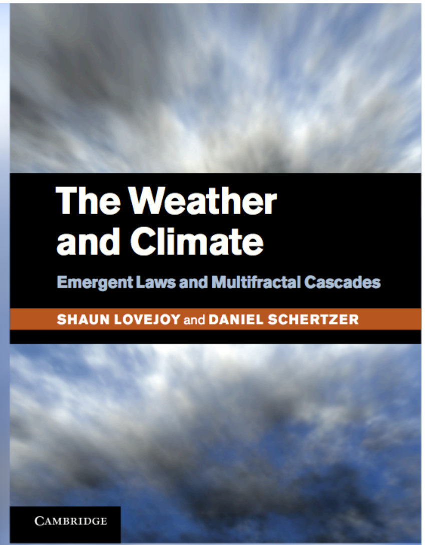


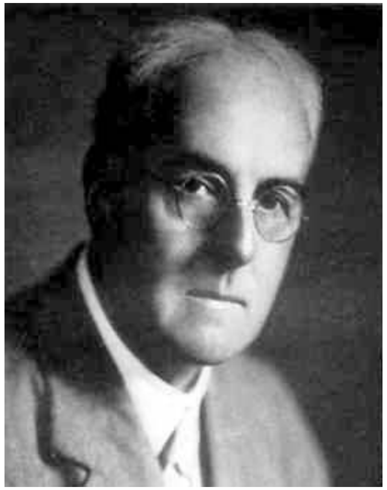
Rocks, clouds, anisotropic multifractals and the unity of geophysics

Multifractal Analysis: From Theory to Applications and Back,
Banff, Alberta, Feb, 25, 2014

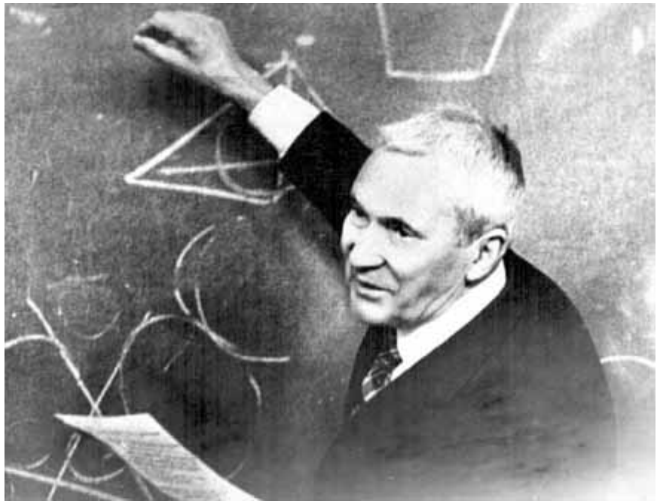
S. Lovejoy McGill, Montreal

Pioneers of turbulence

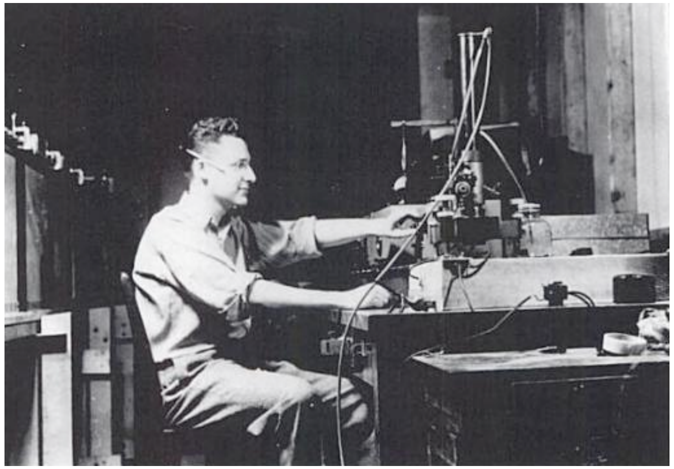
Richardson
1881 - 1953



Kolmogorov
1903 – 1987

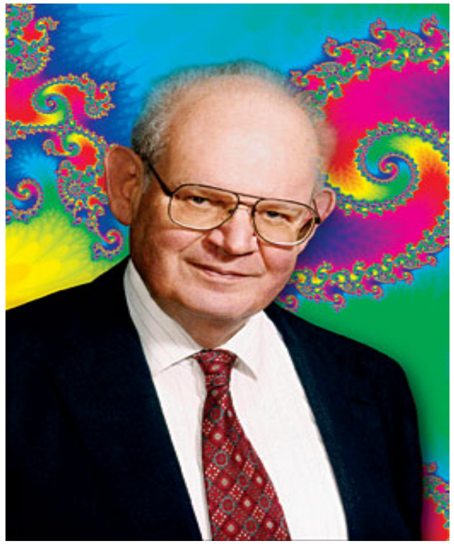


Corrsin
1920 – 1986



Obukhov
1918 – 1989

Mandelbrot
1924-2010



The emergence of atmospheric dynamics

Continuum mechanics

Low level (fundamental)

deterministic

Large Re

Laws of turbulence

Classical:

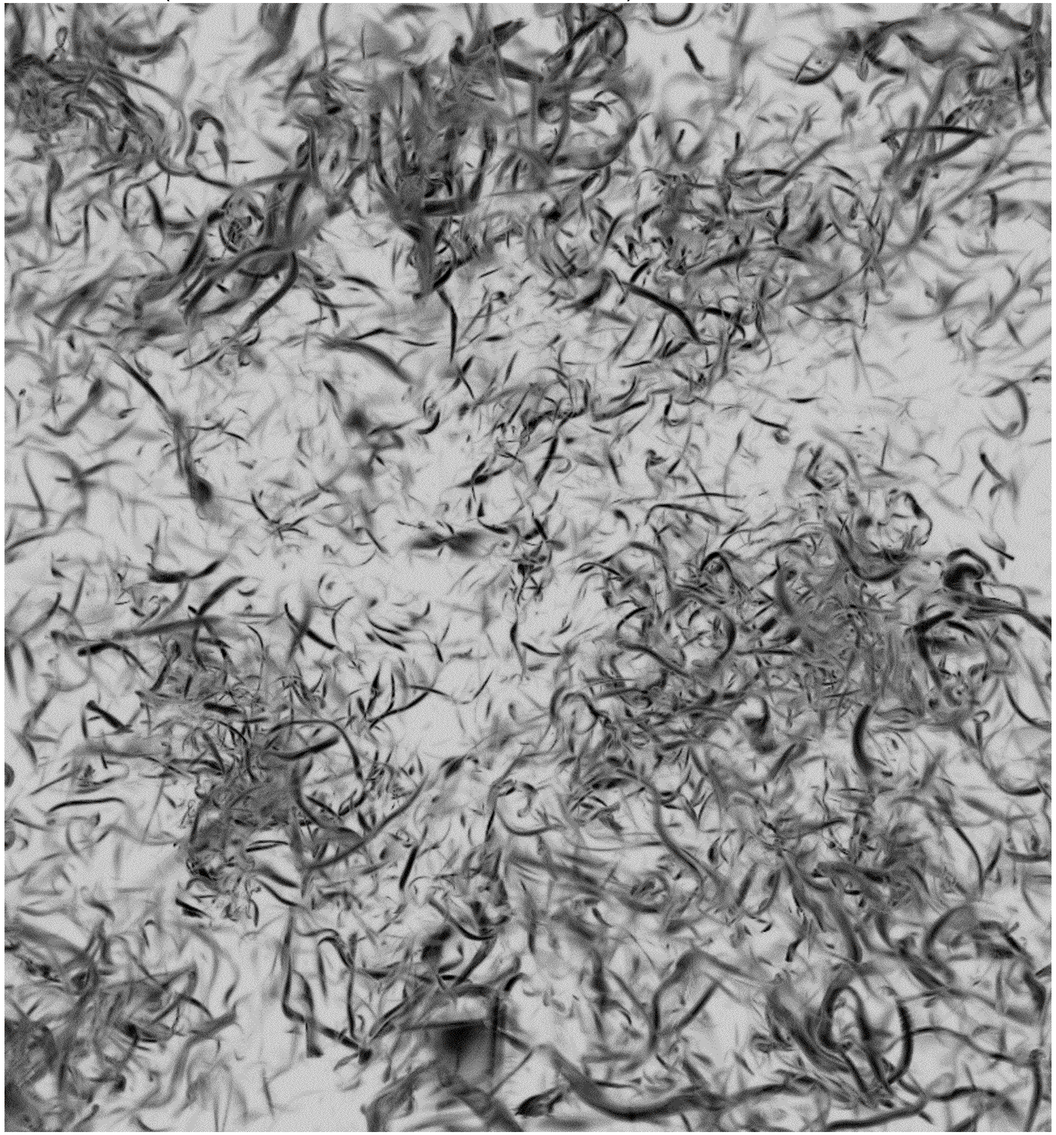
Richardson, Kolmogorov, Corrsin, Obukhov, Bolgiano

High level

stochastic

Vortices in strongly turbulent fluid

(M. Wiczek, numerical simulation, 2010)



$$\Delta v\left(\underline{\Delta r}\right) = \varphi\left|\underline{\Delta r}\right|^{H}$$

e.g. Kolmogorov $\varphi = \varepsilon^{1/3}$, $H = 1/3$

a) $\left|\underline{\Delta r}\right| <\approx 100m$ b) isotropic

c) φ ≈constant, quasi Gaussian

Emergent Turbulence laws

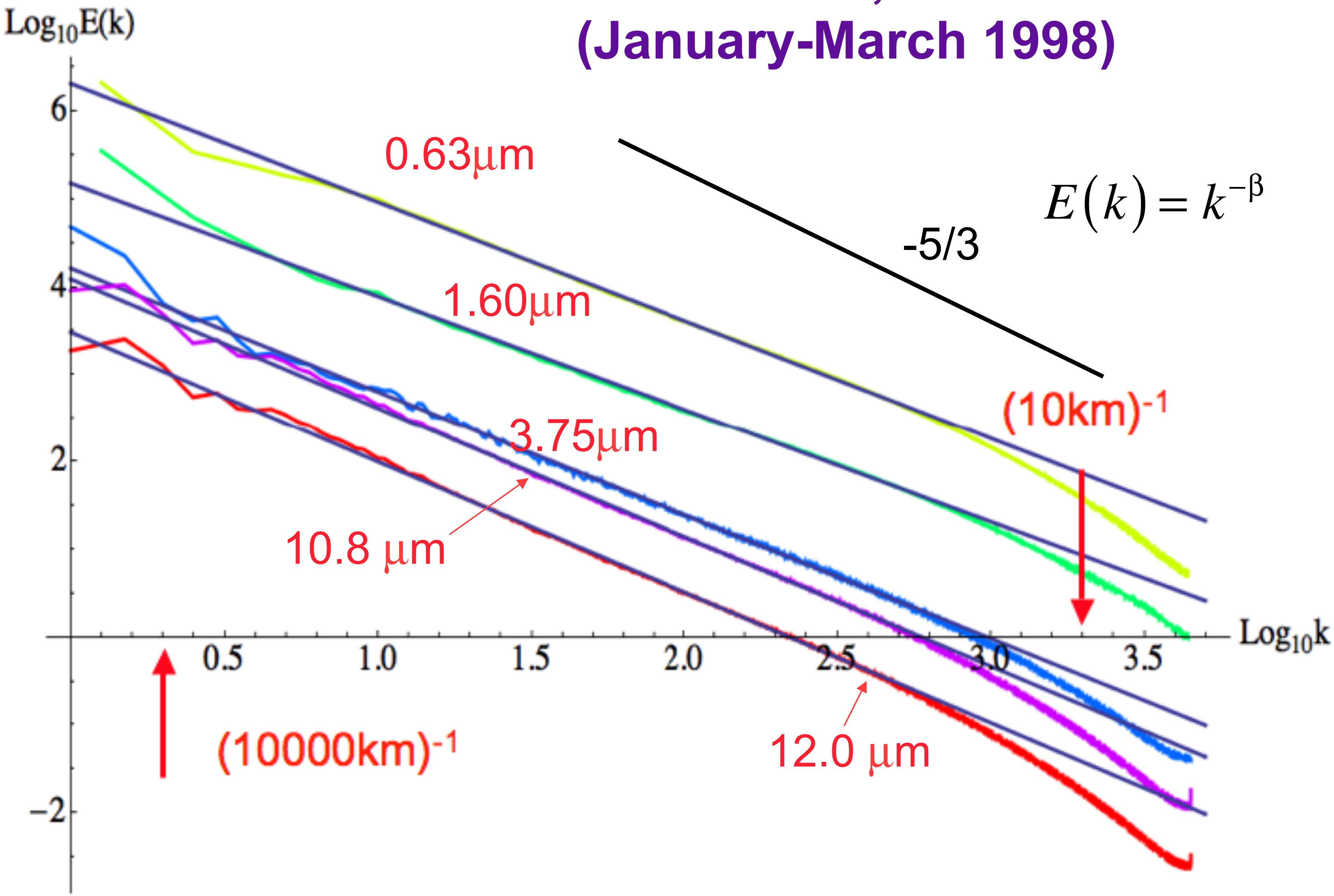
Fluctuations ≈ (turbulent flux) x $(\text{scale})^H$

- Fluctuations: Differences, tendencies, wavelet coefficients
- Turbulent flux: Cascading Turbulent flux
- Scale: Anisotropic Space-time Scale function
- H: Fluctuation exponent

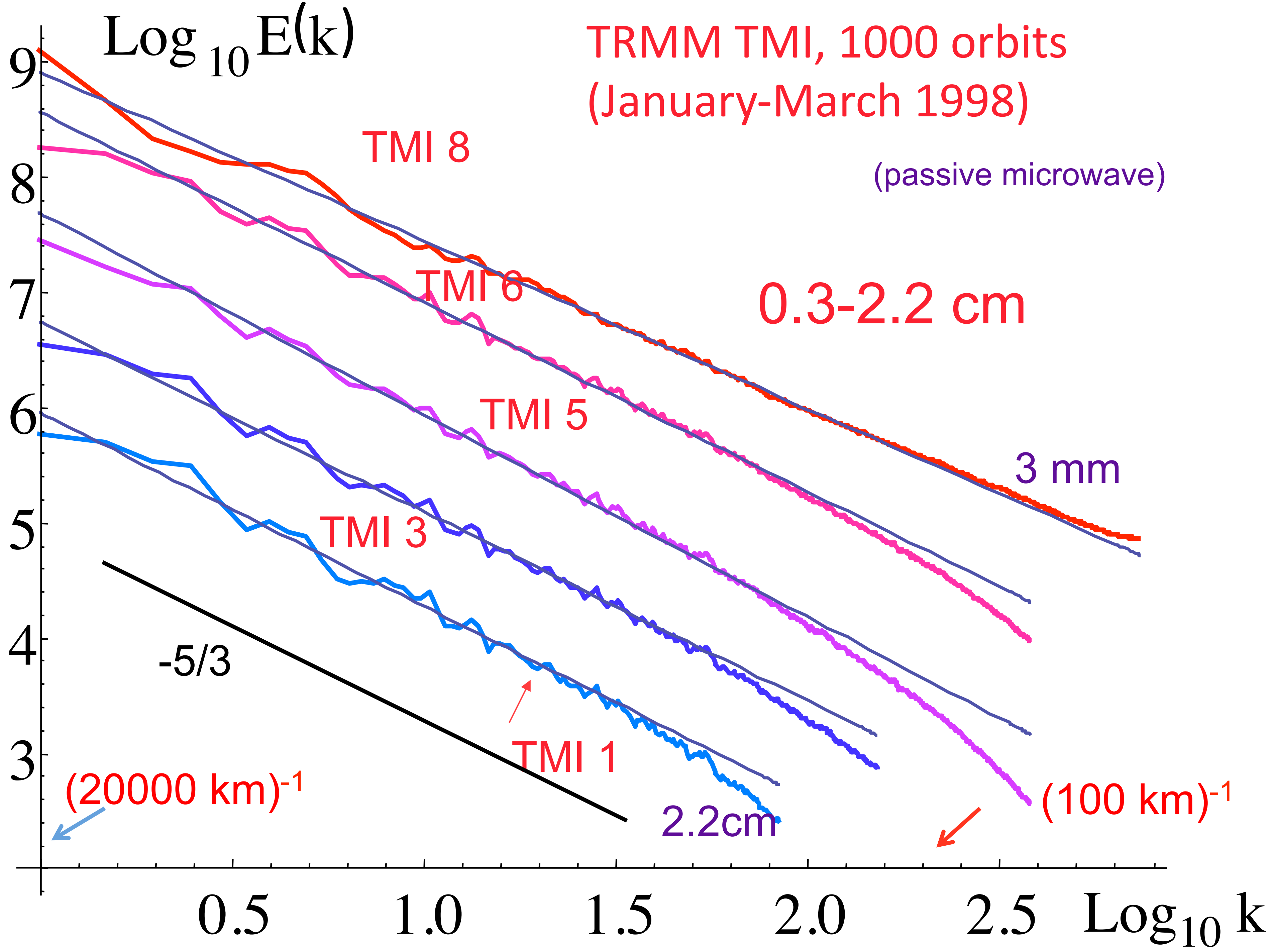
The unity of clouds and rocks:

Scaling

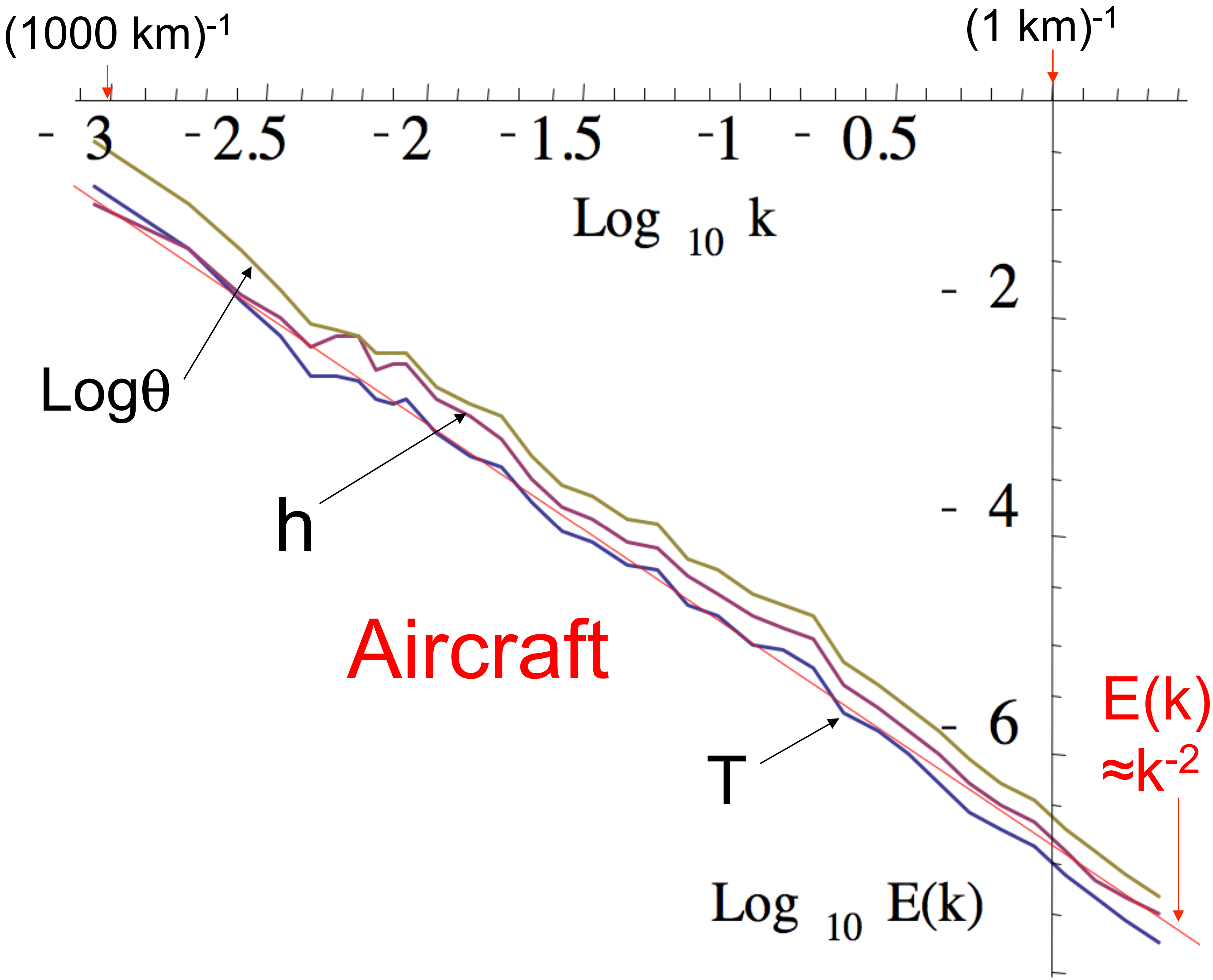
TRMM VIRS, 1000 orbits (January-March 1998)



