### 3. The Signature of Universality

Scaling exponents are usually determined by examining the statistical properties of the fields of interest at different length scales. This can be done either by directly estimating the probability distribution of the field intensities or by looking at the behavior of their statistical moments. However, two important considerations must be taken into account to perform the analysis on simulated or empirical data.

First, the singular behavior of  $\epsilon_\lambda$  as the cascade proceeds to its small scale limit ( $\lambda \rightarrow \infty$ ) requires the introduction of the distinction between "bare"

<sup>10</sup> The expressions "log Lévy" (Brax & Peshanski (1991)) or "log stable" (Kida [1991]) are inexact due to the problem of "dressing" described below (Schertzer & Lovejoy (1987a)).

<sup>11</sup> The upper bound is given by  $C_1 / (1 - \alpha)$  when  $\alpha < 1$ .

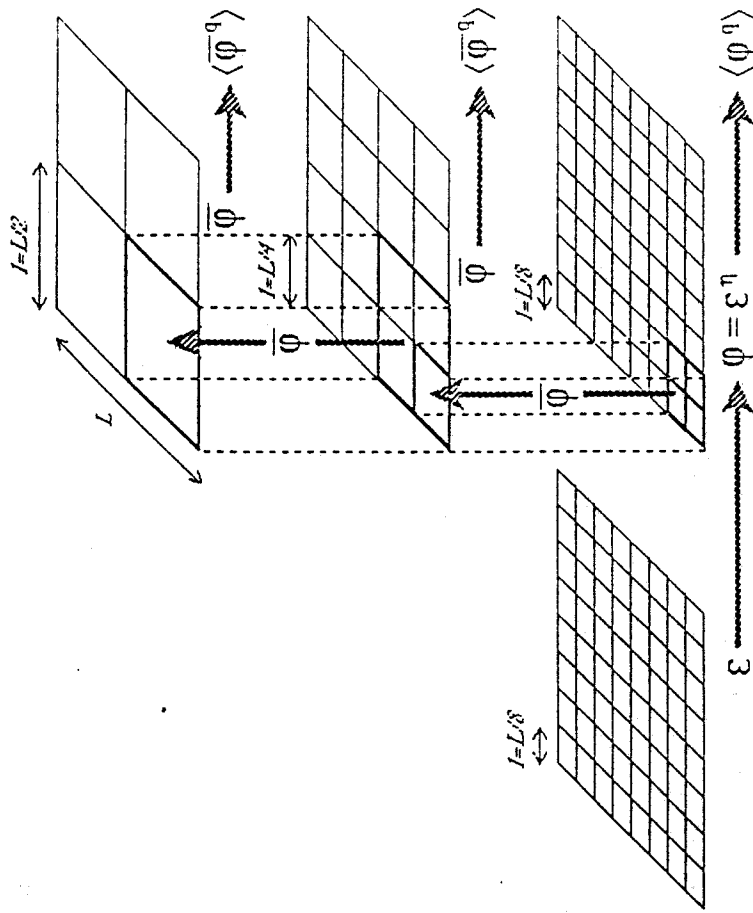


Fig. 3. A schematic diagram illustrating the different averaging scales in the DTM.

and "dressed" quantities (Schertzer & Lovejoy (1987a, b)). The "bare" quantities are essentially theoretical and are obtained after a cascade process has proceeded over a finite range of scales  $\lambda$  (see Fig. 2). The results (2.4) and (2.5) for  $c(\gamma)$ ,  $K(q)$  hold for the "bare" quantities and in the limit  $\lambda \rightarrow \infty$ . The "dressed" quantities are those obtained after integrating a completed cascade<sup>12</sup> ( $\lambda \rightarrow \infty$ ) over a finite scale  $\lambda'$ . Since they are strongly dependent on the singular behavior of the smallest length scale from which the cascade was developed, they are more variable than their "bare" counterparts at the same scale ratio. This dependence on the small scale limit is also responsible for the divergence of the high order statistical moments of the "dressed" quantities (leading to the "pseudo scaling" discussed in Schertzer & Lovejoy (1983), Lavallée (1991) and Schertzer *et al.* (1991b) now understood as a multifractal phase transition). The critical values of moments and

<sup>12</sup> More precisely, quantities are said to be finitely "dressed" when  $\lambda$  is kept finite and infinitely "dressed" when  $\lambda \rightarrow \infty$  (Lavallée (1991)).

singularities contributing to the divergence are  $q_D, \gamma_D$  where  $C(q_D) = D, \gamma_D = K'(q_D)$ . The experimental quantities are best approximated by the "dressed quantities", since they are usually obtained with measuring devices by integrating inhomogeneous fields over scales much larger than the inner scale of the processes observed. Finally, a "dressing" operation is needed to obtain (or transform) the field at various lower resolutions in order to analyze and characterize their scaling properties.

