

Turbulent fluctuations in financial markets: a multifractal approach

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1. Introduction

Due to their common properties with fluid turbulence financial markets are often said to be turbulent: they display huge intermittent fluctuations at all scales. Recently, several papers by physicists have pushed the analogy further, applying some methods of data analysis used in turbulence or geophysics to financial datasets [1-11]. The common feature of these data analyses is scaling: in both turbulence and in finance, over wide ranges the fluctuations have no characteristic scales, leading to scaling statistics. In both cases the first theories proposed were thus scaling and homogeneous, presenting fluctuations at all scales, but no intermittency: Brownian motion in finance [12-13] and the Kolmogorov-1941 law in turbulence [14], which has been modelled by a fractional Brownian motion of order $1/3$ by some authors. Later, intermittency was introduced in turbulence using scaling and intermittent models, ultimately leading to multifractals. In [11] we tested a generic multifractal process, universal multifractals, on Foreign exchange data, in order to validate this model and estimate its parameters; here we recall the basic results we obtained and present some new analysis.

2. Multifractal analysis of the data

2.1 *The data series and the power spectra*

The data we analyze are 5 daily Foreign Exchange rates in French Franc: Swiss Franc (CHF), German Mark (DEM), US Dollar (USD), Great Britain Pound (GBP), Japanese Yen (JPY). Each data series extends from 1 January, 1979 to 30 November, 1993: taking into account only the active days, we have 3680 data points for each series. In Fig. 1 we show two data series: DEM and GBP, showing two qualitatively different behaviours. The DEM fluctuations have a different aspect due to the imposed currency area allowing a maximum of 4.5 p.cent fluctuations between FRF and DEM inside the European Monetary System since 1979. We will see that our framework helps to quantitatively characterize this difference.

For our analysis, we directly studied the fluctuations of our data. In [11] we showed that if one studies the returns $\Delta X_\tau(t) / X(t)$ for a time increment τ (where $X(t)$ is the value of the

exchange rate at time t and $\Delta X_\tau(t) = X(t+\tau) - X(t)$ or the fluctuations $\Delta X_\tau(t)$ themselves, there is no important difference in the scaling exponents. First we performed a Fourier (spectral) analysis: this is a way to estimate the scaling nature of the fluctuations, and also the relative amplitude of each frequency. As is typically obtained, we find the following scaling spectrum:

$$E(f) \propto f^{-\beta} \quad (1)$$

where f is the frequency, ' \propto ' means proportionality and β is the scaling exponent of the power spectrum, which is found to be close to 2 (it is exactly 2 for Brownian motion). This is shown in Fig. 2 for all the spectra, with a straight line of slope -2 for comparison.

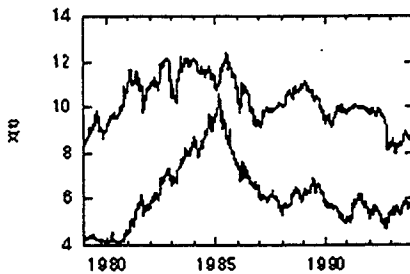


Fig. 1. Two of the foreign exchange time series which are analyzed here: GBP (up) and USD (bottom).

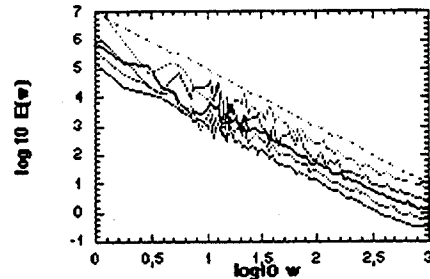


Fig. 2. Power spectra of the time series, in log-log plot; from bottom to top: DEM, CHF, JPY, GBP, USD. For comparison, a continuous line of slope of -2 is also shown.

The power spectrum is only a second order statistic and its slope is not enough to uniquely specify a scaling model: it gives only partial information about the statistics of the process, except when gaussian. One needs the knowledge of the probability distributions of the process, or of its statistical moments other than second order. The usefulness of multifractal analysis is precisely in characterizing all order moments at all scales (and hence the probability distributions at all scales) for the validation of a scaling model. This is done in next section using structure function scaling exponents. In the following analysis, we quickly recall the main results of [11] before considering the GBP fluctuations in two different situations. Finally we discuss the question of the signs of the fluctuations.