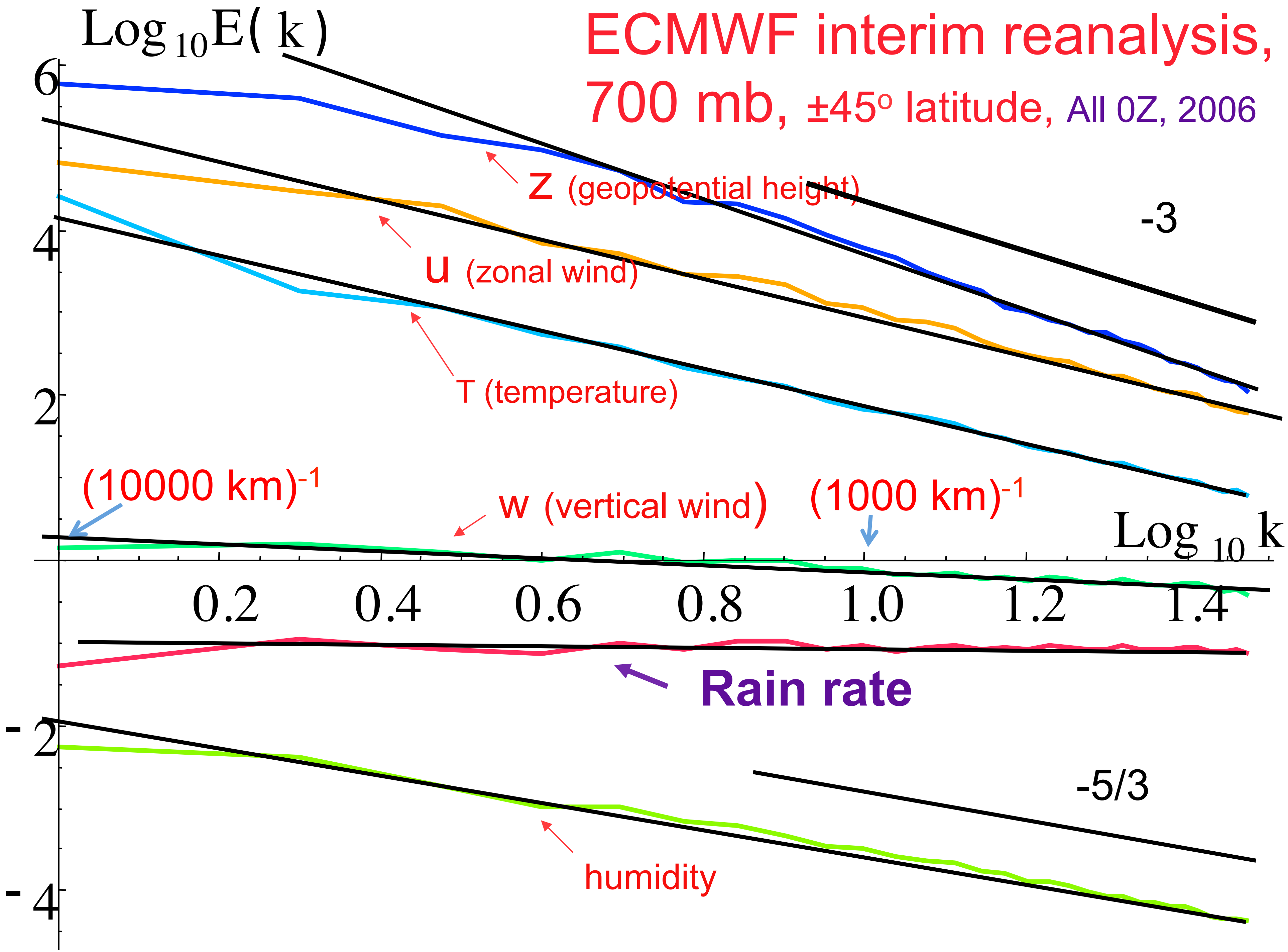
Visible, near infra red, thermal infra red



$Log_{10}E(k)$
TRMM TMI, 1000 orbits
(January-March 1998)
(passive microwave)
TMI 8
TMI 6
0.3-2.2 cm
TMI 5
3 mm
TMI 3
-5/3
TMI 1
$(20000\ km)^{-1}$
2.2cm
$(100\ km)^{-1}$
9
8
7
6
5
4
3
0.5
1.0
1.5
2.0
2.5
$Log_{10}k$

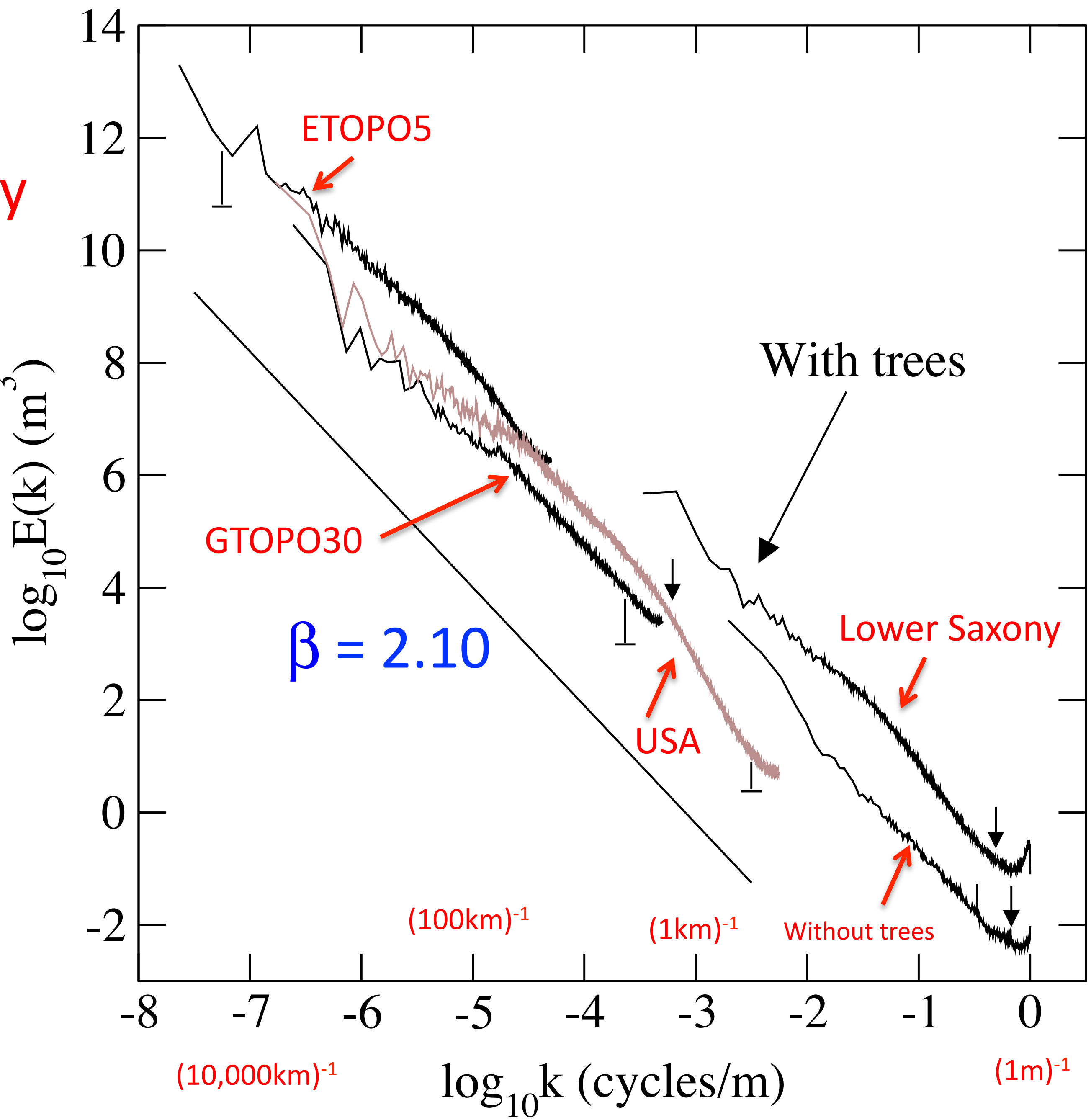


(1000 km)-1
(1 km)-1
-3
-2.5
-2
-1.5
-1
-0.5
Log 10 k
-2
-4
-6
Logθ
h
T
Aircraft
E(k)
≈k-2
Log 10 E(k)



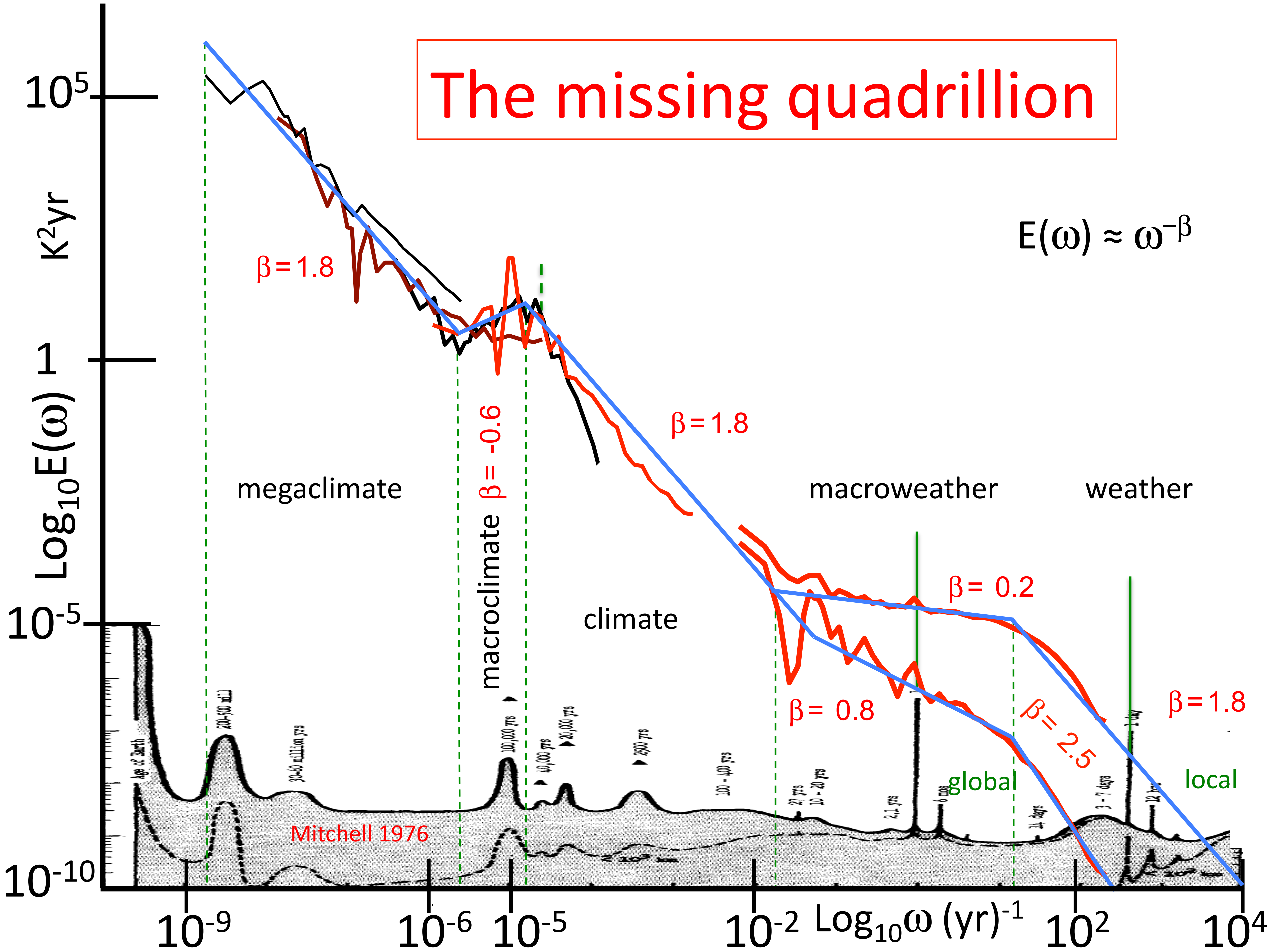
ECMWF interim reanalysis,
700 mb, ±45° latitude, All 0Z, 2006
Log10E(k)
Log10 k
6
4
2
- 2
- 4
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Z (geopotential height)
u (zonal wind)
T (temperature)
w (vertical wind)
Rain rate
humidity
(10000 km)-1
(1000 km)-1
-3
-5/3

Topography

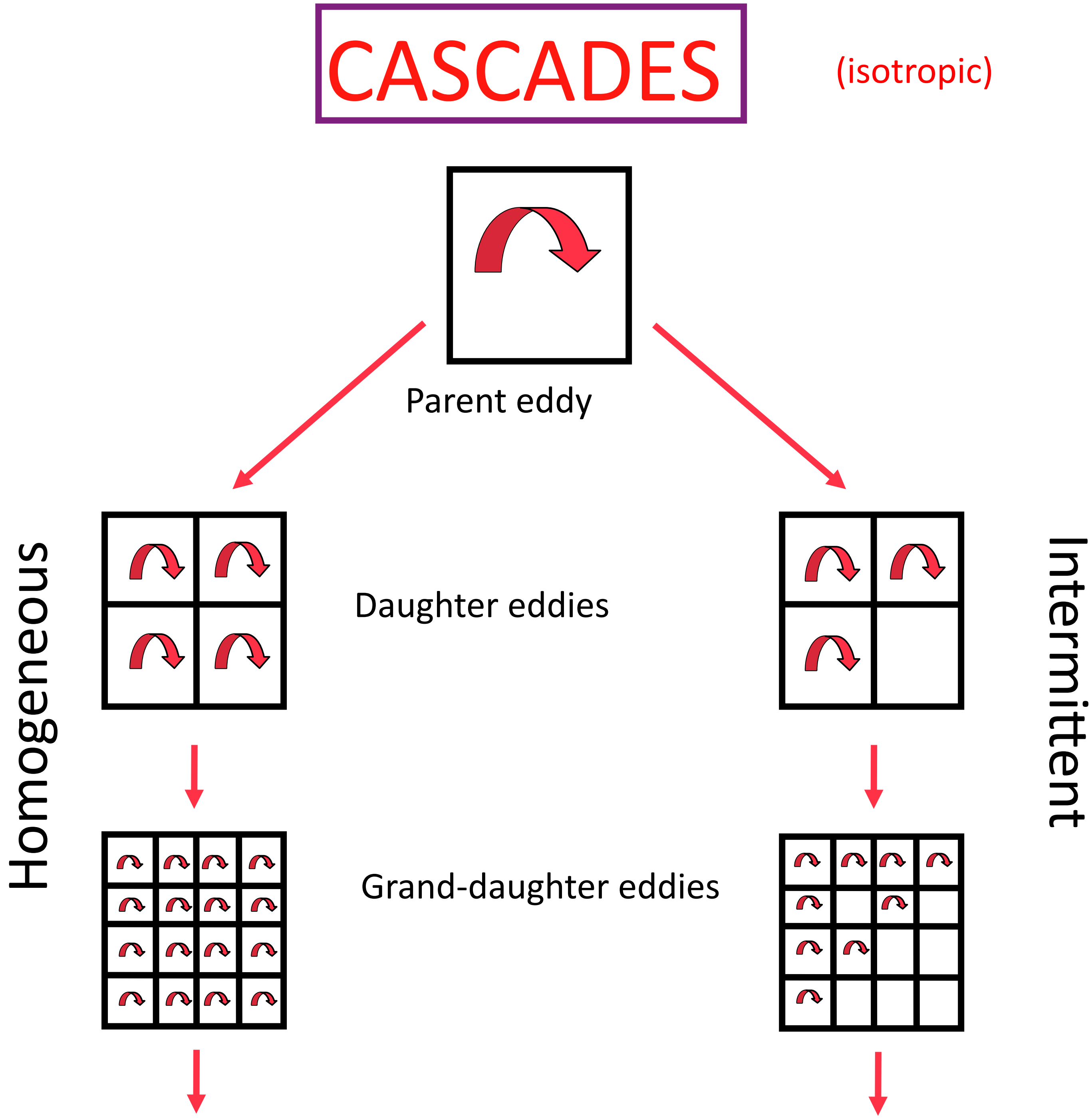


Gagnon, Lovejoy and Schertzer, 2006

Temporal scaling



The missing quadrillion
$E(\omega) \approx \omega^{-\beta}$
K^2yr
$Log_{10}E(\omega)$
10^5
1
10^{-5}
10^{-10}
β = 1.8
megaclimate
macroclimate β = -0.6
climate
β = 1.8
macroweather
weather
β = 0.2
β = 0.8
β = 2.5
β = 1.8
global
local
Mitchell 1976
10^{-9}
10^{-6}
10^{-5}
10^{-2}
$Log_{10}\omega\ (yr)^{-1}$
10^2
10^4


CASCADES
(isotropic)
Parent eddy
Homogeneous
Daughter eddies
Intermittent
Grand-daughter eddies