The exponents  $c(\gamma)$ —or  $K(q)$ —are then given by the scaling behavior of the probability distribution—or of the  $q^{\text{th}}$  statistical moments—of "dressed" quantities as functions of the scale ratio  $\lambda'$  on log-log graphs, for various values of  $\gamma$ —or  $q$ . The scaling exponents determined by such analysis techniques correctly characterize the multifractal processes as long as  $\gamma$ —or  $q$ —did not exceed either  $\gamma_D, q_D$  or the critical values  $\gamma_s$  or  $q_s$ . Introducing the sampling dimension  $D_s$ , related to the number of samples  $N$  by  $D_s \equiv \log_\lambda(N)$ , then the critical value  $\gamma_s$ —solution of  $c(\gamma_s) \approx D + D_s$ —is the maximum order of singularities that can be observed in the sample size  $N$ . When  $q > q_c = \min(q_s, q_D)$  the asymptotic behavior of  $K(q)$  is then linear and given by (Lavallée *et al.* (1991a)):

$$K(q) \approx q\gamma_c - c(\gamma_c), \quad q > q_c. \quad (3.1)$$

The  $q^{\text{th}}$  moment  $q_s$  corresponding, by the Legendre transform, to  $\gamma_s$  is given by the following relation:

$$q_s = \left. \frac{dc(\gamma)}{d\gamma} \right|_{\gamma=\gamma_s} = \left( \frac{D + D_s}{C_1} \right)^{1/\alpha}. \quad (3.2)$$

For small  $N, q_s < q_D$ , hence increasing the values of  $D$  or  $D_s$  provides a larger interval of values  $c(\gamma)$  or  $K(q)$  to estimate their universal parameters  $\alpha$  and  $C_1$  until the second critical values  $\gamma_D$  or  $q_D$  is reached (this cannot be avoided when the Lévy index  $\alpha$  is greater than 1). In any case the determination of the universal parameters from  $c(\gamma)$  or  $K(q)$  will depend on nonlinear regression techniques and on the ability to apply them to the appropriate range of values of  $\gamma$  or  $q$ .

The DTM method allows a direct estimate of the values of  $\alpha$ . The main idea behind it is to study how the scaling properties of fields are modified when a transformation or operation is performed on them, and in particular, to determine the functional behavior of the scaling exponents and their dependence on  $\alpha$  and  $C_1$ . The operation considered here consists in taking the  $\eta^{\text{th}}$  power<sup>13</sup> of  $\varepsilon_\lambda$  at the largest scale ratio  $\lambda'$  (smallest size). Then a dressing operation is performed on  $\varepsilon_\lambda^{\eta}$ , and various  $q^{\text{th}}$  order statistical moments are estimated at decreasing values of the scale ratio  $\lambda \leq \lambda'$  (see

<sup>13</sup> This corresponds to the change  $\gamma \rightarrow \eta\gamma$ .

also Fig. 3). The  $q, \eta$  DTM at resolution  $\lambda'$  and  $\lambda$  has the following multiple scaling behavior :

$$T\tau_\lambda \left[ \varepsilon_\lambda^\eta \right]^q = \left\langle \sum_i \left( \int_{B_{\lambda_i}} d^D x \varepsilon_\lambda^\eta \right)^q \right\rangle \approx \lambda^{K(q,\eta) - (q-1)D}, \quad (3.3)$$