2.2 Scaling of the structure functions, convexity and multifractality

Structure function analysis consists in studying the scaling behavior of fluctuations for different time increments τ . This is done by estimating the statistical moments of these fluctuations, which depend only on the time increment in a scaling way (see [15]):

$$\langle |\Delta X_\tau|^q \rangle = C_q \left(\frac{\tau}{T} \right)^{\zeta(q)} \quad (2)$$

where T is the fixed larger time scale of the system, $C_q = \langle |X(t+T) - X(t)|^q \rangle$ is a constant (in τ), ' $\langle \rangle$ ' denotes ensemble average, q is the order of the moments (we take here $q > 0$), and $\zeta(q)$ is the scale invariant structure functions' exponent. The average of the fluctuations corresponds to $q=1$, and $H = \zeta(1)$ is the parameter characterizing the non-conservation of the mean. For Brownian motion, $\zeta(1) = 1/2$. The second moment is essentially the Fourier transform of the spectrum and is linked to the slope β of the power spectrum: $\beta = 1 + \zeta(2)$.

We estimated $\zeta(q)$ as the slope of $\langle |\Delta X_\tau|^q \rangle$ vs. τ in log-log plot for all moments between 0.1 and 4.0 with a 0.1 increment. The resulting curves for the 5 data series are presented in Figs. 3 and 4: as we discussed in [11] their nonlinear shape is a signature of multifractality. In particular this invalidates additive models for which this function is linear, or bi-linear (two portions of straight lines): $\zeta(q) = q/2$ for Brownian motion; $\zeta(q) = q(h-1/2)$ for a fractional integration of order h ($0 < h < 1$) of a Gaussian noise (see [16]). For Lévy and fractional Lévy motion, we showed in [11] the following relations:

$$\zeta(q) = \begin{cases} q(h-1+1/\alpha) & q < \alpha \\ q(h-1-D_s/\alpha) + 1 + D_s & q \geq \alpha \end{cases} \quad (3)$$

where α is the characteristic exponent of the Lévy noise ($0 \leq \alpha \leq 2$; for $\alpha=2$ one recovers Gaussian noise) which is fractionally integrated by h ($0 < h \leq 1$; $h=1$ for 'Lévy motion'), D_s is a 'sampling dimension' [17] such that $N_s = (T/\tau)^{D_s}$ is the number of realisations studied. The change of slope between the two straight lines is $(1+D_s)/\alpha$. Here we have 1 sample and hence $D_s=0$. Even if the empirical nonlinear shapes we obtain in Fig. 3 for $\zeta(q)$ could be (poorly) approximated by two lines in order to test this model, the change of slope occurs generally at a moment larger than 2, and the empirical changes in slope are smaller than $1/2$, therefore invalidating this model for which this change ($=1/\alpha$) must be larger than $1/2$.

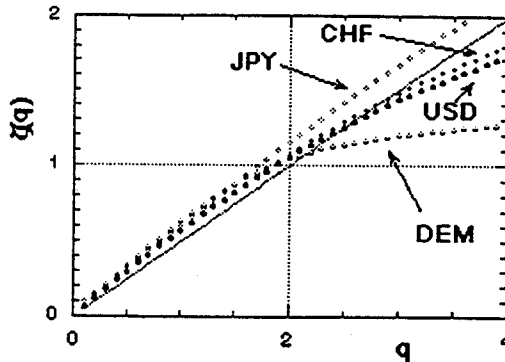


Fig. 3. The functions $\zeta(q)$ obtained for several times series. Their nonlinearity indicates multifractality. The line corresponds to $\zeta(q) = q/2$, obtained for Brownian motion.

Figure 4 shows a portion of the GBP time series: when outside the European Change Mechanism (ECM) from 1979 to October 1990 the fluctuations have a different appearance than those when inside the ECM from October, 1990 to October, 1992 with the imposed currency area allowing a maximum of 12 % fluctuations between FRF and GBP inside the European Monetary System. We quantify here this visual difference using $\zeta(q)$ for both portions of the GBP series. In these two contexts, the fluctuations are still scaling, but with scaling exponents clearly different for moments larger than 1, as shown by Fig. 5. The fluctuations when inside the ECM are more intermittent (more pronounced convexity of $\zeta(q)$, smaller value of $H=\zeta(1)$); this may be due to 'outside' intervention from central banks in order to keep the currencies inside the imposed margins: these interventions correspond to superimposed intermittency, in addition to the 'natural' intermittency which would be obtained if the system had no interference. The same behavior is visible for DEM (for which there is a maximum of 4.5 % fluctuations since 1979): its function $\zeta(q)$ has a more convex shape than of CHF or USD.