Multiplicative Cascades

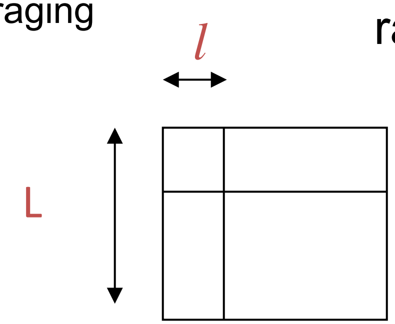
Generic statistical behaviour:

scaling — Scale invariant

$$\left\langle \varepsilon_{\lambda}^{q} \right\rangle \approx \lambda^{K(q)}$$

Statistical averaging

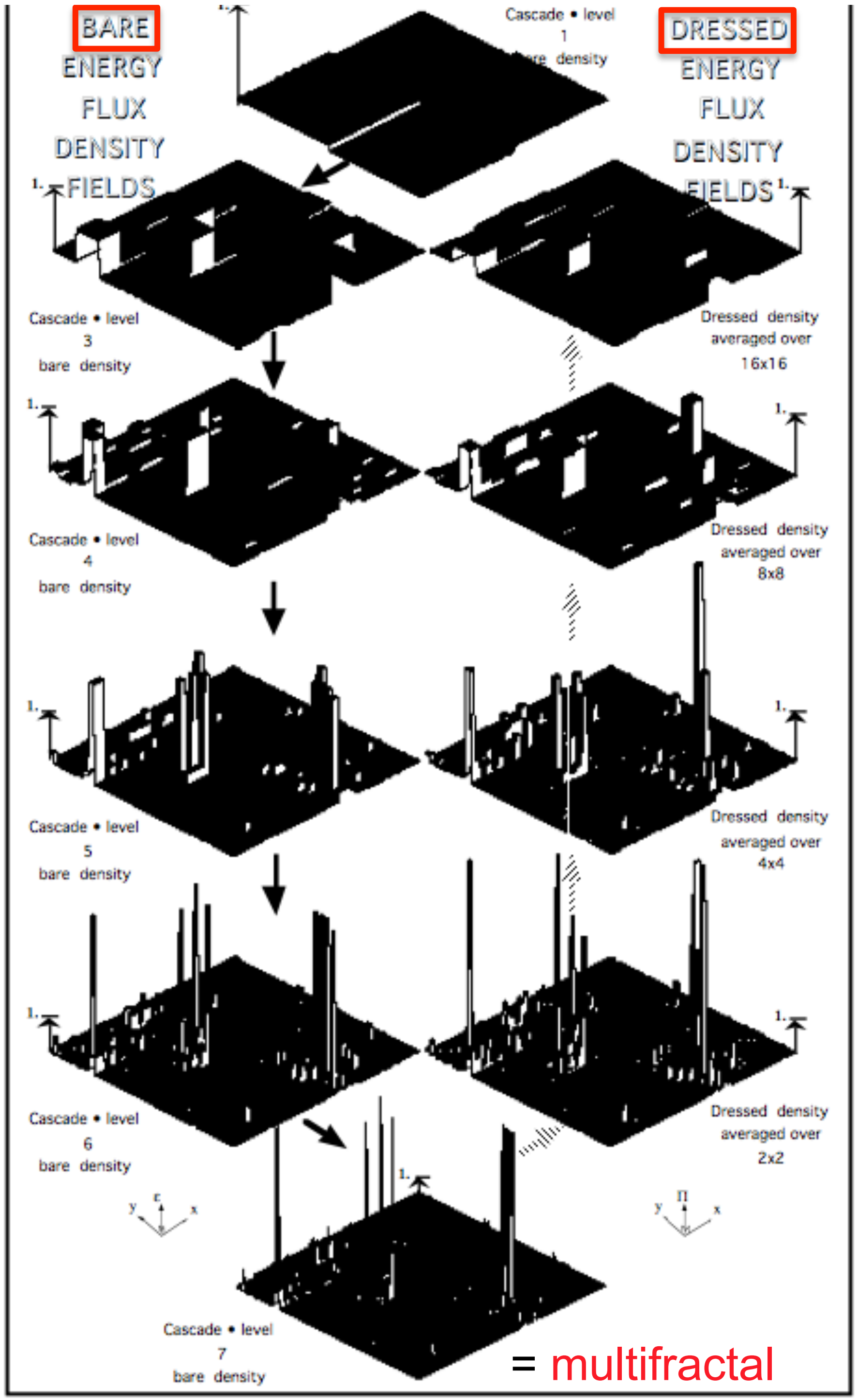
Resolution: ratio $\lambda = L/l$



Probabilities:

$$\Pr\left(\varepsilon_{\lambda} > \lambda^{\gamma}\right) \approx \lambda^{-c(\lambda)}$$

No pointwise convergence - no Hölder exponents (hence bare-dressed)



Codimension and dimension multifractal formalisms

Codimension

(densities of measures, stochastic)

Singularities

$$\varepsilon_\lambda = \lambda^\gamma$$

Dimension

(measures, deterministic) $l = \lambda^{-1}$ $vol(B_\lambda) = \lambda^{-d}$

$$\Pi_\lambda = \int_{B_\lambda} \varepsilon_\lambda d^d \underline{x} = \lambda^{-\alpha_d}$$

$$\alpha_d = d - \gamma$$

$$\Pi_\lambda = \varepsilon_\lambda vol(B_\lambda) = \lambda^{\gamma - d}$$

Probabilities

Probabilistic definition

$$\Pr(\varepsilon_\lambda = \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

$$Number(\Pi_\lambda = \lambda^{-\alpha_d}) = \lambda^{f_d(\alpha_d)}$$

$$Number = \lambda^d \Pr$$

($c \geq 0$)

$$f_d(\alpha_d) = d - c(\gamma)$$

($d \geq f_d \geq 0$)

Statistical Moments

$$\sum_{i=1}^{\lambda^d} \Pi_{\lambda,i}^q = \lambda^{-\tau_d(q)}$$

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$$

$$\sum_{i=1}^{\lambda^d} \Pi_{\lambda,i}^q = \left\langle \sum_{i=1}^{\lambda^d} \left(\lambda^{-d} \varepsilon_\lambda\right)^q \right\rangle = \lambda^{d(q-1)} \langle \varepsilon_\lambda^q \rangle = \lambda^{K(q) - d(q-1)}$$

$$\tau_d(\alpha_d) = d(q-1) - K(q)$$

$$c(\gamma) \overset{L.T.}{\leftrightarrow} K(q); \quad f_d(\alpha_d) \overset{L.T.}{\leftrightarrow} \tau(q)$$

Multiplicative cascade processes

The process

(canonical conservation of flux)

$$\varepsilon_\lambda = e^{\Gamma_\lambda}$$

Multifractal process over scale range λ= Multiplicative cascade

The generator= Additive process

Dimension of space

Auxiliary variable α'

$$\frac{1}{\alpha'}+\frac{1}{\alpha}=1$$

$$\Gamma_\lambda(\underline{r}) \propto C_1^{1/\alpha}\gamma_\alpha(\underline{r}) * \left|\underline{r}\right|_{[1,\lambda^{-1}]}^{-d/\alpha'}$$

$$\tilde{\Gamma}_\lambda(\underline{k}) \propto \tilde{\gamma}_\lambda(\underline{k})\left|\underline{k}\right|^{-d/\alpha}$$

The codimension of the mean= Amplitude of the generator, sparseness of ε

i.i.d. Levy noise index α= subgenerator

Convolution with singularity over range 1 to λ^{-1}

Fourier space= power law filter

The statistics

$$\left\langle \varepsilon_\lambda^q \right\rangle = \lambda^{K(q)}$$

General multifractal statistics, convex K(q)

$$K(q) = \frac{C_1}{\alpha-1}\left(q^\alpha - q\right)$$

$$q < q_D \quad 0 \le \alpha \le 2$$

Universal multifractals ("multiplicative central limit theorem", Schertzer Lovejoy 1987)

$$\Pr(\varepsilon_\lambda > s) \approx s^{-q_D}$$

$$s \gg 1$$

Extremes: "Fat tails"
Mandelbrot 1974

Fractionally Integrated Flux (FIF) model (both additive and multiplicative)

The process

$$I(\underline{r}) = \varepsilon_\lambda(\underline{r}) * |\underline{r}|^{-(d-H)} \longleftrightarrow \tilde{I}(\underline{k}) = \tilde{\varepsilon}_\lambda(\underline{k}) |\underline{k}|^{-H}$$

Convolution= fractional integration order H

Fourier space= power law filter

The statistics

$$S_q(\Delta r) = \left\langle \Delta I(\Delta r)^q \right\rangle = \left\langle \varepsilon_\lambda^q \right\rangle |\Delta r|^{qH} = |\Delta r|^{\xi(q)} \qquad \xi(q) = qH - K(q)$$

q^{th} order structure function (S_q)

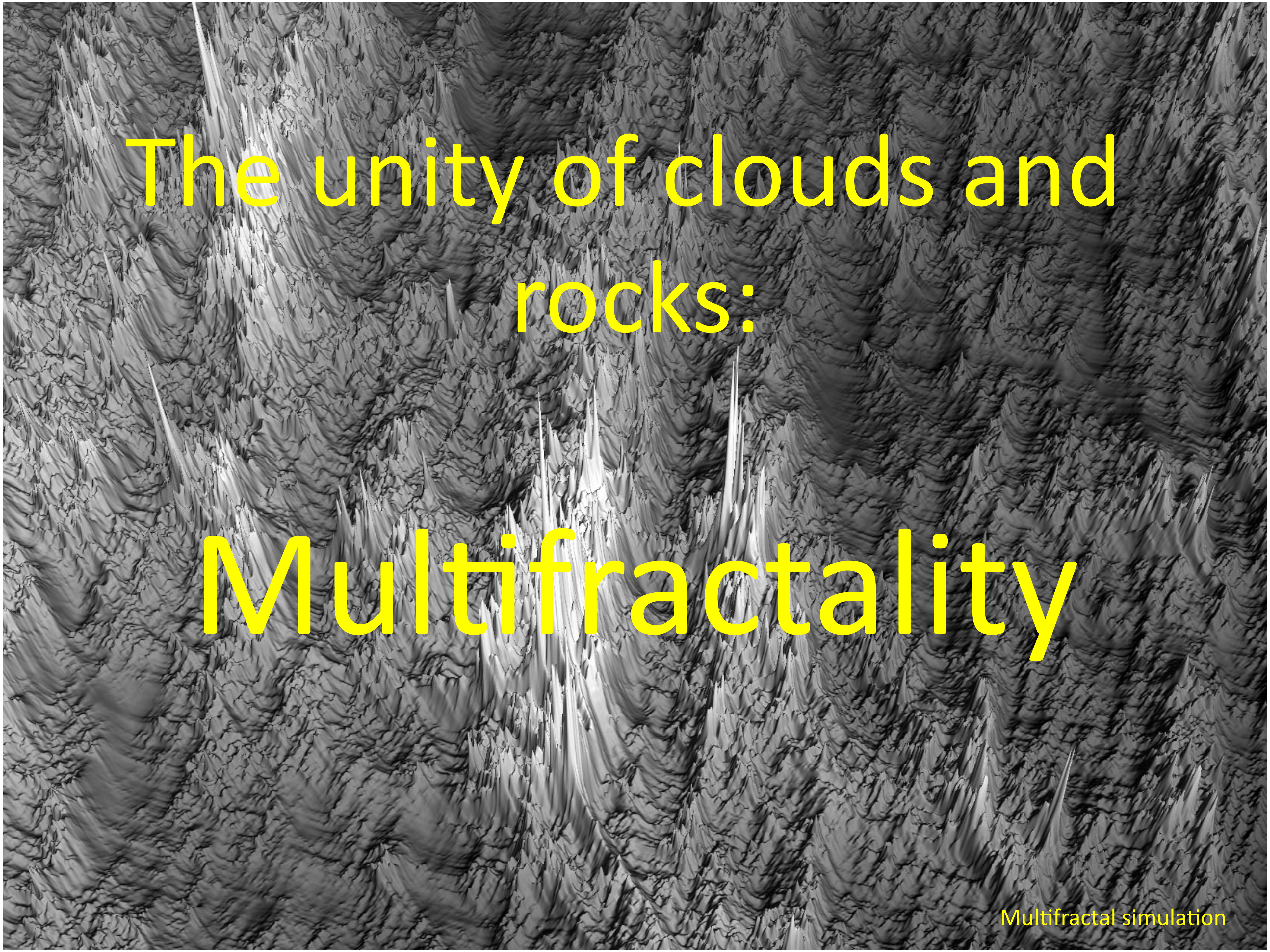
fluctuation (ΔI)

Note:
$\lambda = L / |\Delta r|$
$\left\langle \varepsilon_\lambda^q \right\rangle = \lambda^{K(q)}$

structure function exponent ($\xi(q)$)

The unity of clouds and rocks:

Multifractality

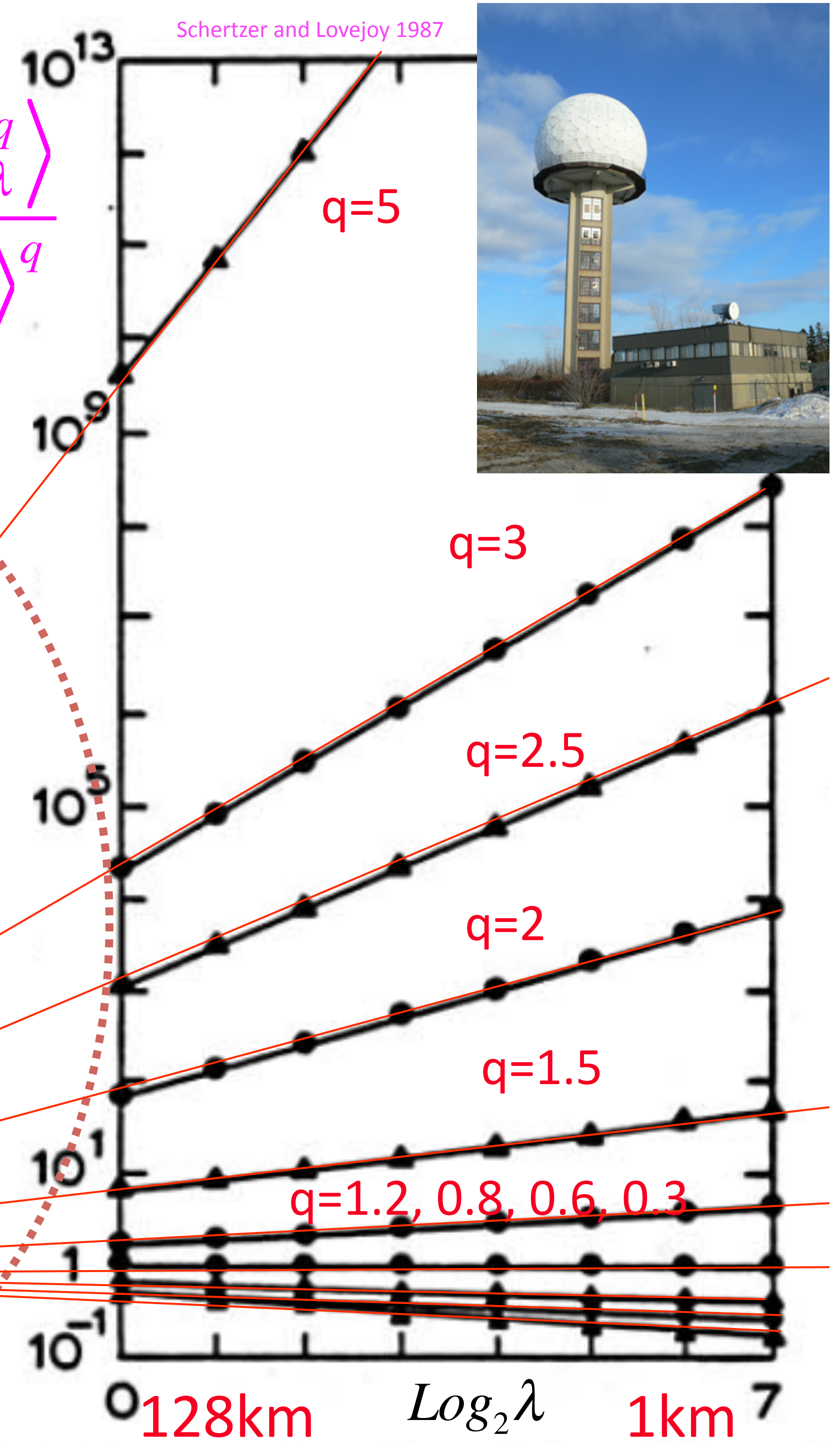


Multifractal simulation

Early evidence of cascades: **Precipitation 1987**

(70 Radar Scans, Montreal, horizontal 3 weeks of rain data)

$$M = \frac{\langle Z_\lambda^q \rangle}{\langle Z \rangle^q}$$



Large scales

?

?

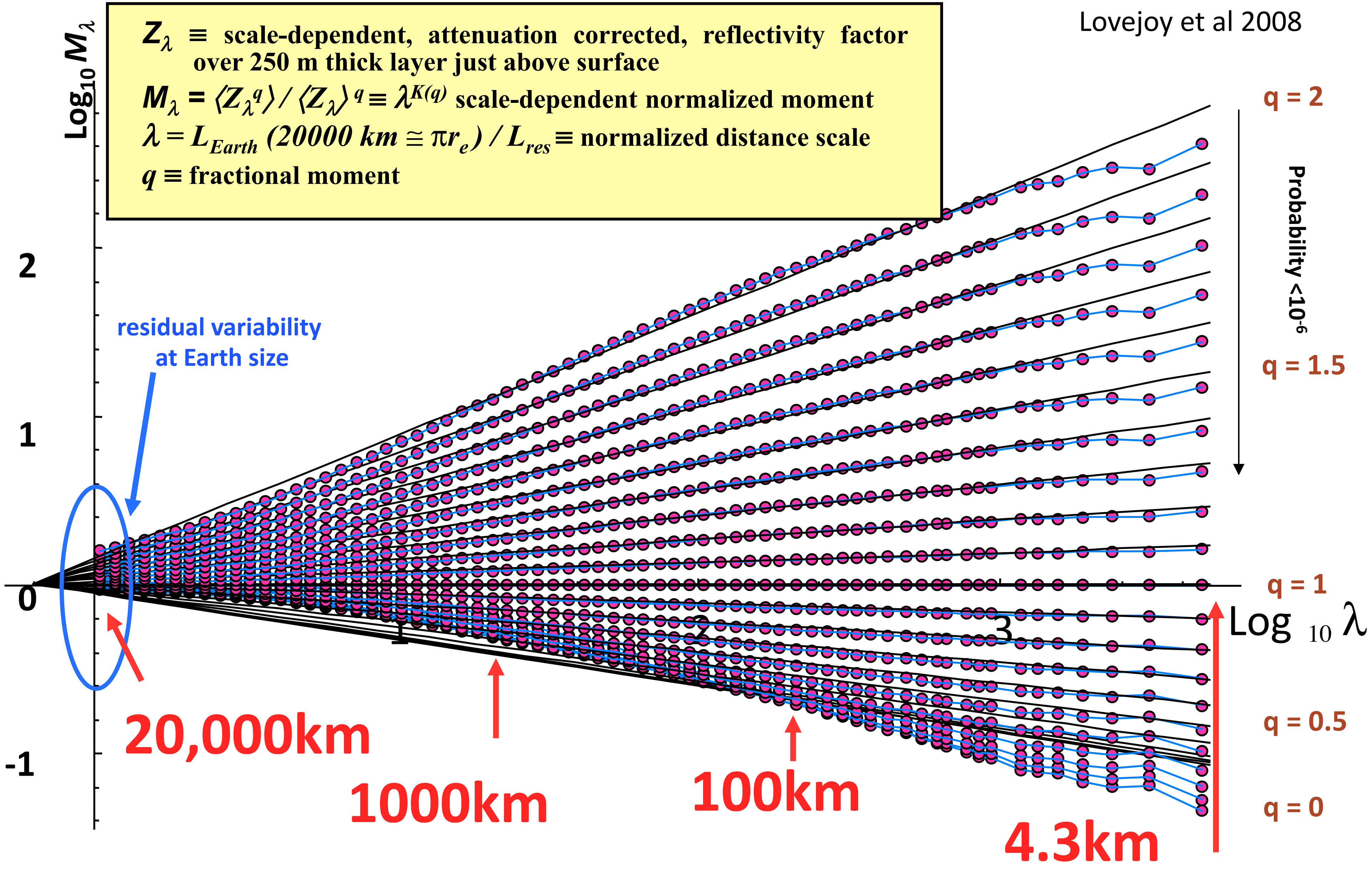
32,000km

Cascade prediction:

$$\langle Z_\lambda^q \rangle / \langle Z_1 \rangle^q = \lambda^{K(q)}$$

$$\lambda = L_{eff} / L_{res}$$

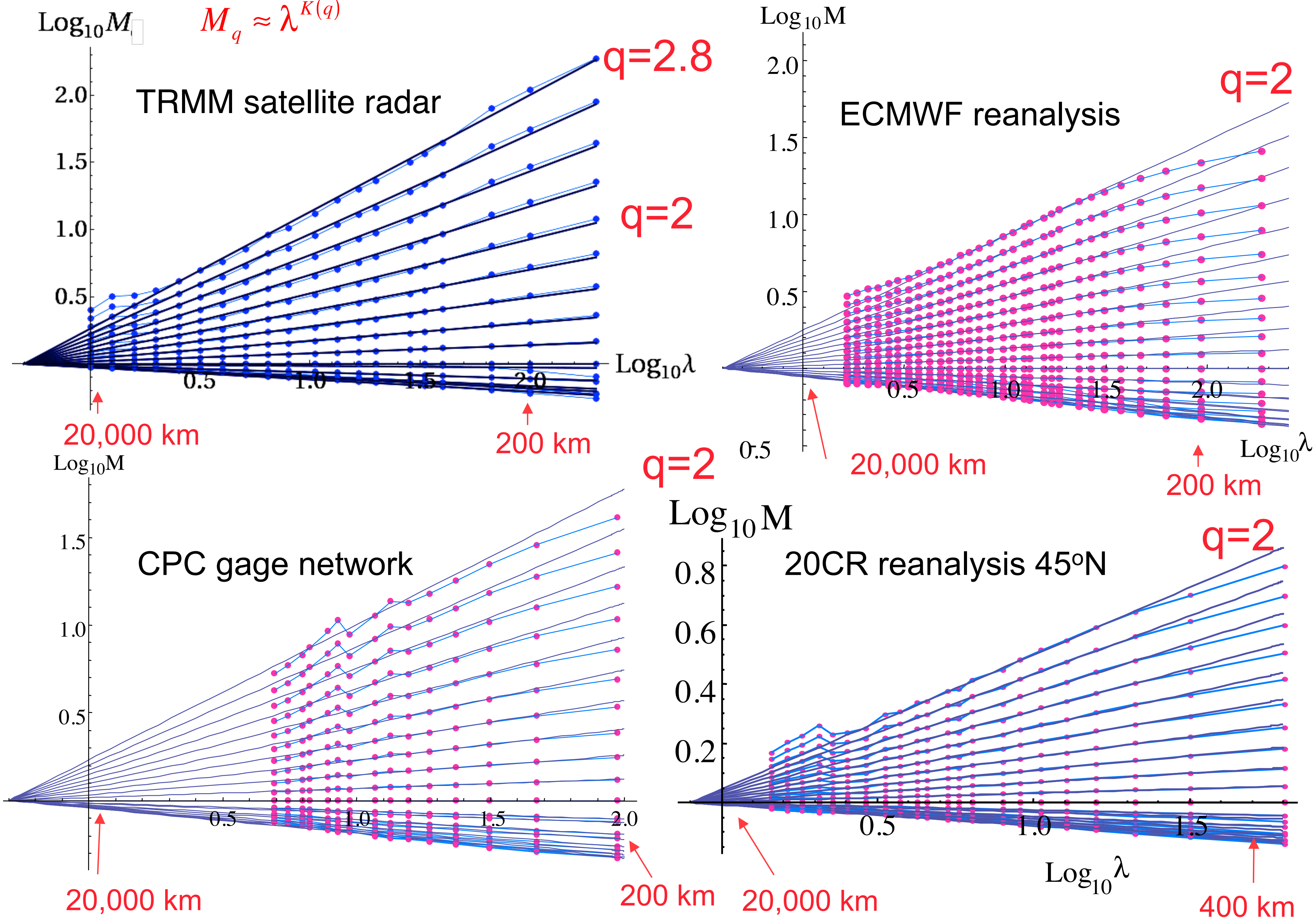
Scale-dependent TRMM PR Attenuation Corrected Reflectivity Factor [Z_λ] (1176 consecutive orbits -- ~70 days)