where the sum is over all the events  $\varepsilon_\lambda^\eta$ , in the disjoint boxes  $B_{\lambda_i}$  at scale  $\lambda$  (of volume  $\lambda^{-D}$ ) and corresponds to the "dressing" operation. This is equivalent to averaging the fields over the boxes  $B_\lambda$ . The sum is over all the disjoint boxes  $B_\lambda$  needed to completely cover the field. For  $\eta = 1$  the right hand side of (3.3) reduces to the usual trace moment (the ensemble average of the usual partition function). The scaling exponent  $K(q, \eta)$  is related to the usual scaling exponent  $K(q, 1) \equiv K(q)$  by the following relation (Lavallée (1991) and Lavallée *et al.* (1991b, c)):

$$K(q, \eta) = K(q\eta) - qK(\eta). \quad (3.4)$$

The first term on the right side of (3.4) is the usual scaling exponent of  $\langle \varepsilon_\lambda^{q\eta} \rangle$  (see (2.4)), corresponding to the scaling properties of the "bare" quantities at the same scale length  $\lambda$ . The second term assures the preservation of the two scaling symmetries characterizing the operation performed to "dress" the fields at different scale ratios  $\lambda'$ . First, the "dressing" operation conserves the first moment<sup>14</sup>,  $q = 1$ , of the  $\varepsilon_\lambda^\eta$  which is constant (scale independent), this implies that  $K(1, \eta) = 0$ . Second, for  $q = 0$ , the statistical moments must be independent of the multifractal processes<sup>15</sup> at any scale ratio  $\lambda$ , hence  $K(0, \eta) = 0$ . Use of the universality classes in (2.5) gives the expression for  $K(q, \eta)$ :

$$K(q, \eta) = \eta^\alpha K(q, 1) = \begin{cases} \frac{C_1}{\alpha-1} \eta^\alpha (q^\alpha - q), & \alpha \neq 1; \\ C_1 \eta q \log(q), & \alpha = 1, \end{cases} \quad (3.5)$$

with  $0 \leq \alpha \leq 2$  and  $q > 0$  for  $\alpha \neq 2$ .

The main feature of this relation is that  $K(q, \eta)$  factorizes into the product of two functions, one for each of the independent variables  $\eta$  and  $q$ . The  $\eta$ -dependent part has a simple dependence on  $\alpha$ . Thus, by keeping  $q$  fixed (but different from the special values 0 or 1) the slope of the curve of  $|K(q, \eta)|$  as a function of  $\eta$  on a log-log graph gives the value of the parameter  $\alpha$ . Universality assures us that for different values of  $q$  these curves

<sup>14</sup> The sum of averaged  $\varepsilon_\lambda^\eta$  (over boxes of different sizes) is constant. For  $q = 1$  the sum and the integral commutes in (3.3).

<sup>15</sup> In this case the dependence in  $\lambda$  is only through the (multiple) scaling properties of the support (dimension  $D$ ) of the processes.

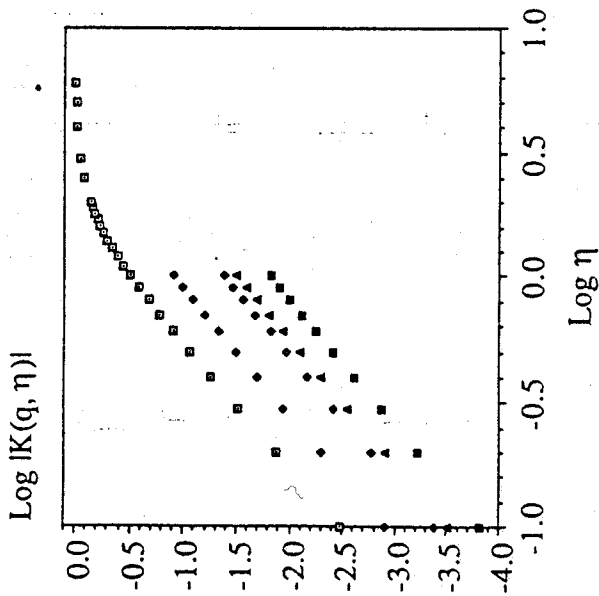


Fig. 4. From top to bottom, the curves of the  $\log |K(q, \eta)|$  as functions of the  $\log \eta$ , with  $\alpha = 2$  (lognormal) and  $C_1 = 0.18$ , are given for the following values of  $q$ : 2., 1.5, 0.5, 0.25 and 0.9. All the curves are parallel as predicted by (3.5). When their slopes are used to estimate  $\alpha$ , with  $\eta$  taking values between 0.1 and 1, we obtained respectively: 1.97, 1.99, 2.00, 1.99 and 2.00. The values of  $C_1$ , obtained by solving the expression for the  $\log \eta = 0$  intercept given by the  $\log |K(q, 1)|$  and using (3.5), are respectively: 0.16, 0.17, 0.17 and 0.17. For  $q = 2$  and  $\eta$  large enough the curve  $K(2, \eta)$  becomes constant. Other numerical results are discussed in Lavallée (1991) and Lavallée *et al.* (1991b).