2.3 Universal multifractal parameters

Multiplicative cascade models [17] are rather generic multifractal processes. If one densifies (in scales) the cascade, one obtains continuous cascade models [17]; recently the term infinitely divisible cascades (referring to the infinitely divisible probability distributions) has also been used [18]. Choosing different infinitely divisible laws, the following models were obtained in the turbulence literature: log-normal [19-21], log-Lévy [17, 22-25], log-Gamma [26], log-Poisson [27, 18]. Although all these models share in common a weak universality (i.e. few relevant parameters define the infinite hierarchy of exponents), only those having a Lévy generator correspond to a strong universality (as discussed in [25]): within the multiplicative framework, they are the only stable (and attractive) models, i.e. the only models which are stable under the operation of raising to various powers, or convolution with different realizations of similar processes. Multiplicative models are mathematically more precise formulations of the (vague) law of "proportionnal effects" which was once believed to generally lead to lognormal distributions. The development of multifractals has shown that the universality class is larger; encompassing Lévy generators, with gaussian only as a special case. The following form is then obtained for $\zeta(q)$:



Fig. 4. A portion of the GBP time series showing the period of two years during which it was inside the ECM. This period ends with an abrupt fall.

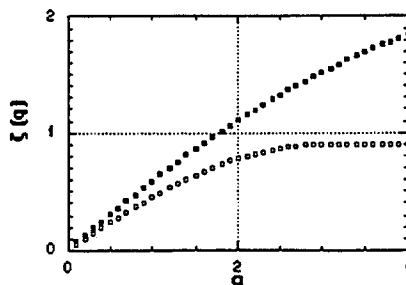


Fig. 5. The function $\zeta(q)$ obtained for GBP when outside the ECM (full dots) or inside the ECM (open dots).

$$\zeta(q) = qH - \frac{C_1}{\alpha - 1}(q^\alpha - q) \quad (4)$$

where C_1 is the fractal codimension of the mean of the process ($0 \leq C_1 \leq d=1$ for a 1-D dataset), and α is the Lévy index (of the generator, which is the log of the multiplicative process); we have $0 \leq \alpha \leq 2$. The log-normal model corresponds to $\alpha=2$. We may emphasize here the difference of this model with additive Lévy models: for universal multifractals, the Lévy distribution is not assumed for the difference of the price, but for its generator, the log of the (absolute) difference. We estimated these 3 parameters in [11], using for α a technique taking into account the non-analyticity of $\zeta(q)$ at $q=0$ (see Table 1).

The values reported in Table 1 show that $H=0.60 \pm 0.03$ is quite stable for the different series, as well as $\beta=2.10 \pm 0.05$ (the latter is given for information and is not a fundamental parameter of the model) whereas C_1 is more variable: $C_1=0.05 \pm 0.02$ and α seems to be less precisely estimated, being approximately $\alpha=1.5 \pm 0.35$. This last parameter needs in fact more data (with for example intraday Foreign exchange data) to be more accurately estimated; with more data, it is possible that the estimates of this parameter for different currencies will be closer to the mean value 1.5.

Series	$H=\zeta(1)$	$\beta=1+\zeta(2)$	C_1	α
CHF	0.56	2.07	0.03	1.75
DEM	0.63	2.05	0.08	1.6
GBP	0.60	2.13	0.04	1.26
JPY	0.60	2.15	0.05	1.15
USD	0.58	2.06	0.05	1.87

Table 1. The parameters H , C_1 and α estimated for the five time series, with a method explained in Schmitt et al (1997). The value of $\beta=1+\zeta(2)$ is also given, as estimated from the slope of the power spectrum (this last value is linked to the three others).