Rainrates East-West

$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$

$$M_q \approx \lambda^{K(q)}$$



Satellite radiances:

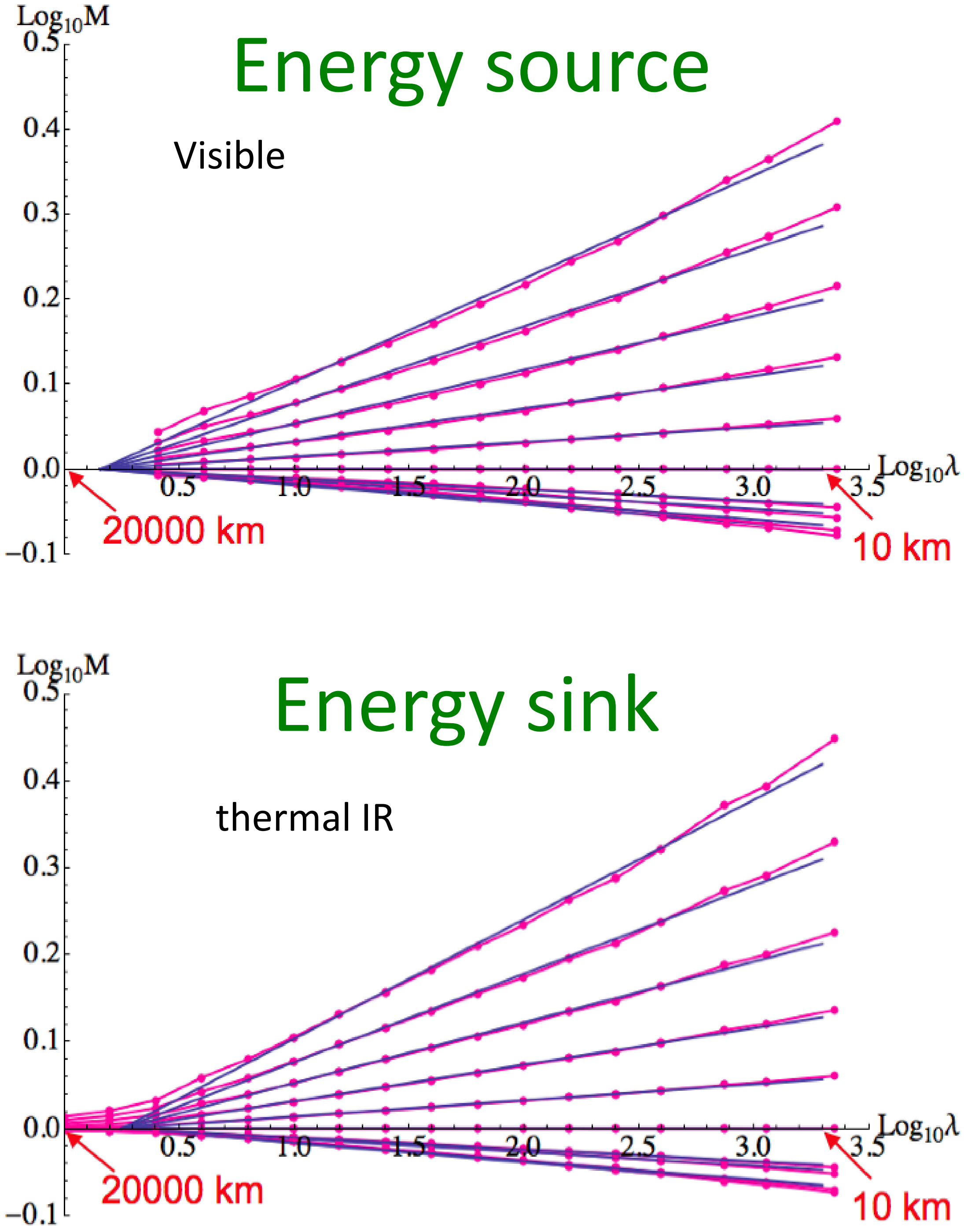
Energy budget

TRMM satellite data, ≈1000 orbits

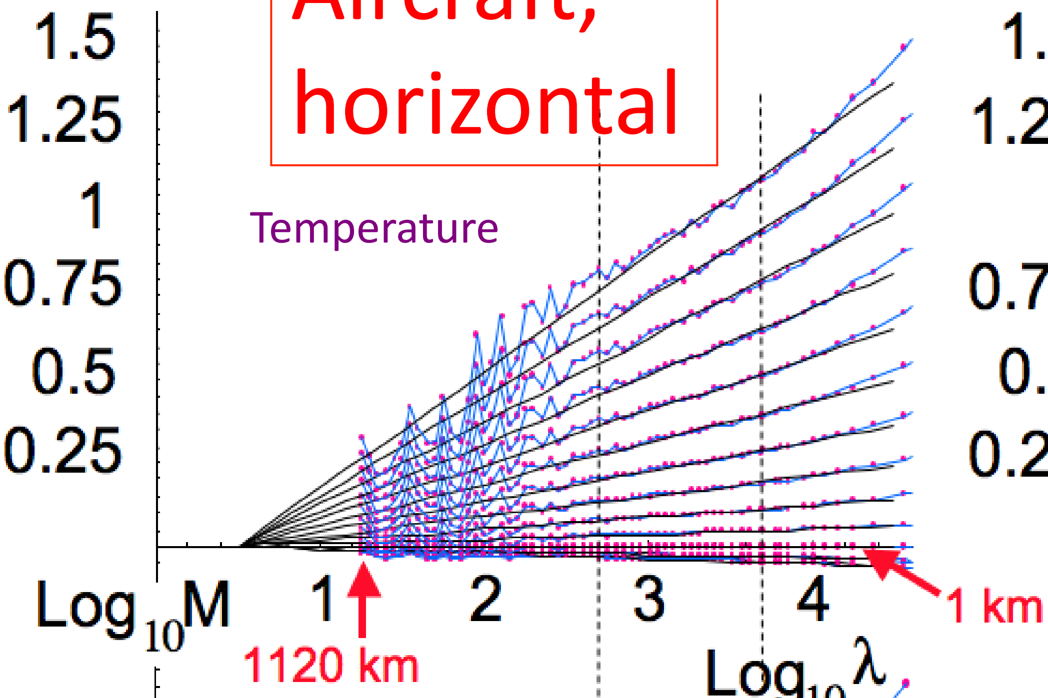
$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$

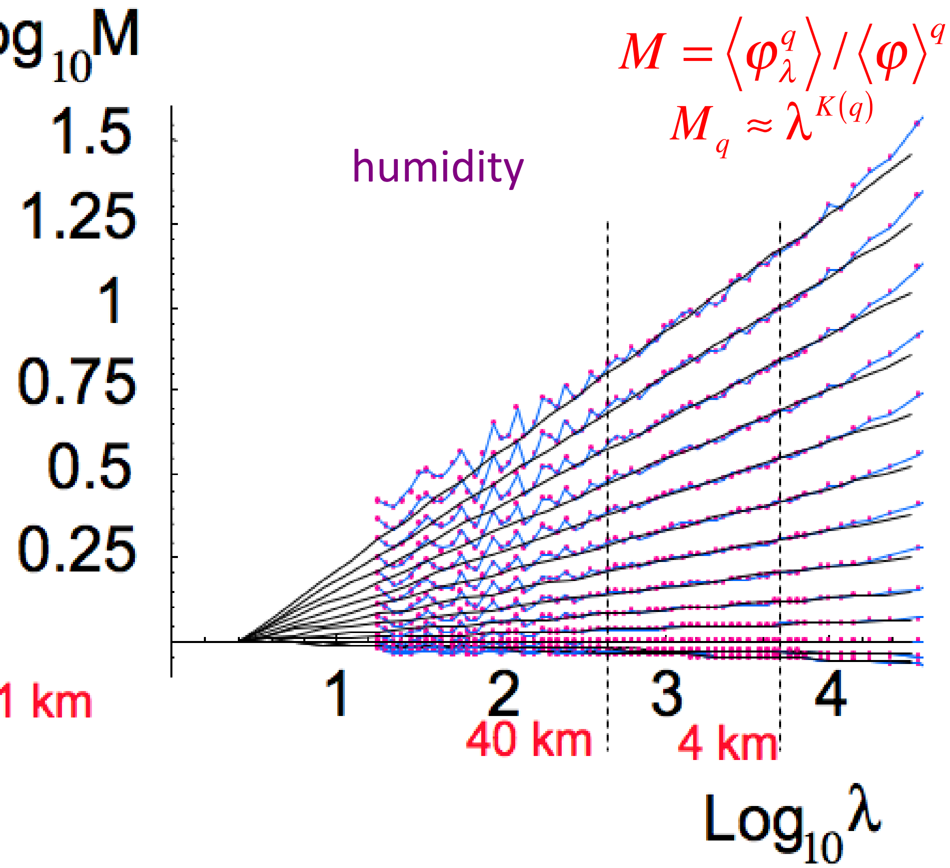
$$M_q \approx \lambda^{K(q)}$$

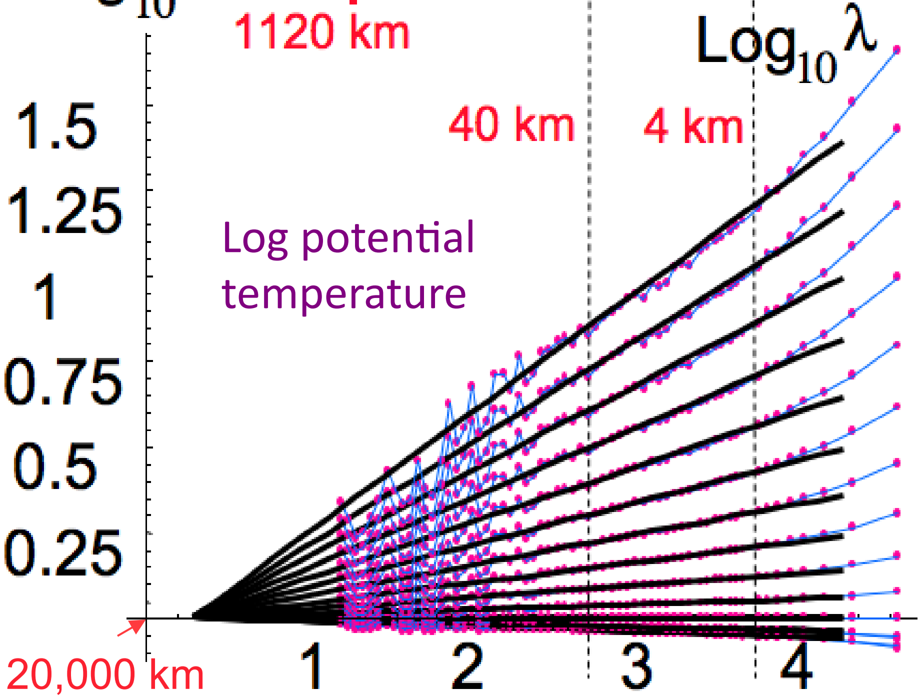
Lovejoy et al 2009



Aircraft, horizontal

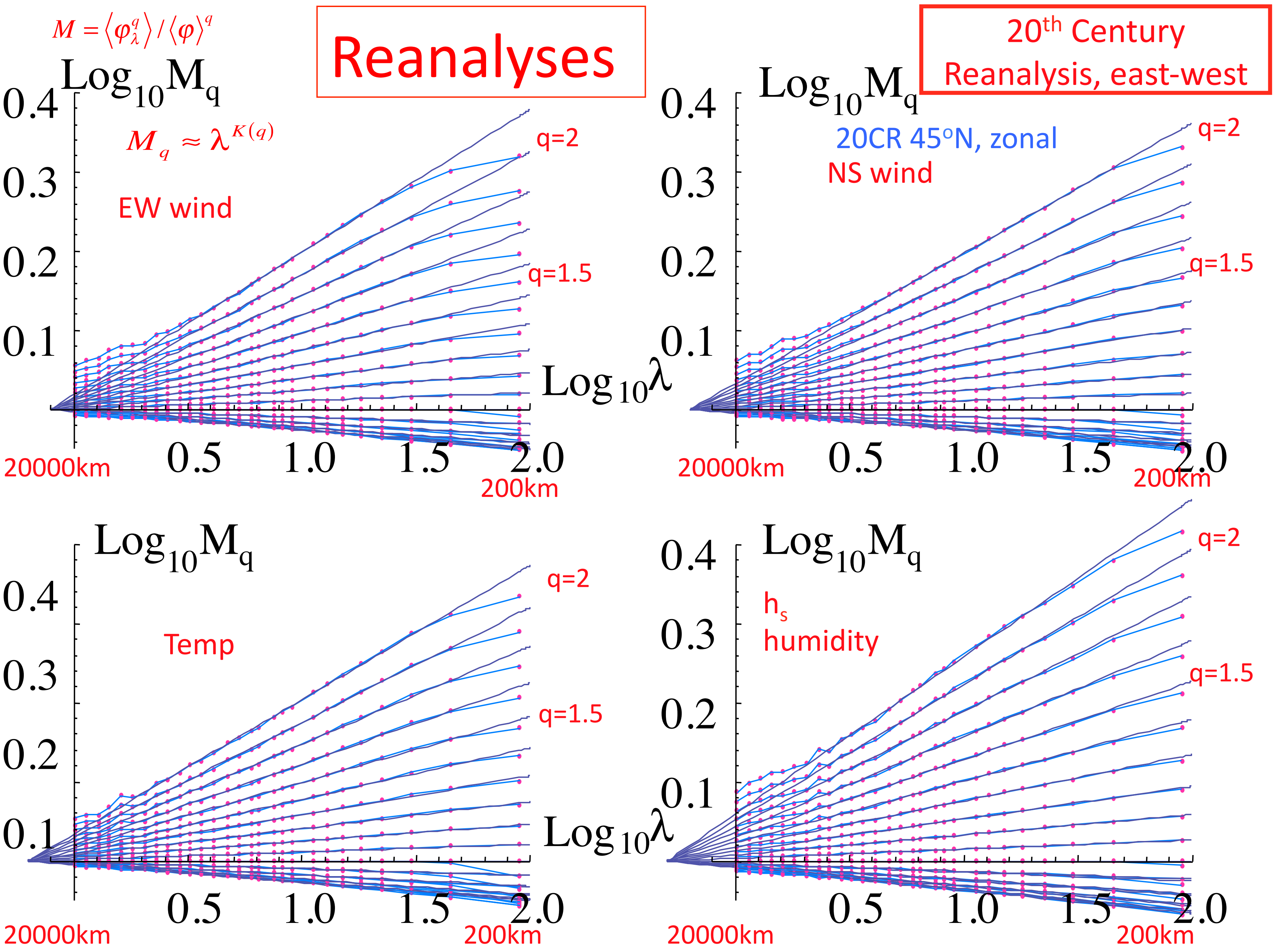






From 24 aircraft legs (altitude 11-13km)

Fields that are relatively unaffected by the trajectories



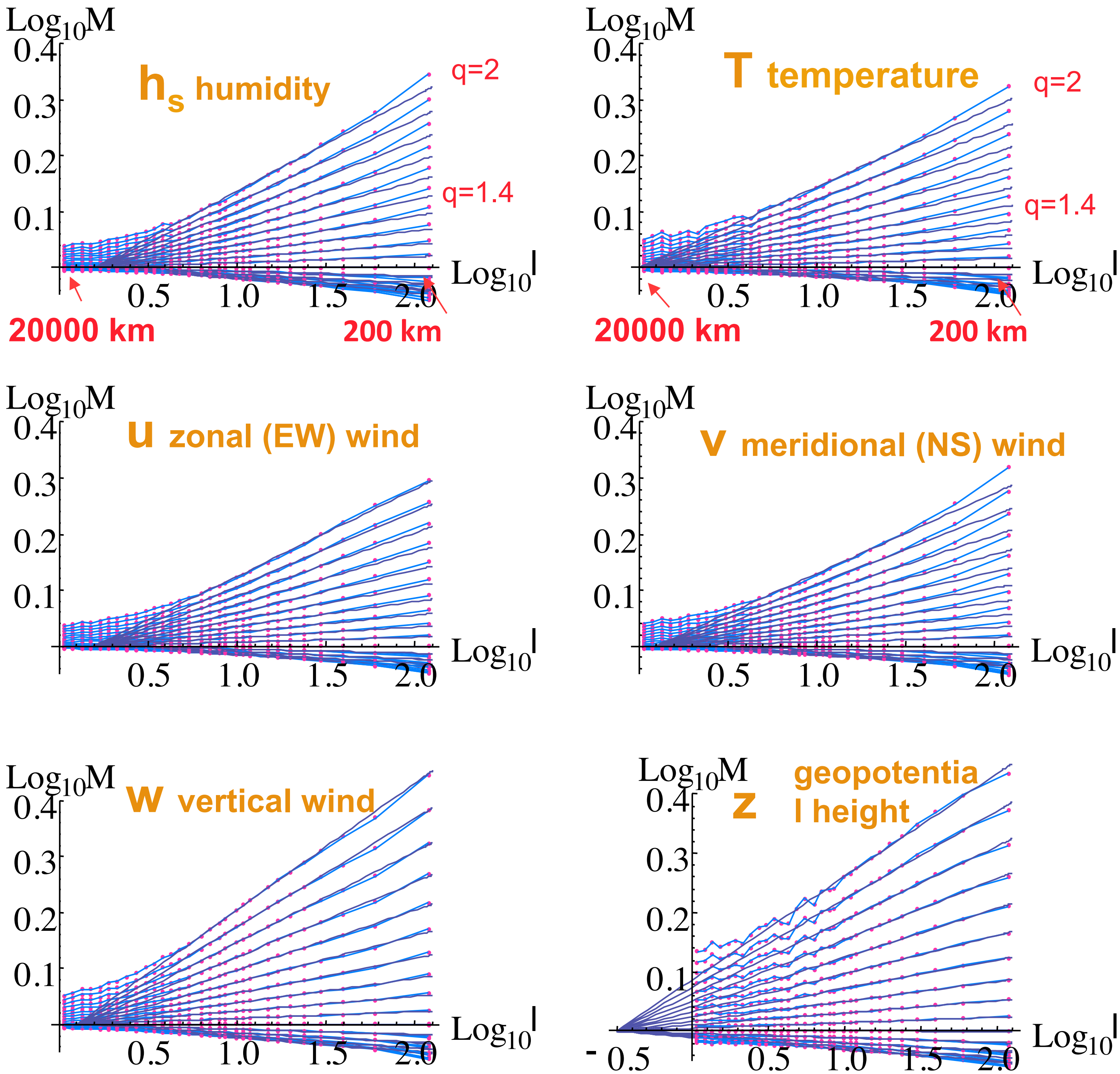
Reanalyses
20th Century Reanalysis, east-west
$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$
$M_q \approx \lambda^{K(q)}$
$Log_{10}M_q$
EW wind
q=2
q=1.5
$Log_{10}\lambda$
20000km
200km
20CR 45°N, zonal
NS wind
Temp
h_s
humidity
0.1
0.2
0.3
0.4
0.5
1.0
1.5
2.0