will be parallel and will have the same slopes. Thus, this typical behavior of the  $K(q, \eta)$  is the signature of universality as illustrated in Fig. 4. On the log-log plot the values of  $C_1$  can be estimated by solving the expression for the intercept, which is a function of  $C_1$  and  $\alpha$ , or alternatively by solving the expression of the slope of  $K(q, \eta)$  as a function of  $\eta^\alpha$  on a log-log graph. It is important to notice that in both cases the accuracy of the estimates of  $C_1$  will depend on the accuracy of the estimated  $\alpha$ . Whenever  $\max(q, q\eta) > \min(q, qD)$  expression (3.4) breaks down and the scaling exponent  $K(q, 1)$  becomes a linear function of  $q$  and (3.4) indicates then that  $K(q, \eta)$  becomes independent of  $\eta$  (see also Fig. 4).

The particular scaling behavior of the DTM provides two advantages when compared with the existing multifractal analysis methods. First, the estimated scaling exponent  $K(q, \eta)$  is independent of the normalization<sup>16</sup> of

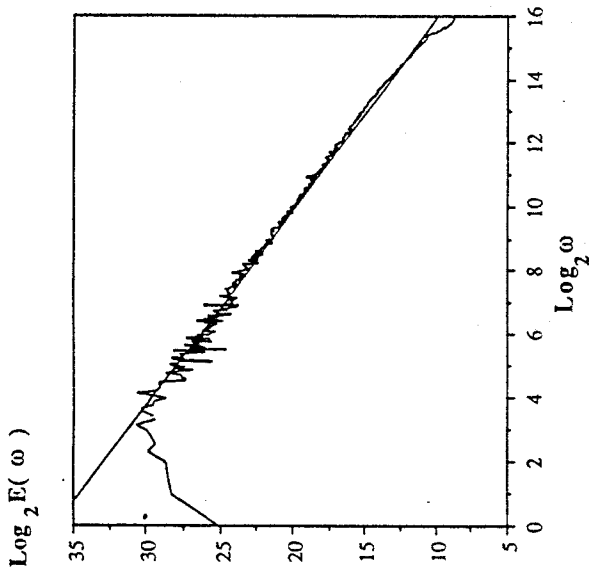


Fig. 5. The velocity spectrum averaged over  $2^9$  samples each of length  $2^{13}$  times the finest resolution (which corresponds to 5kHz). The spectrum  $E(\omega) \approx \omega^{-\beta}$  ( $\omega$  is the frequency) with  $\beta \approx 1.65$  is shown for comparison.

the data (when  $\gamma \rightarrow \gamma + b$ ). The second is that when a multiplicative change of  $\gamma$  is made ( $\gamma \rightarrow a\gamma$ ) the scaling exponent is transformed as  $K(q, \eta) \rightarrow K(q, a\eta) = a^\alpha K(q, \eta)$  (where  $a$  corresponds to a contraction in the  $\gamma$  space, but is also equivalent to the raising of the fields  $\epsilon_N$  to an unknown power  $a$  by the experimental apparatus). This implies that the determination of  $\alpha$  will also be independent of the power  $a$  to which the process is raised.<sup>17</sup> In other words the universality has been exploited to give a method to determine  $\alpha$  which is invariant under the affine transformation  $\gamma \rightarrow a\gamma + b$ . Since (3.5) is invariant under the transformation  $\eta \rightarrow a\eta$  and  $C_1 \rightarrow a^{-\alpha} C_1$ , if  $a$  is unknown, the parameter  $C_1$  can only be determined up to a multiplicative constant  $a^\alpha$ .