2.4 The sign of the fluctuations

Up until now our analysis contains information on the statistics (at all scales) of the amplitude of the fluctuations in the time series. In this short section we focus on the sign of these fluctuations, an important quantity if we want to model financial time series. Lie cascades [28] or at least complexe cascades should be used in order to properly introduce the sign information into a multiplicative cascade. Here as a first step, we simply transform our series into a series of signs +1 or -1 of the (non-overlapping) fluctuations: $s_\tau(t) = \text{sign}[\Delta X_\tau(t)]$. We then test here mainly some possible correlations in these series of signs, using the correlation function $C_\tau(T) = \langle s_\tau(t) s_\tau(t+T) \rangle$. We compute this function for each series and for various values of the increment τ (not too large because there are N/τ different signed data, where N is the original size of the series). A significant correlation in the signs (a power law or exponential decrease for example) indicates a structure and the necessity to modeling the signs with a well suited algorithm. This is shown in Fig. 6 for the USD series, for $\tau=1$ and 3: there is a flat correlation function oscillating randomly around 0, which seems to indicate that we can model the signs in a simple random non-correlated manner. The same is obtained for the other time series, except JPY which contains a structure (see Fig. 7) which therefore need more detailed analysis to be properly modeled.

3. Conclusion and discussion

We analyzed with multifractal analysis techniques several foreign exchange rates data which we compared and contrasted with geophysical turbulence. Some of the currencies are very rarely traded as JPY/FRF, others belong to the European Monetary System and some are frequently traded ('liquid') as USD/FRF and GBP/FRF. Despite these differences, all the series showed multifractality in their statistics, with nonlinear and convex shape of their structure function exponents $\zeta(q)$. The currencies belonging to ECM (DEM/FRF and GBP/FRF during two years) displayed a rather different shape for $\zeta(q)$ than other currencies which have no limit in their variation: this corresponds to new intermittency introduced in the system in order to keep the fluctuations inside some margins. The fluctuations of the different series rather correspond to a universal multifractal model: the parameters obtained in this framework are not far from each other, which may indicate a unique limit process corresponding to the 'trading currency' process. This, with the sign information we obtained here, can be used to produce 'synthetic' simulated multifractal financial time series, with an algorithm which is numerically implemented in [29].

We must finally underline an important point behind this analysis: multifractal fields present long range correlations, coming from the embedding structure of multiplicative cascades which are used to produce them. Therefore the time series are not Markovian (more precisely they do not correspond to a finite Markov chain) and the long range interrelations which they display can be used for predictability purposes (see [30] in another related context): the general approach corresponds to use the past of the process in order to compute in an optimal manner the generator defined by this past, as well as its statistical influence on the future.

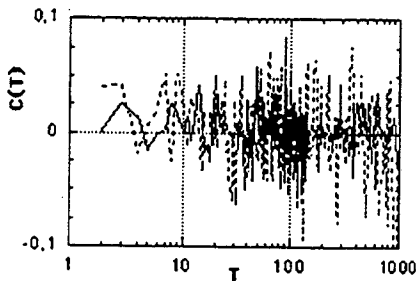


Fig. 6. Correlation function of the signs of USD time series, for $\tau=1$ (continuous line) and $\tau=3$ (dotted line): no trend is visible.

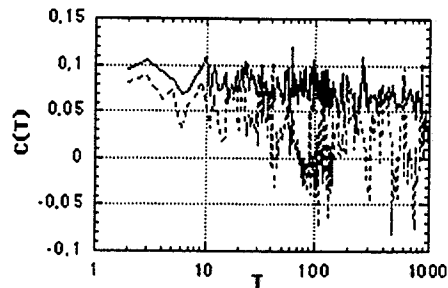


Fig. 7. Correlation function of the signs of JPY time series, for $\tau=1$ (continuous line) and $\tau=3$ (dotted line): a weak trend is visible.