Reanalyses

ECMWF reanalysis

East-West

(2006, 0Z, 700 mb)



$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$

$$M_q \approx \lambda^{K(q)}$$

Weather models:

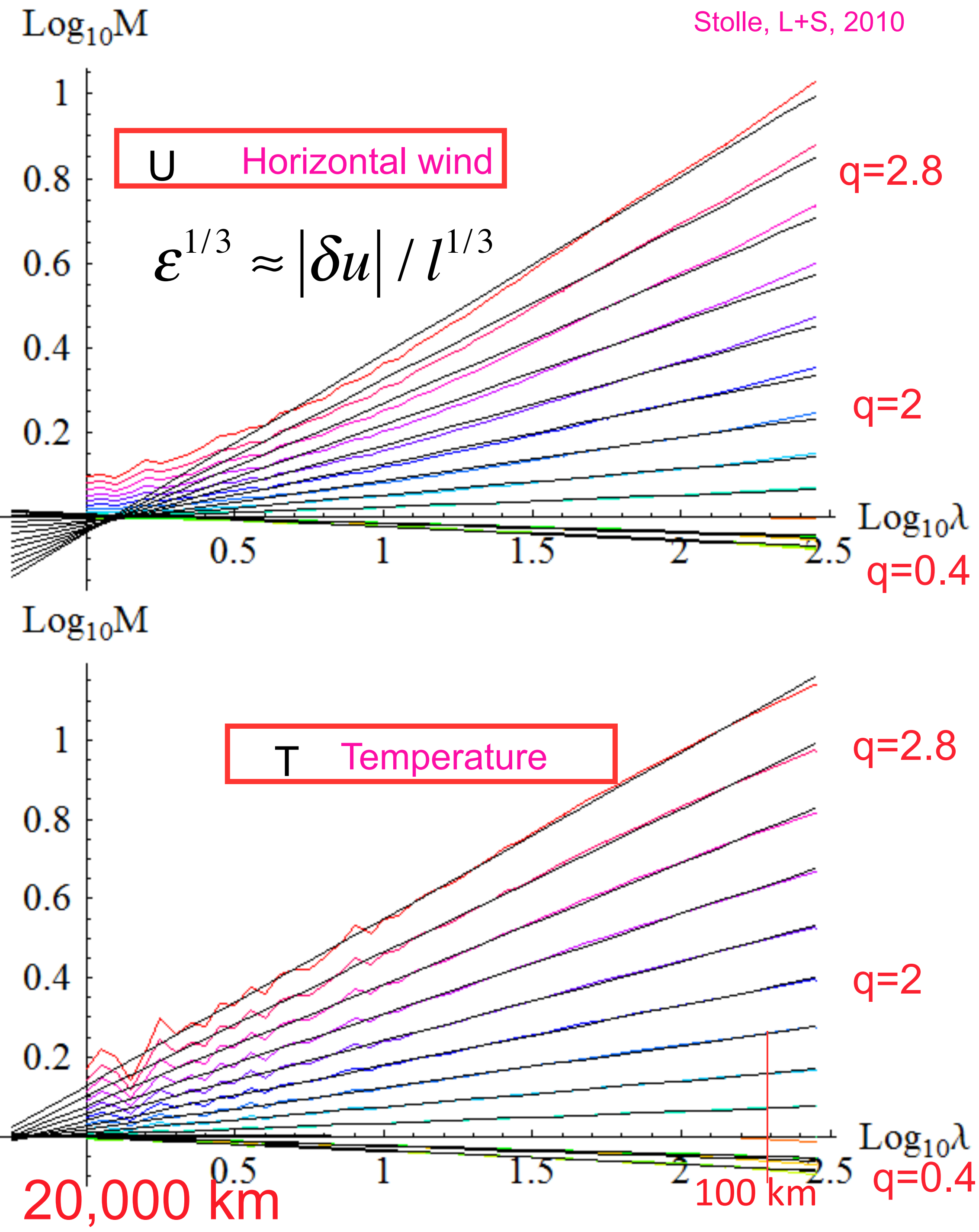
Global GEMS Model 00h

Analysis of four months
U,T at 1000 mb

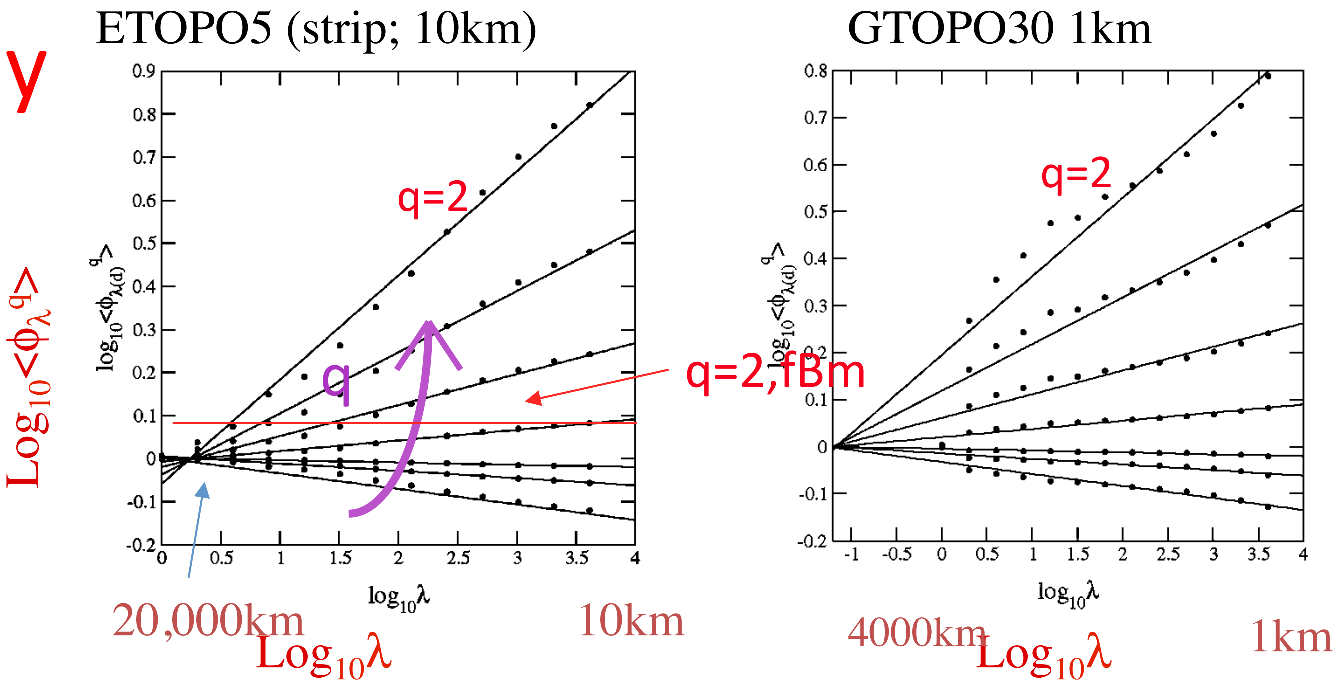
(48 h forecasts are
almost the same)

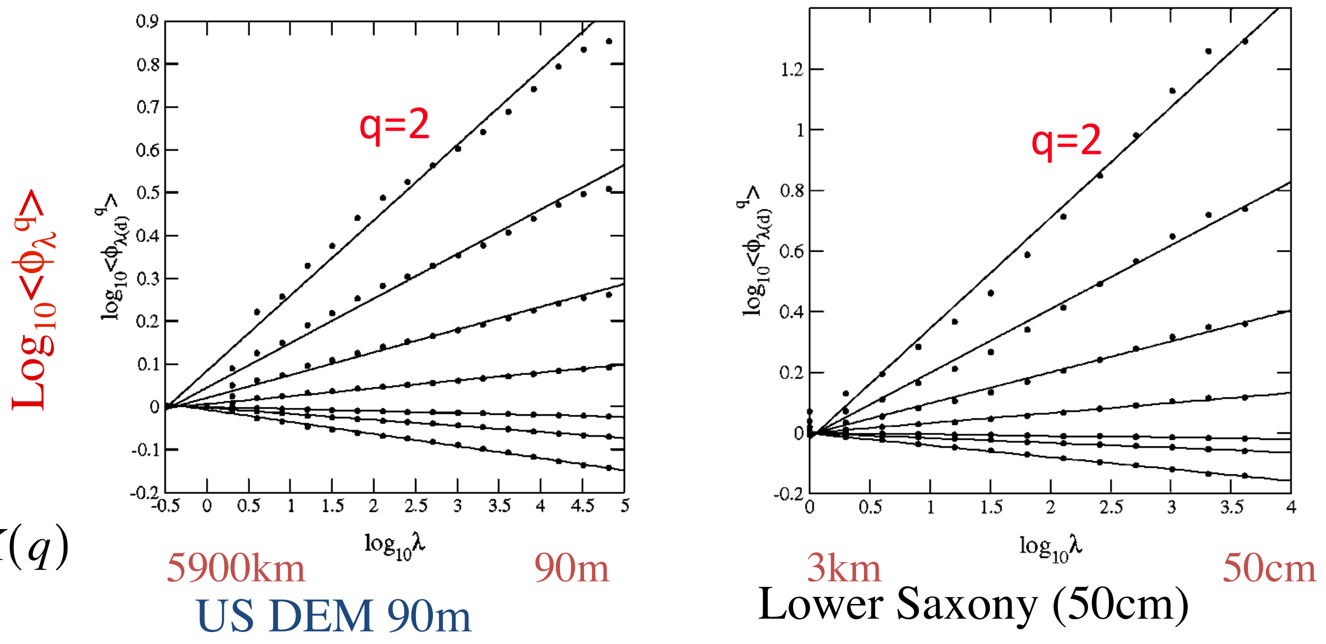
$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$

$$M_q \approx \lambda^{K(q)}$$



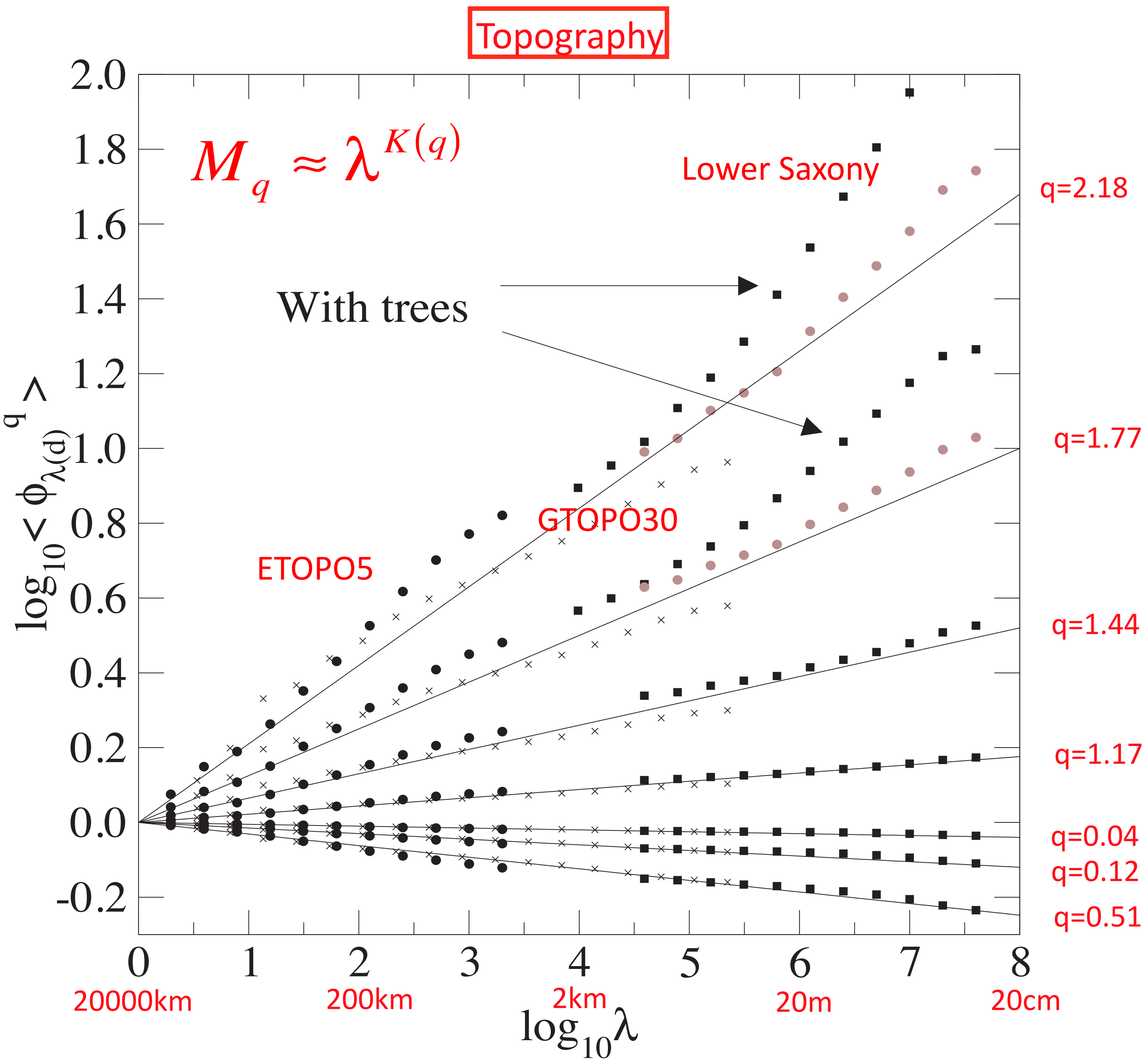
Topography





$$\phi_\lambda = \left|\Delta h_\lambda\right|$$

$$\left\langle \phi_\lambda^q \right\rangle = \lambda^{K(q)}$$



Topography
$M_q \approx \lambda^{K(q)}$
Lower Saxony
With trees
GTOPO30
ETOPO5
q=2.18
q=1.77
q=1.44
q=1.17
q=0.04
q=0.12
q=0.51
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
$\log_{10} < \phi_{\lambda(d)}^{q} >$
0
1
2
3
4
5
6
7
8
20000km
200km
2km
20m
20cm
$\log_{10}\lambda$

Multifractal parameters of geophysical fields

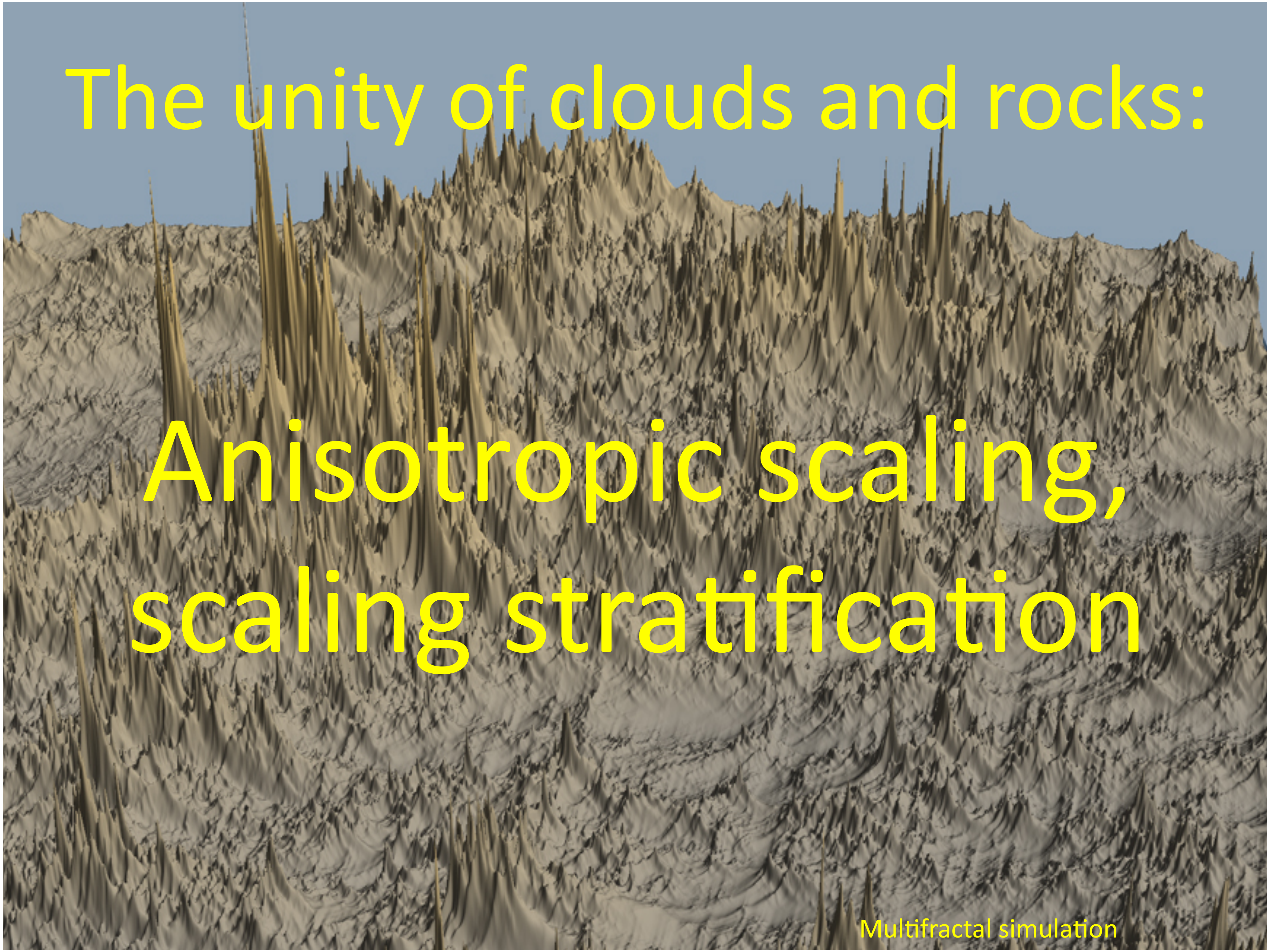
$$S_q(\Delta x) = \left\langle \Delta v(\Delta x)^q \right\rangle = \left\langle \varphi_{\Delta x}^q \right\rangle \Delta x^{qH} \approx \Delta x^{\xi(q)}; \quad \left\langle \varphi_{\Delta x}^q \right\rangle = \left(\frac{L_{eff}}{\Delta x} \right)^{K(q)} ; \quad \xi(q) = qH - K(q)$$

With universality: $$K(q) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right)$$

i.e. we seek H, C_1, α

$$\xi(q) = qH - K(q) = qH - \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right)$$