#### 4. Universality in Turbulent Flows

The velocity measurements were performed by Gagne at the ONERA wind tunnel, and by Modane with a high resolution hot wire anemometer, sampling at 5kHz. The number of samples analyzed is  $N = 2^9$ , each of scale partition function.

<sup>17</sup> The notation  $a$  is reserved to the factors of the  $\alpha$  generated in the cascade process or

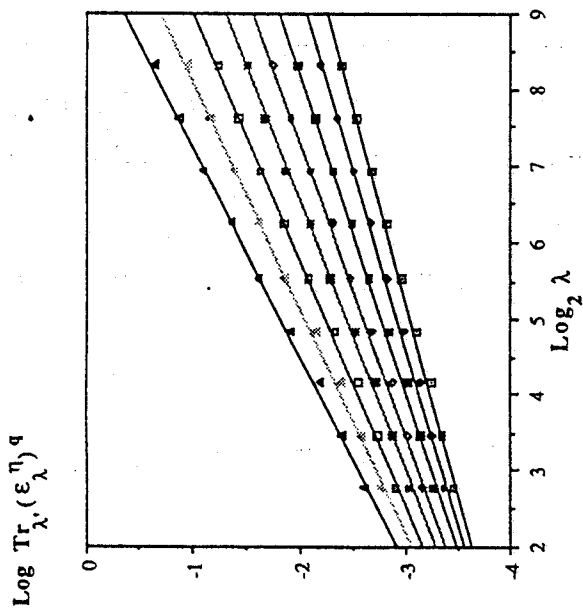


Fig. 6. Log of the DTM as a function of  $\lambda$  for various values of  $q$ ,  $\eta$ , showing that the scaling is well respected. The extreme four octaves which are part of the dissipation range were not used.

ratio  $\approx 2^9$  in the inertial (scaling) regime and  $\approx 2^4$  in the dissipation region (hence  $D_s = 1$ ). The time series had empirical energy spectra of the form  $E(\omega) \approx \omega^{-\beta}$ , with  $\beta \approx 1.65$ ,  $\omega$  is the frequency as displayed in Fig. 5. It was then power law filtered by  $\omega^{1/3}$ . This removes the  $\lambda^{-1/3}$  Kolmogorov scaling yielding the roughly stationary quantity  $\varepsilon_\lambda^{1/3}$  related to the velocity by:

$$\Delta v_\lambda \approx \varepsilon_\lambda^{1/3} \lambda^{-1/3}. \quad (4.1)$$

The amount of filtering required to yield an exactly stationary process is not exactly  $\omega^{1/3}$ . It also depends on the exponent involved in the statistical space/time transformation required to transform from temporal to spatial statistics (in the atmosphere the usual Taylor's hypothesis needs an anisotropic generalization), but also on the (initially unknown) values of  $C_1$ ,  $\alpha$ , as well as on possible deviations from the theoretical behavior. Fortunately, the DTM technique is not adversely affected by the precision of the filtering (Lavallée (1991)).

The DTM technique is applied to the series—transformed as indicated above—for various values of the parameters  $q$ ,  $\eta$ . The results are shown in Fig. 6. As long as  $\eta$  and  $q\eta$  are below  $q_s \approx 5$ , the plots of  $\log |K(q, \eta)|$  vs.

Log  $K(q, \eta)$

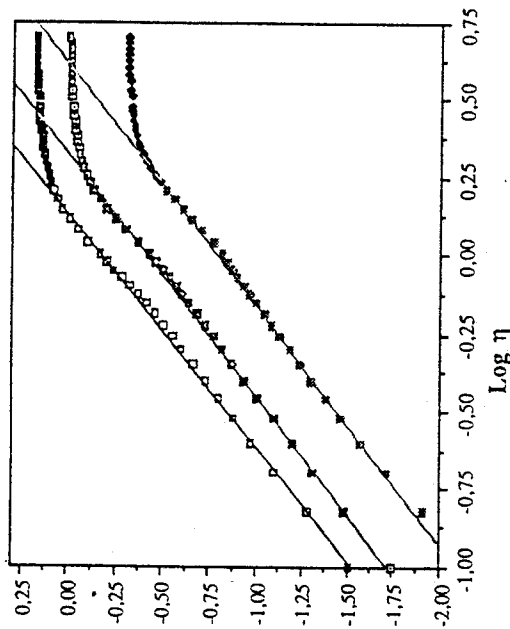


Fig. 7. Curves of  $\log |K(q, \eta)|$  vs  $\log \eta$ , for  $q = 2.5, 2.0, 1.5$  (top to bottom). The curves are nearly parallel and for  $\eta < q_s \approx 5$  have slopes  $\alpha \approx 1.3 \pm 0.1$ . Furthermore, the values of  $\log |K(q, 1)|$  give  $C_1 \approx 0.25 \pm 0.05$ .