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Bibliography

- 1 Vassilicos, J. C., A. Demos and F. Tata, "No Evidence of Chaos But Some Evidence of Multifractals in the Foreign Exchange and the Stock Markets", in *Applications of Fractals and Chaos*, A. J. Crilly et al. eds., Berlin, Springer-Verlag, 1993, pp.249-265.
- 2 Evertsz, C., "Fractal geometry of financial time series", *Fractals*, 3, 609-616 (1995).
- 3 Mantegna R. and H. Stanley, "Scaling behaviour in the dynamics of an economic index", *Nature*, 376, 46-49 (1995).
- 4 Mantegna R. and H. Stanley, "Turbulence and financial markets", *Nature*, 383, 587-588 (1996).
- 5 Mantegna R. and H. Stanley, "Stock Market dynamics and turbulence: parallel analysis of fluctuation phenomena", *Physica A*, 239, 255-266 (1997).
- 6 Ghashghaie S. et al, "Turbulent cascades in foreign exchange markets", *Nature*, 381, 767-770 (1996).
- 7 Bouchaud, J.-P., D. Sornette and M. Potters, "Option pricing in the presence of extreme fluctuations", *Proceedings of the Newton Institute Session on Mathematical Finance*, M. Dempster and S. Pliska eds., Springer, 1997.
- 8 Sornette, D., A. Johansen and J.-P. Bouchaud, "Stock Market Crashes: Precursors and Replicas", *J. Phys. I France*, 6, 167-175 (1996).
- 9 Stanley, M. H. et al, "Can statistical physics contribute to the science of economics?", *Fractals*, 4, 415-425 (1996).
- 10 Vandewalle, N. and M. Ausloos, "Coherent and random sequences in financial fluctuations", *Physica A* (1997, in press).
- 11 Schmitt, F., D. Schertzer and S. Lovejoy, "Multifractal analysis of foreign exchange data", submitted to *Appl. Stoch. Models Data Analys.* (1997).
- 12 Bachelier, L., "Theory of Speculation" (Translation of 1900 French edn), in *The random character of stock market prices*, Cootner, P.H. editor, Cambridge, The MIT Press, 1964, pp.17-78.
- 13 Osborne, M. F., "Brownian Motion in the Stock Market", *Opns. Res.*, 7, 145-173 (1959).
- 14 Kolmogorov, A. N., "The local structure of turbulence in incompressible viscous fluid for very large Reynolds number", *C. R. Acad. Sci. URSS*, 30, 299 (1941).
- 15 Monin, A. S. and A. M. Yaglom, *Statistical Fluid Mechanics*, MIT Press, 1975.
- 16 Mandelbrot B. and Van Ness, "Fractionnal Brownian motions, fractionnal noises and applications", *SIAM Rev.*, 10, 422 (1968).
- 17 Schertzer, D. and S. Lovejoy, "Physical modeling and analysis of rain and clouds by anisotropic scaling multiplicative processes", *J. Geophys. Res.*, 92, 9693-9721 (1987).
- 18 She, Z. S. and E. Waymire, "Quanticized energy cascade and Log-Poisson statistics in fully developed turbulence", *Phys. Rev. Lett.*, 74, 262-265 (1995).
- 19 Kolmogorov, A. N., "A refinement of previous hypothesis concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number", *J. Fluid Mech.*, 13, 82 (1962).
- 20 Obhukov, "Some specific features of atmospheric turbulence", *J. Fluid Mech.*, 13, 77 (1962).
- 21 Yaglom, A., "The influence of fluctuations in energy dissipation on the shape of turbulent characteristics in the inertial range", *Sov. Phys. Dokl.*, 2, 26 (1966).
- 22 Schertzer, D. and S. Lovejoy, "Nonlinear variability in geophysics: multifractal simulations and analysis", in *Fractals: Physical Origin and Consequences*, L. Pietronero ed., Plenum, New York, 1989, pp.49-77.
- 23 Kida, S., "Log-stable distribution and intermittency of turbulence", *J. Phys. Soc. Japan*, 60, 5 (1991).
- 24 Schertzer, D., S. Lovejoy and F. Schmitt, "Structures in turbulence and multifractal uni-

- versality", in *Small scale structures in 3D hydro and MHD turbulence*, M. Meneguzzi et al. eds., Springer, New York, 1995, pp.137-144.
- 25 Schertzer, D. and S. Lovejoy, "Universal Multifractals do exist!", *J. Appl. Meteor.* 36, 1296 (1997).
- 26 Saito, I. "Log-Gamma distribution model of intermittency in turbulence", *J. Phys. Soc. Jpn.*, 61, 403 (1992).
- 27 Dubrulle, B., "Intermittency in fully developed turbulence: log-Poisson statistics and generalized scale covariance", *Phys. Rev. Lett.*, 308, 151-154 (1994).
- 28 Schertzer, D. and S. Lovejoy, "From scalar cascades to Lie cascades: joint multifractal analysis of rain and cloud processes", in *Space and Time scale variability and interdependencies in hydrological processes*, R. Feddes ed., Cambridge University Press, New York, 1995, pp.153-172.
- 29 Pecknold, S. et al, "The simulation of universal multifractals", in *Cellular Automata: Prospects in Astronomy and Astrophysics*, J.M. Perdang and A. Lejeune eds., World Scientific, Singapore, 1993, pp.228-267.
- 30 Marsan D., D. Schertzer and S. Lovejoy, "Causal space-time multifractal processes: Predictability and forecasting of rain fields", *J. Geophys. Res.*, 101, 26333 (1996).