		C_1	α	H	β	L_{eff}
State variables	u, v	0.09	1.9	1/3, (0.77)	1.6, (2.4)	(14 000)
	w	(0.12)	(1.9)	(−0.14)	(0.4)	(15 000)
	T	0.11, (0.08)	1.8	0.50, (0.77)	1.9, (2.4)	5000 (19 000)
	h	0.09	1.8	0.51	1.9	10 000
	z	(0.09)	(1.9)	(1.26)	(3.3)	(60 000)
Precipitation	R	0.4	1.5	0.00	0.2	32 000
Passive scalars	Aerosol concentration	0.08	1.8	0.33	1.6	25 000
Radiances	Infrared	0.08	1.5	0.3	1.5	15 000
	Visible	0.08	1.5	0.2	1.5	10 000
	Passive microwave	0.1–0.26	1.5	0.25–0.5	1.3–1.6	5000–15 000
Topography	Altitude	0.12	1.8	0.7	2.1	20 000
Sea surface temperature	SST (see Table 8.2)	0.12	1.9	0.50	1.8	16 000


The unity of clouds and rocks:
Anisotropic scaling,
scaling stratification
Multifractal simulation

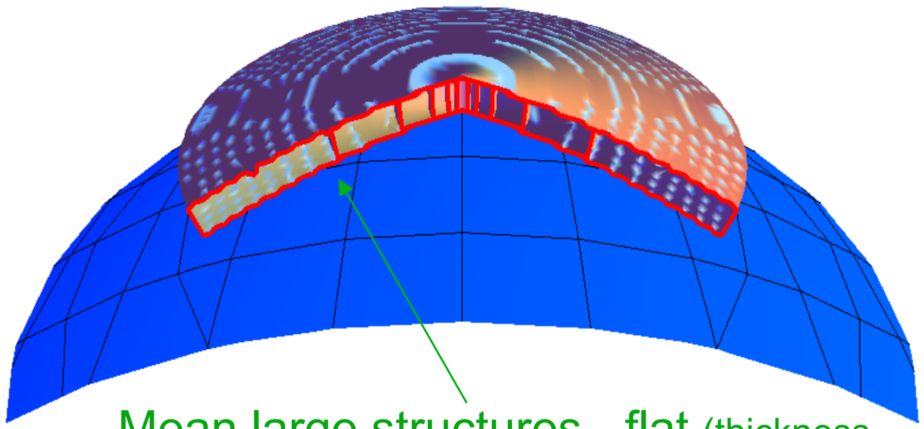
The Standard (2D/3D) Model

"Weather"

Size notion:

$$\left|(\Delta x, \Delta y)\right| = \left(\Delta x^2 + \Delta y^2\right)^{1/2}$$

Large scale 2D

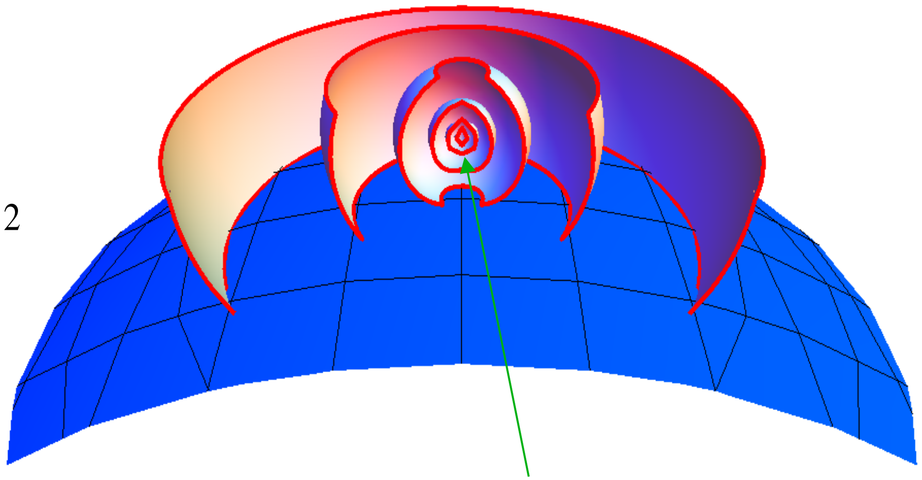


"Turbulence"

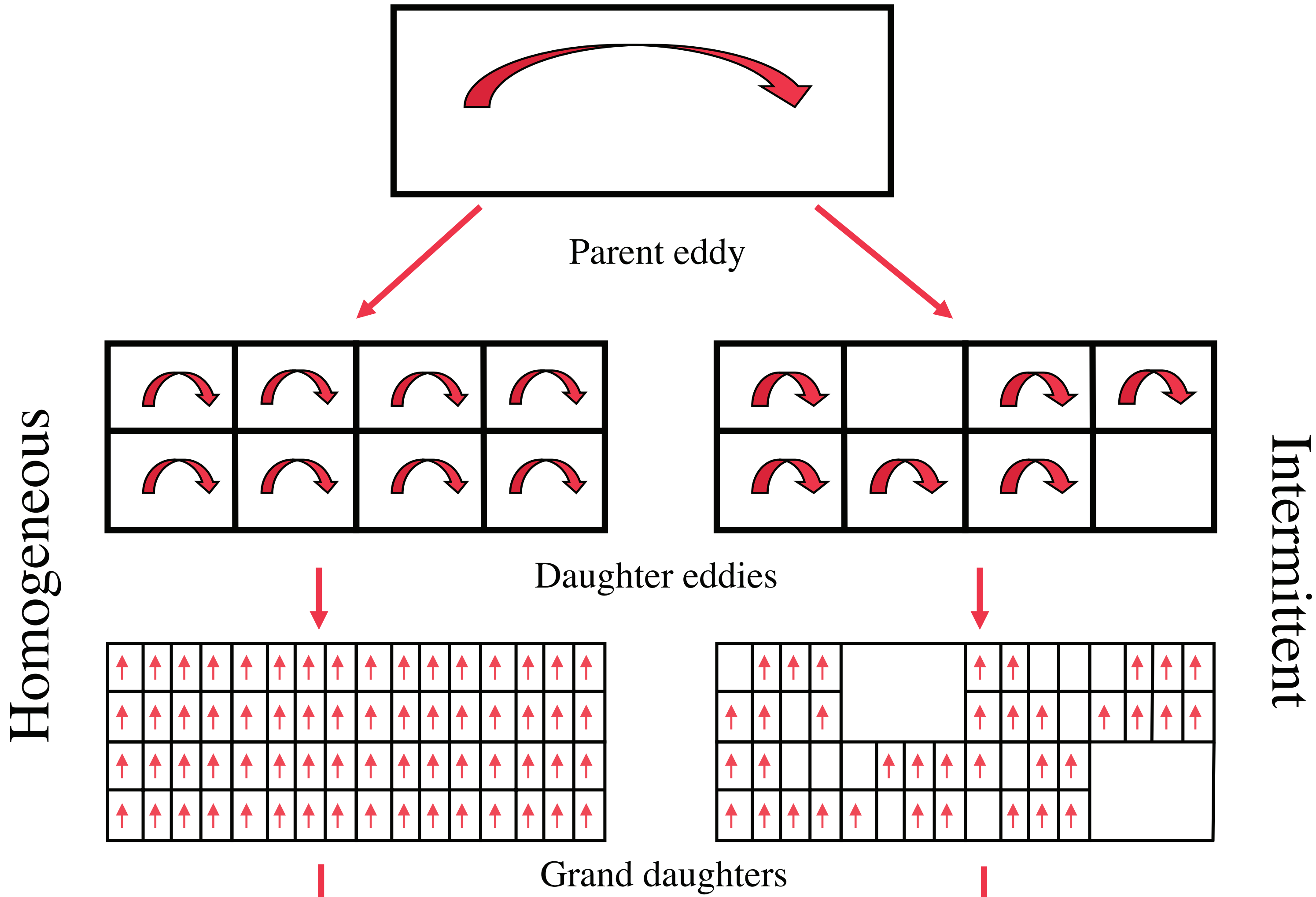
Size notion:

$$\left|(\Delta x, \Delta y, \Delta z)\right| = \left(\Delta x^2 + \Delta y^2 + \Delta z^2\right)^{1/2}$$

Small scale 3D



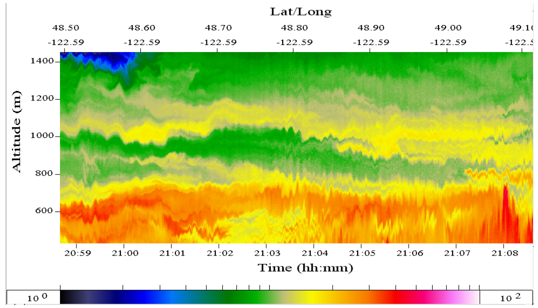
Stratified CASCADES



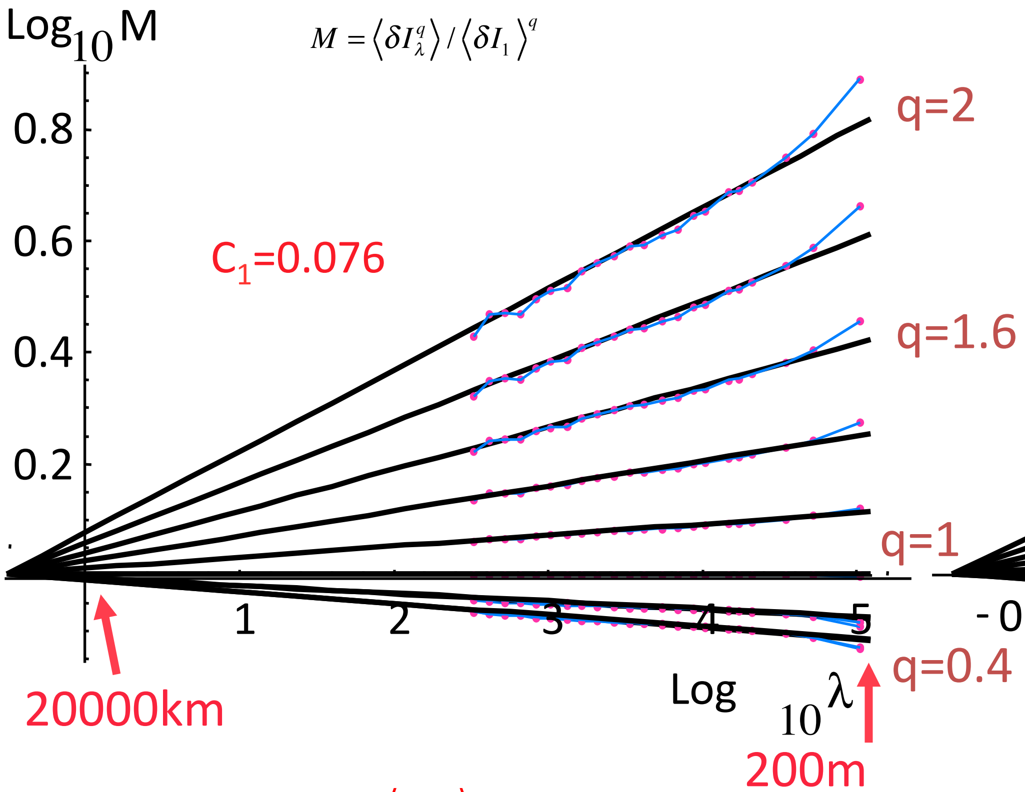
$$D = \frac{Log8}{Log4} = \frac{3}{2}$$

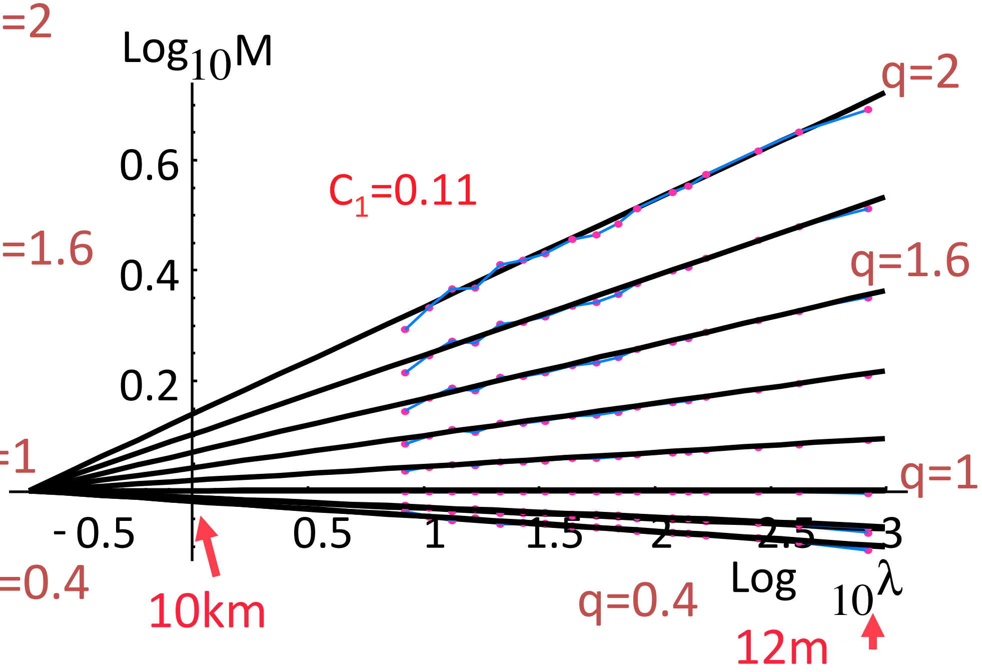
$$D = \frac{Log6}{Log4} = 1.29...$$

Vertical cascades:
lidar backscatter



From 10 airborne lidar cross-sections near Vancouver B.C.





$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$

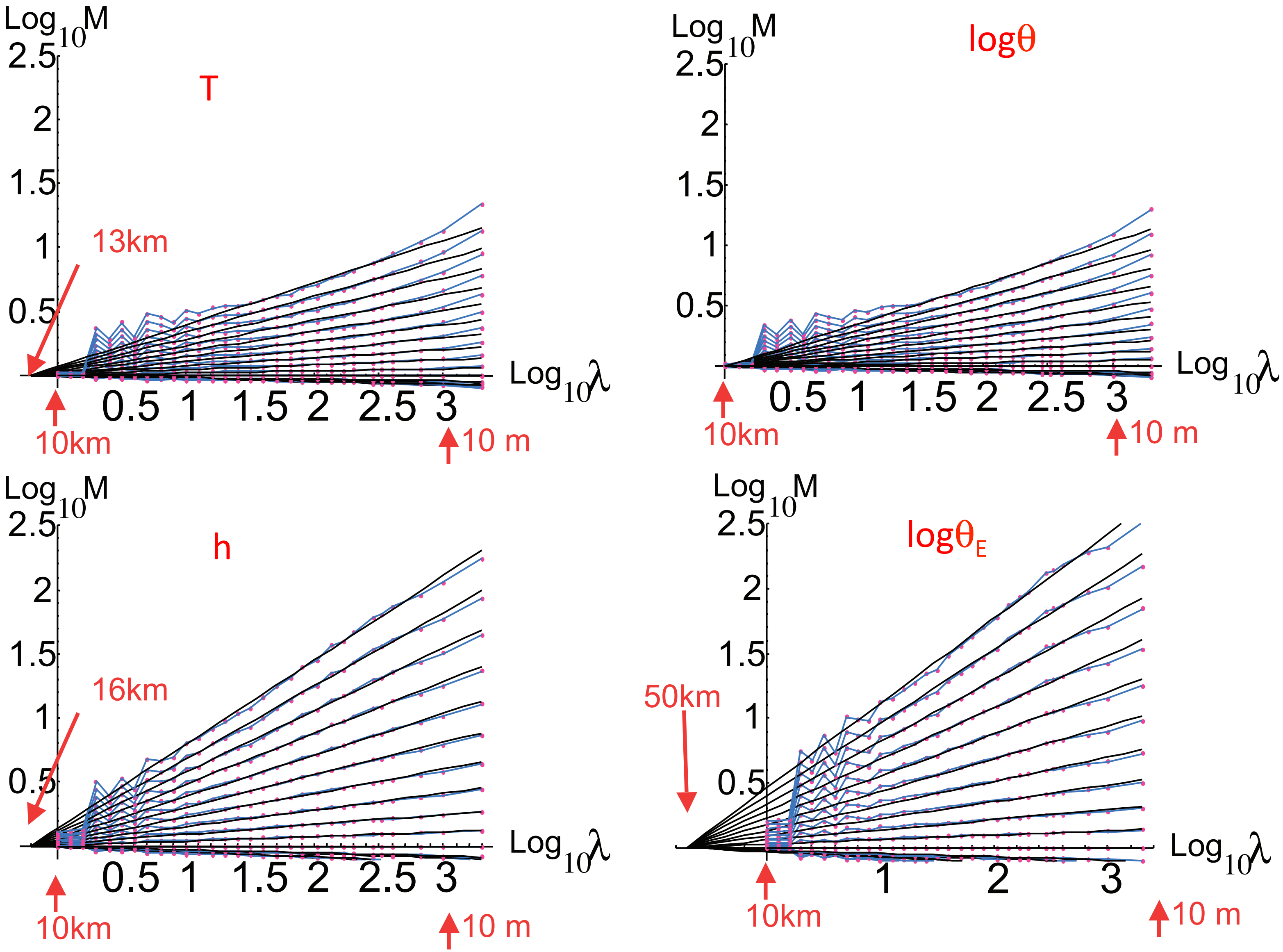
$$M_q \approx \lambda^{K(q)}$$

q=0, 0.2, 0.4..., 2

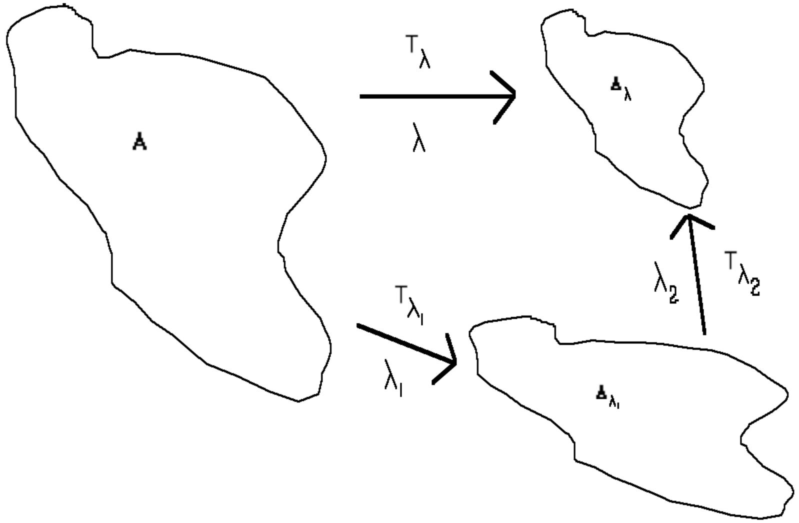
Vertical cascades:

Thermodynamic fields (drop sondes)

$$M = \left\langle \varphi_\lambda^q \right\rangle / \left\langle \varphi \right\rangle^q$$
$$M_q \approx \lambda^{K(q)}$$



Generalized Scale Invariance (GSI)
The scale changing operator T_λ



T_λ is the rule relating the statistical properties at one scale to another and involves only the scale ratio. This implies that T_λ has certain properties. In particular, if and only if $\lambda_1\lambda_2 = \lambda$, then:

$$B_\lambda = T_\lambda B_1 = T_{\lambda_1\lambda_2} B_1 = T_{\lambda_1} B_{\lambda_2} = T_{\lambda_2} B_{\lambda_1}$$

it is also commutative $$T_\lambda = T_{\lambda_2} T_{\lambda_1} = T_{\lambda_1} T_{\lambda_2}$$

This implies that T_l is a one parameter multiplicative group with parameter λ

The Elements of (GSI)

T_λ is a generalized contraction on a vector space E, it is a one-parameter (semi-) group for the positive real scale ratio λ ($\lambda \geq 1$ for a semi-group), i.e.:

$$\forall \lambda, \lambda' \in R^+ : T_{\lambda'} \circ T_\lambda = T_{\lambda'\lambda} \qquad \text{hence} \qquad \boxed{T_\lambda = \lambda^{-G}}$$

and admits a generalized scale denoted $\|\underline{r}\|$ (double lines to distinguish it from the usual Euclidean metric $|\underline{r}|$), which in addition to being nonnegative, satisfies the following three properties:

i) *Nondegeneracy*:

$$\|\underline{r}\| = 0 \Leftrightarrow \underline{r} = \underline{0}$$

ii) *Linearity* with the contraction parameter 1/λ:

$$\forall \underline{x} \in E, \forall \lambda \in R^+ : T_\lambda \|\underline{r}\| \equiv \|\underline{\underline{T}}_\lambda \, \underline{r}\| = \lambda^{-1} \|\underline{r}\|$$

iii) *Strictly decreasing balls*: the balls defined by this scale

$$B_\ell = \{\underline{r} \,|\, \|\underline{r}\| \leq \ell\}$$ (used to define anisotropic Hausdorff measures)

must be strictly decreasing with the contraction: $\forall L \in R^+, \forall \lambda > 1 : B_{L/\lambda} \equiv T_\lambda(B_L) \subset B_L$

and therefore: $\forall L \in R^+, \forall \lambda' \geq \lambda \geq 1 : B_{L/\lambda'} \subset B_{L/\lambda}$

A generalized blow-down with increasing of the acronym "NVAG". If $G = I$, we would have obtained a standard reduction, with all the copies uniformly reduced converging to the centre of the reduction. Here the parameters are

$$G = \begin{pmatrix} 1.3 & -1.3 \\ 0.3 & 0.7 \end{pmatrix}$$

and each successive reduction is by 28%.