$\alpha \approx 1.3 \pm 0.1$ , and intercepts yielding  $C_1 \approx 0.25 \pm 0.05$ . For comparison with other empirical results, the standard intermittency parameter<sup>18</sup>  $\mu$  is calculated. For the lognormal model ( $\alpha = 2$ ),  $\mu = 2C_1$ , whereas for the  $\beta$ -model ( $\alpha = 0$ ),  $\mu = C_1$ . Here, with  $\alpha = 1.3$ , we obtain  $\mu \approx 1.55C_1 = 0.35 \pm 0.1$  which is exactly in the middle of the conventionally accepted range 0.2 (Meneveau & Sreenivasan (1987b)) to 0.5 (Anselmetti *et al.* (1984)). Using (3.2) we obtained the critical values  $q_s \approx 5.0$  which is in agreement with the breakdown of (3.5) for large  $q$  found in Fig. 7.

A summary of the estimated  $\alpha$  and  $C_1$ , when using the DTM to study and characterize the scaling properties of a wide variety of experimental data, are given in table I.

## 5. Conclusion

The results discussed above confirm the importance of scale invariance in characterizing the extremely variable nature of turbulence. Because  $\alpha \geq 1$  turbulence is an unconditionally "hard" multifractal process: i.e., high enough order statistical moments will diverge when energy fluxes are averaged over

<sup>18</sup> This is the autocorrelation exponent for  $\varepsilon_\lambda$ ,  $\mu = K(2, 1)$

| The empirical data                   |                      | $\alpha$ | $C_1$    | References                      |
|--------------------------------------|----------------------|----------|----------|---------------------------------|
| Temperature in turbulent flows       |                      | 1.2      | 0.35     | Schertzer <i>et al.</i> (1991)  |
| Wind velocity in turbulent flows     |                      | 1.3      | 0.25     | Schmittet <i>et al.</i> (1991a) |
| Rain§                                | vertical direction   | 1.35     | 0.15     | Tessier <i>et al.</i> (1991)    |
|                                      | time                 | 0.5      | 0.6      | Tessier <i>et al.</i> (1991)    |
| Rain§§                               | horizontal direction | 1.35     | 0.15     | Tessier <i>et al.</i> (1991)    |
|                                      | time                 | 0.5      | 0.6      | Tessier <i>et al.</i> (1991)    |
| Hadron jets                          |                      | 0.5-0.7  | 0.5-0.25 | Ratti <i>et al.</i> (1991)      |
| Sea surface infrared reflectivities  |                      | 1.75     | 0.15     | Lavallée <i>et al.</i> (1991d)  |
| Landscape topography                 |                      | 1.8      | 0.05     | Lavallée <i>et al.</i> (1991c)  |
| Cloud radiances (visible wavelength) |                      | 1.35     | 0.10     | Tessier <i>et al.</i> (1991)    |
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§§ Daily rainfall accumulation on the entire earth for 1983

TABLE I

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spaces with arbitrary dimensions (as long as the average is over scales much larger than the dissipation scale). This is in qualitative accord with empirical evidence from atmospheric data<sup>19</sup> (Schertzer & Lovejoy (1985b)). The "hard" nature of the turbulence is also confirmed by an independent estimate  $\alpha \approx 1.65$  by Kida (1991).<sup>20</sup> The result given in section 4 holds for length scales much smaller than the resolution of the data set (i.e., as long as the scaling symmetry is preserved) and hence provide a complete statistical description of the hierarchy of singularities of the Navier-Stokes equations.

<sup>19</sup> The atmospheric data suggested  $q_D \approx 5$ , not  $q_a$ . It is possible that the discrepancy is due to different  $C_1$  values in the atmosphere and wind tunnels. Experiments are currently underway to check this possibility.

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