The scale function equation

The basic scale function equation is:

$$\left\| T_\lambda \underline{r} \right\| = \lambda^{-1} \left\| \underline{r} \right\|$$

With group generator: $\left\| \lambda^{-G} \underline{r} \right\| = \lambda^{-1} \left\| \underline{r} \right\|$

In terms of the infinitesimal generator g(r) = Gr we have:

$$\left(\underline{g}\left(\underline{r} \right) \cdot \nabla \right) \left\| \underline{r} \right\| = \left\| \underline{r} \right\|$$

Nonlinear GSI: anisotropy = scale and position dependent

In the case of linear GSI, G is a matrix and we have:

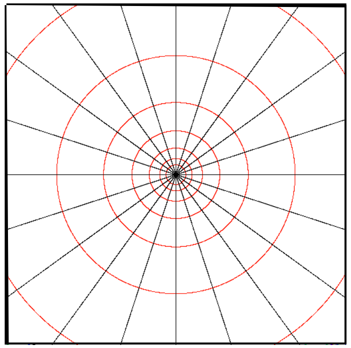
$$\left(\underline{r}^T \cdot \underline{G} \cdot \nabla \right) \left\| \underline{r} \right\| = \left\| \underline{r} \right\|$$

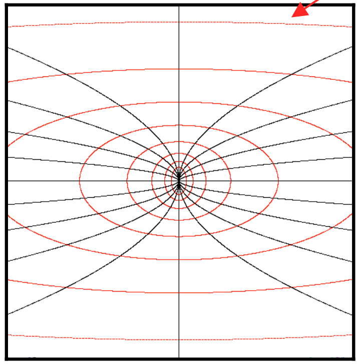
Linear GSI: anisotropy = scale dependent only

Scale functions in linear GSI (position independent)

Scale isolines in red

Isotropic
(self similar)





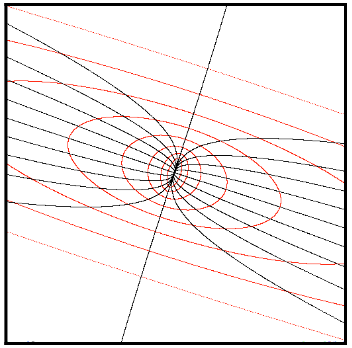
Self-affine

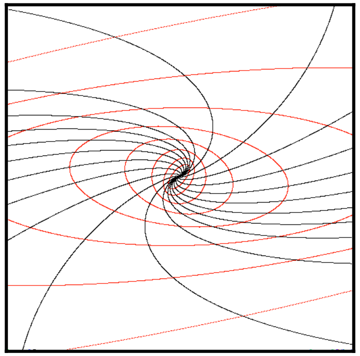
$$T_\lambda = \lambda^{-G}$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & 0 \\ 0 & 0.65 \end{pmatrix}$$

Stratification dominant (real eigenvalues)





Rotation dominant (complex eigenvalues)

$$G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}$$

The physical scale function and differential scaling

$$|\underline{\Delta r}| \rightarrow \|\underline{\Delta r}\|$$

Usual distance (=vector norm)

Scale function (scale notion)

Scale symmetry $\quad \left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\|$

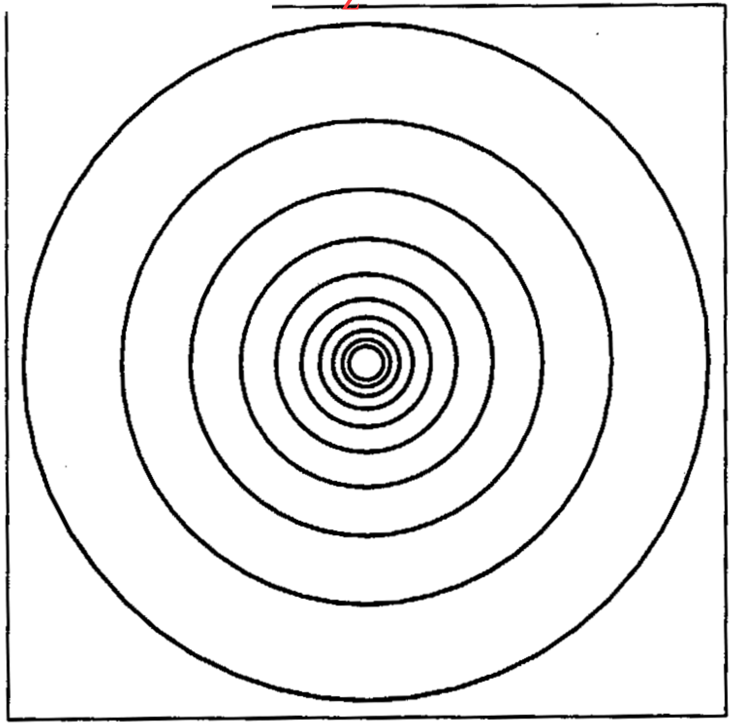
"canonical" scale function:

$$\|(\Delta x, \Delta z)\| = l_s\left(\left(\frac{\Delta x}{l_s}\right)^2 + \left(\frac{\Delta z}{l_s}\right)^{2/H_z}\right)^{1/2}$$

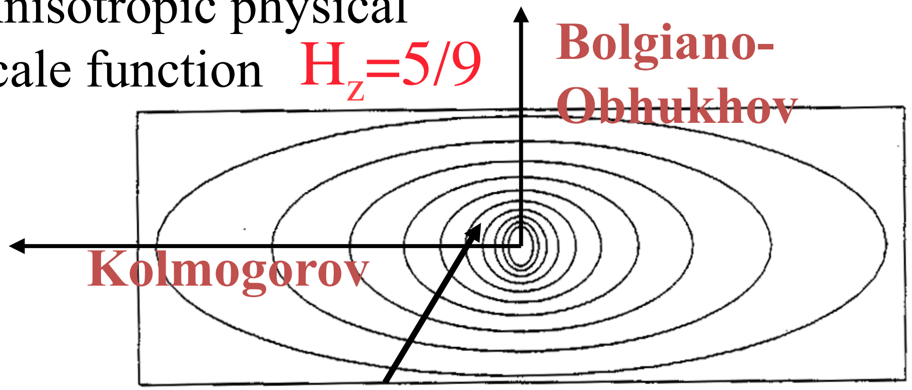
$$G = \begin{pmatrix} 1 & 0 \\ 0 & H_z \end{pmatrix}$$

Vertical sections

Isotropic function $H_z=1$

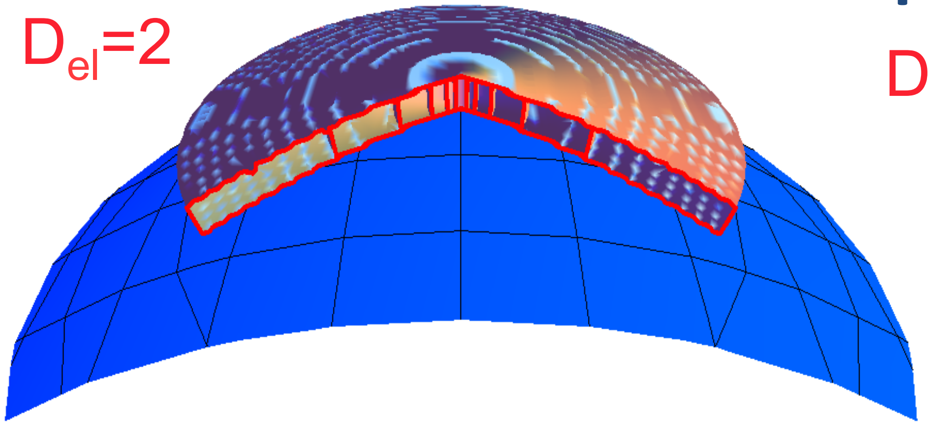


Anisotropic physical scale function $H_z=5/9$

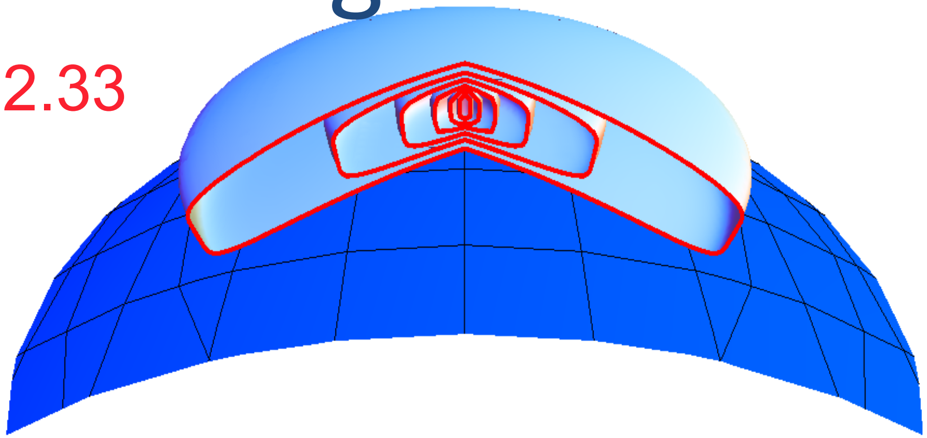


Anisotropic Scaling

$D_{el}=2$

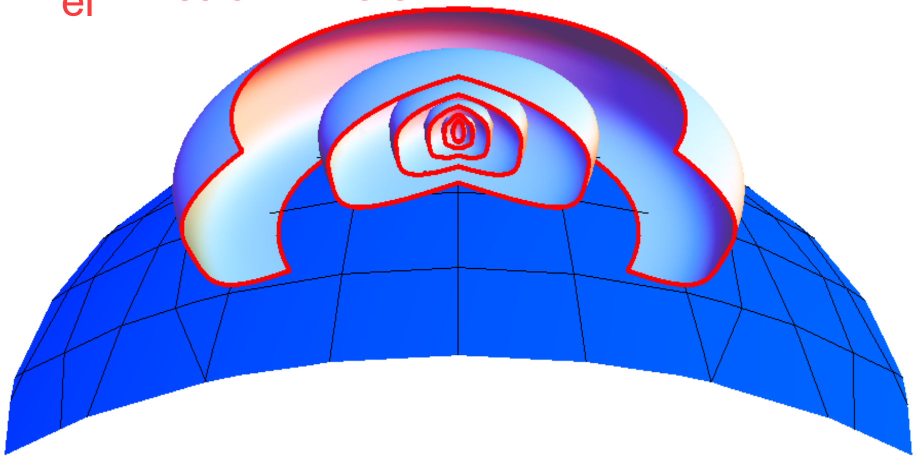


$D_{el}=2.33$

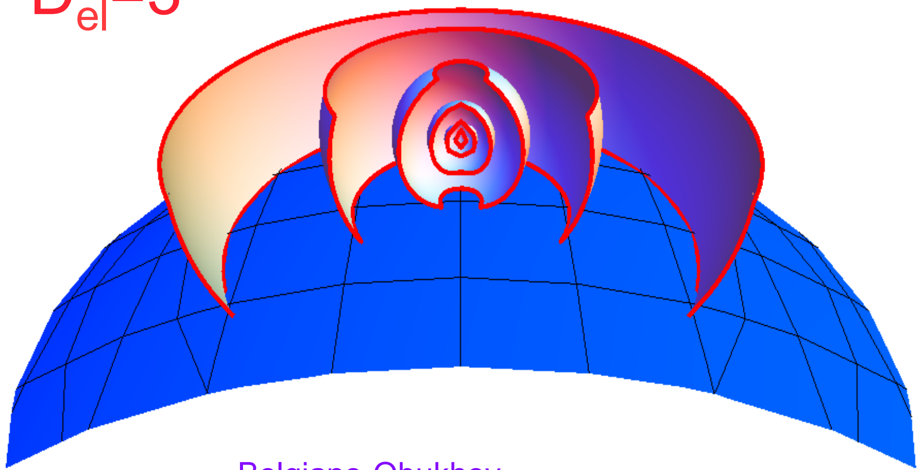


$D_{el}=23/9=2.55$

c.f. empirical: 2.57



$D_{el}=3$



The 23/9D model:

Bolgiano-Obukhov

$$\Delta v(\Delta x) = \varepsilon^{1/3}\Delta x^{1/3}; \quad \Delta v(\Delta z) = \phi^{1/5}\Delta z^{3/5}$$

$H_z=(1/3)/(3/5)=5/9$

Kolmogorov

Volume $\approx L \times L \times L^{Hz} \approx L^{Del}$

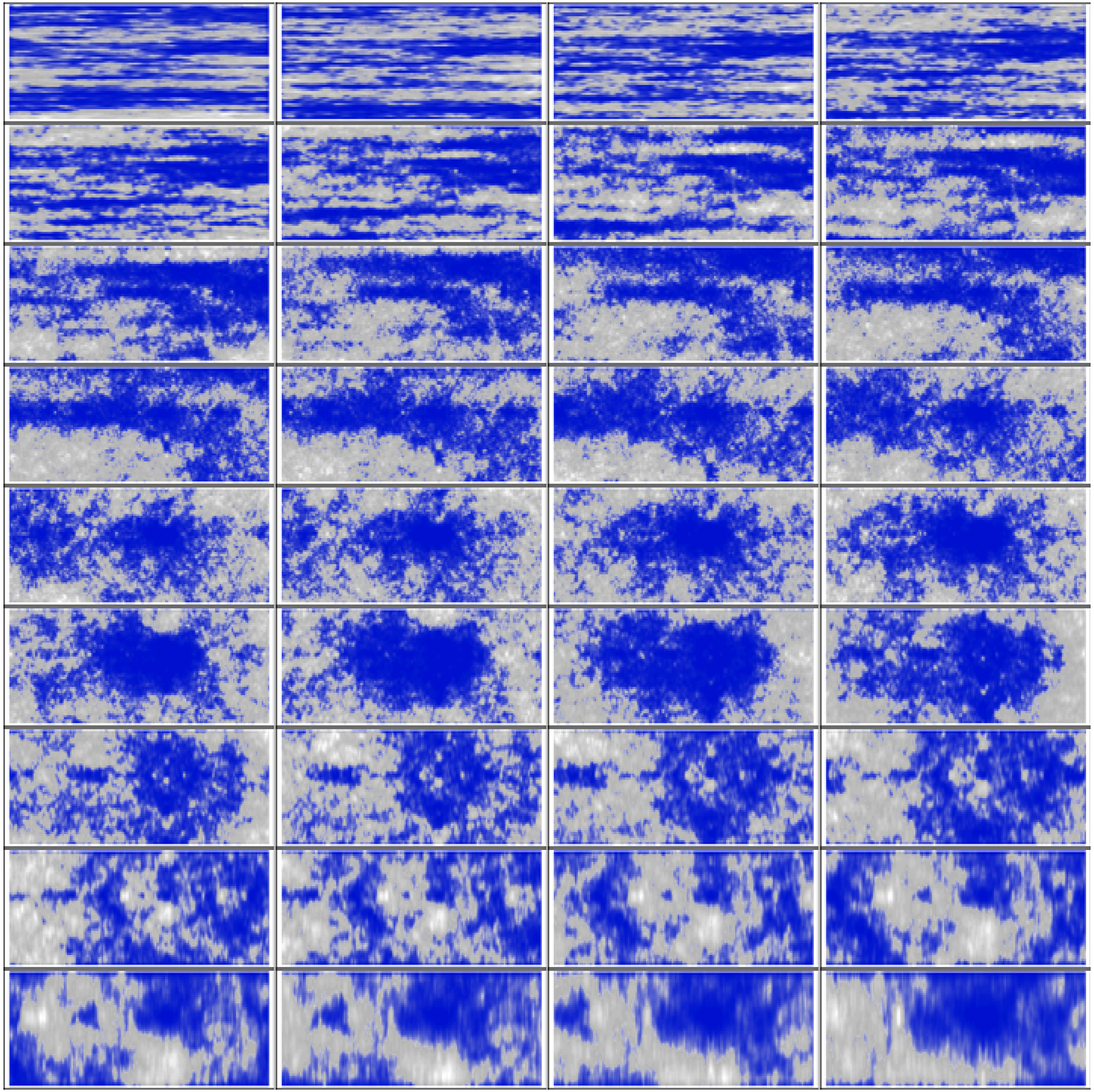
$D_{el}=2+H_z=23/9$

Overall

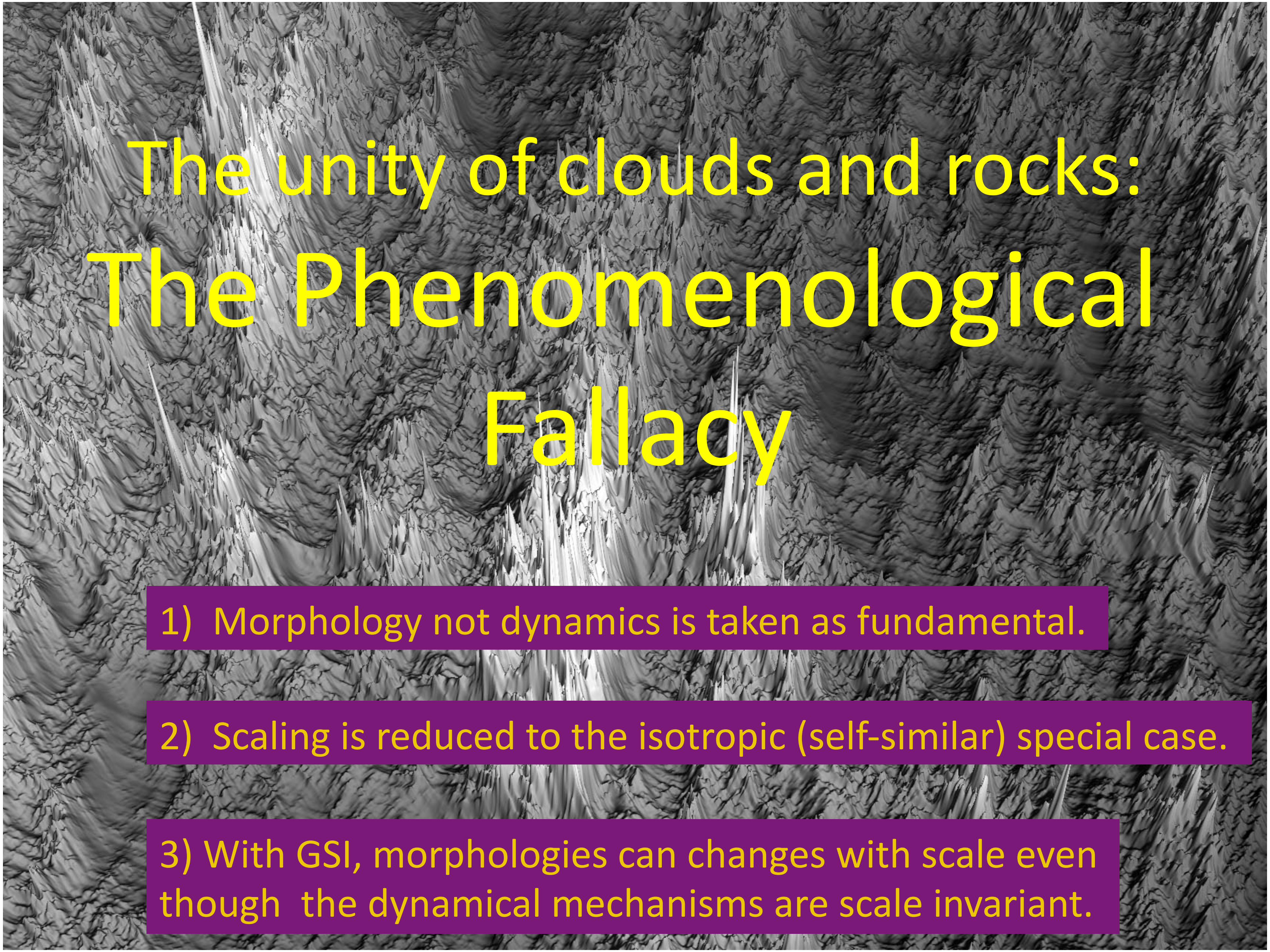
Isotropy ⟶ **anisotropy**

$$\left|\underline{x}\right| \to \left\|\underline{x}\right\|; \quad D \to D_{el}$$

Zoom factor 1000



Vertical cross-section



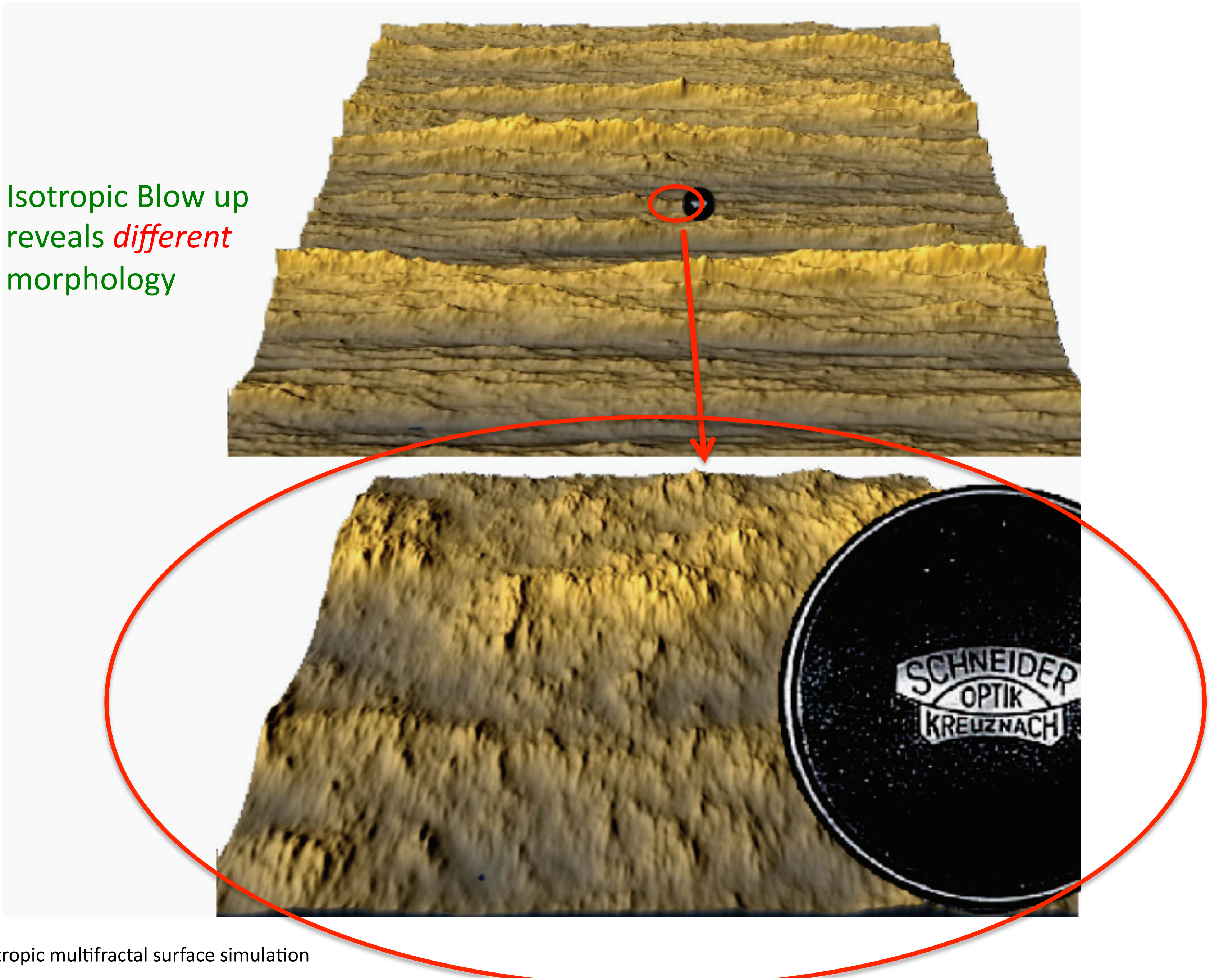
The unity of clouds and rocks:

The Phenomenological Fallacy

1) Morphology not dynamics is taken as fundamental.

2) Scaling is reduced to the isotropic (self-similar) special case.

3) With GSI, morphologies can changes with scale even though the dynamical mechanisms are scale invariant.



Anisotropic multifractal surface simulation

Illustrating the effect of varying *G* and the unit ball with multifractal simulations

The basic morphologies don't depend on the orientation or size;
it suffices to consider *d* =1, *r* = *f*, *c* = 0, i.e. to only consider matrices of the form:

$$G = \begin{pmatrix} 1 & r-e \\ r+e & 1 \end{pmatrix}$$

In order to explore the possible morphologies, the last element we need is therefore a specification of the unit ball. A convenient one-parameter parametrization is:

Polar coordinate representation of the unit ball:

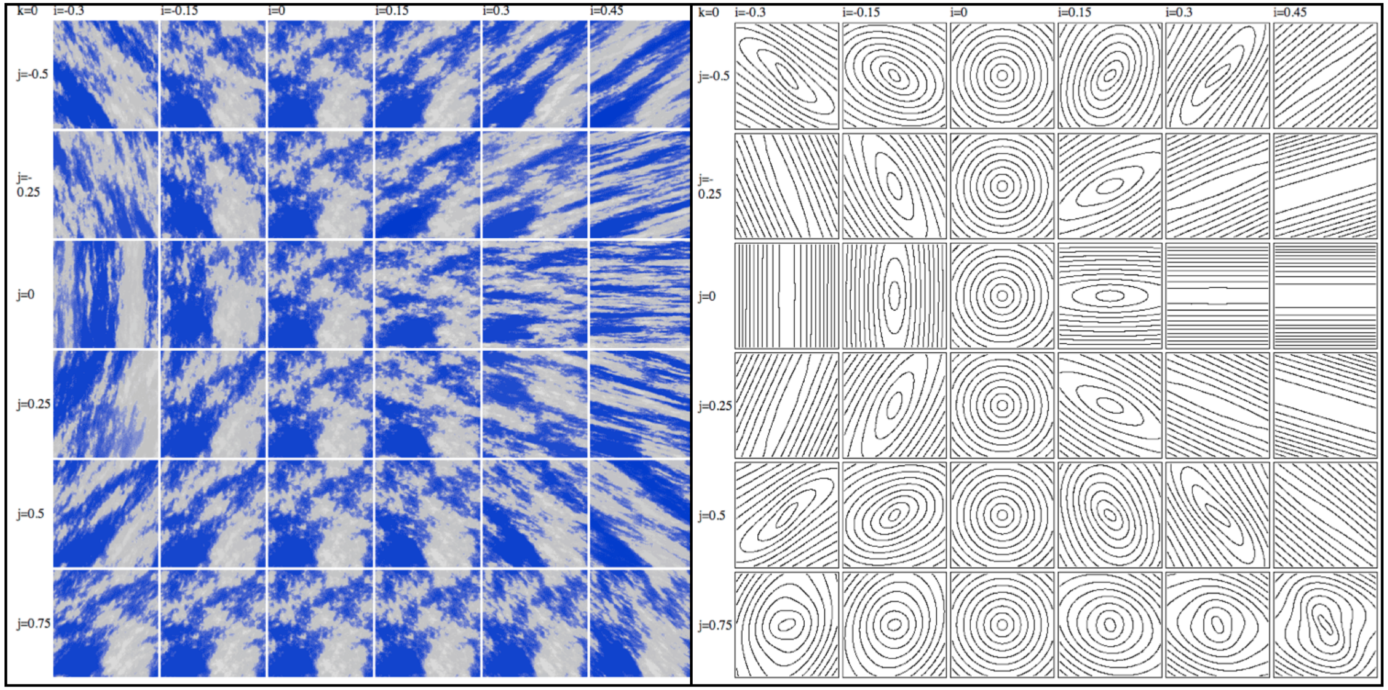
$$r(\theta'') = 1/\Theta(\theta'')$$

$$\Theta(\theta'') = 1 + \frac{1-2^{-k}}{1+2^{-k}}\cos\theta''$$

k is the $\log_2$ of the ratio of the largest to smallest scale on the unit ball http://www.physics.mcgill.ca/~gang/multifrac/index.htm

Roundish unit ball

r

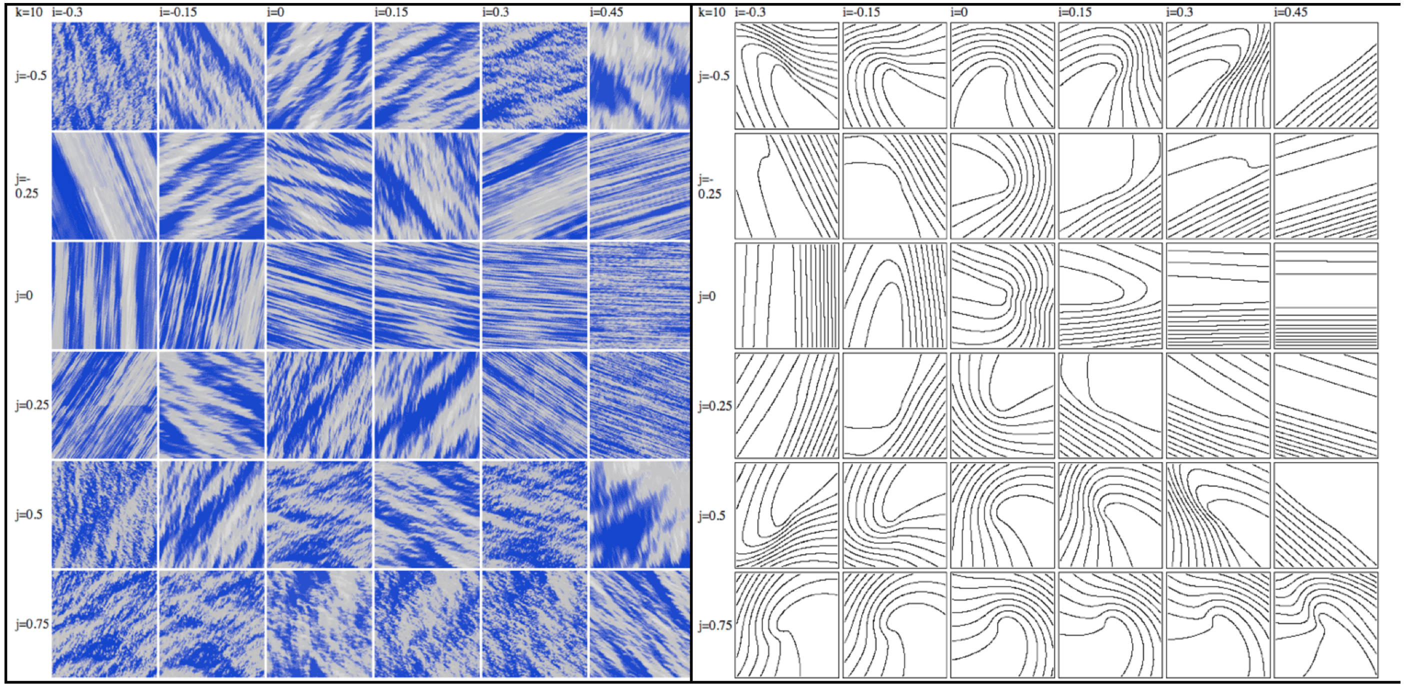


e

k =0: we vary *r* (denoted *i*) from -0.3, -0.15, …0.45 left to right and *e* (denoted *j*) from -0.5, -0.25, …0.75 top to bottom. On the right we show the contours of the corresponding scale functions.

$$G=\begin{pmatrix} 1 & r-e \\ r+e & 1 \end{pmatrix}$$

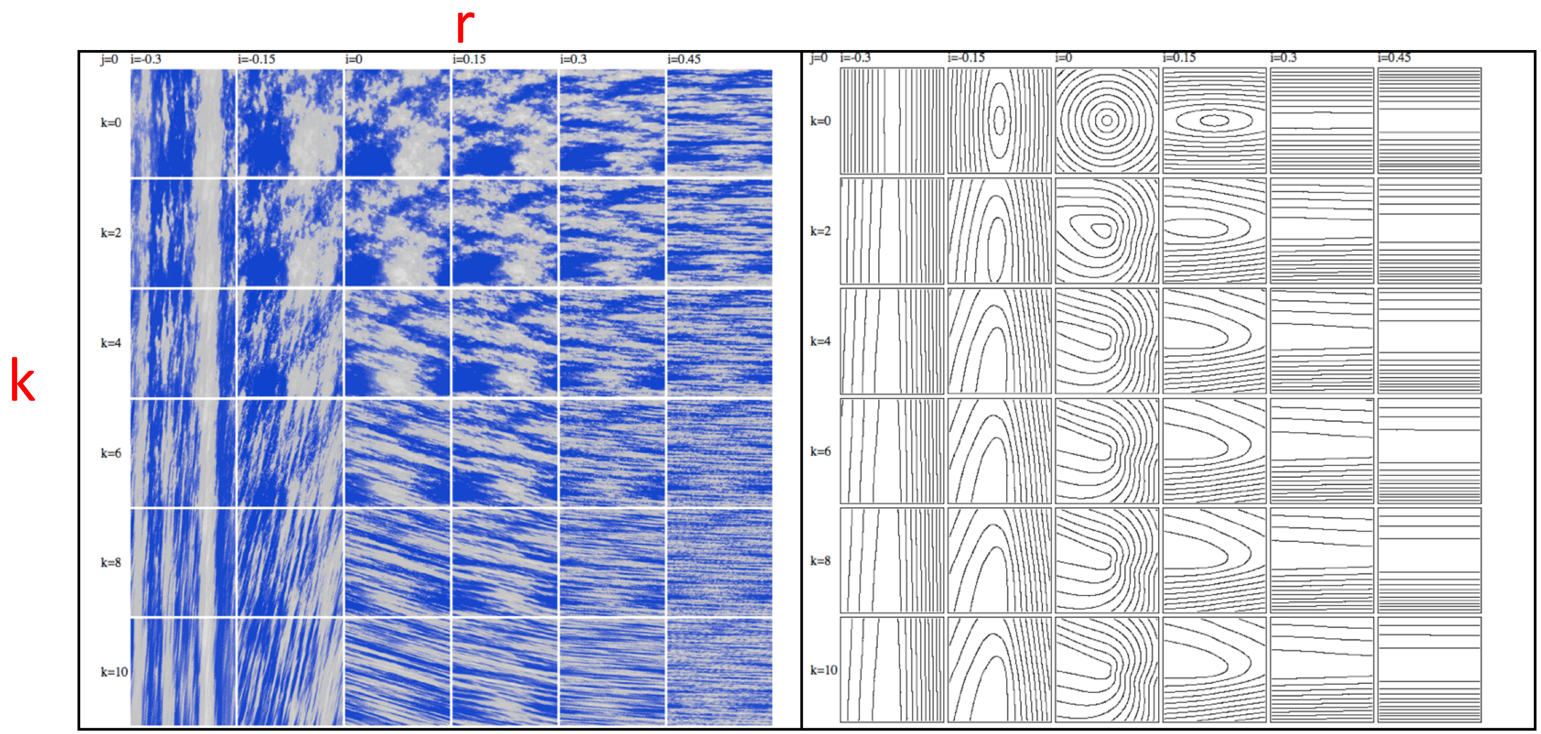
Highly anisotropic unit ball: *k* =10



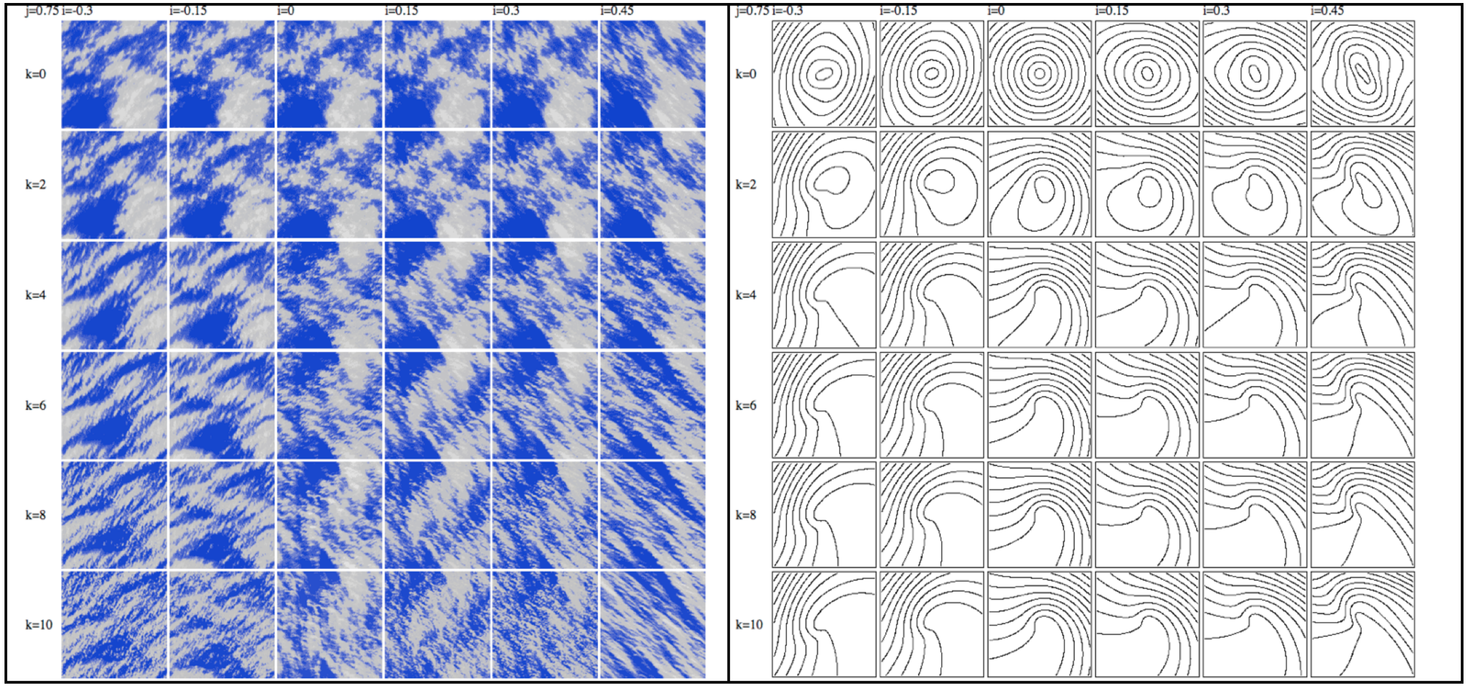
$$\Theta(\theta'')=1+\frac{1-2^{-k}}{1+2^{-k}}\cos\theta''$$

e = 0

r is increased from -0.3, -0.15, …0.45 left to right, from top to bottom, *k* is increased from 0, 2, 4,..10.



e = 0.75

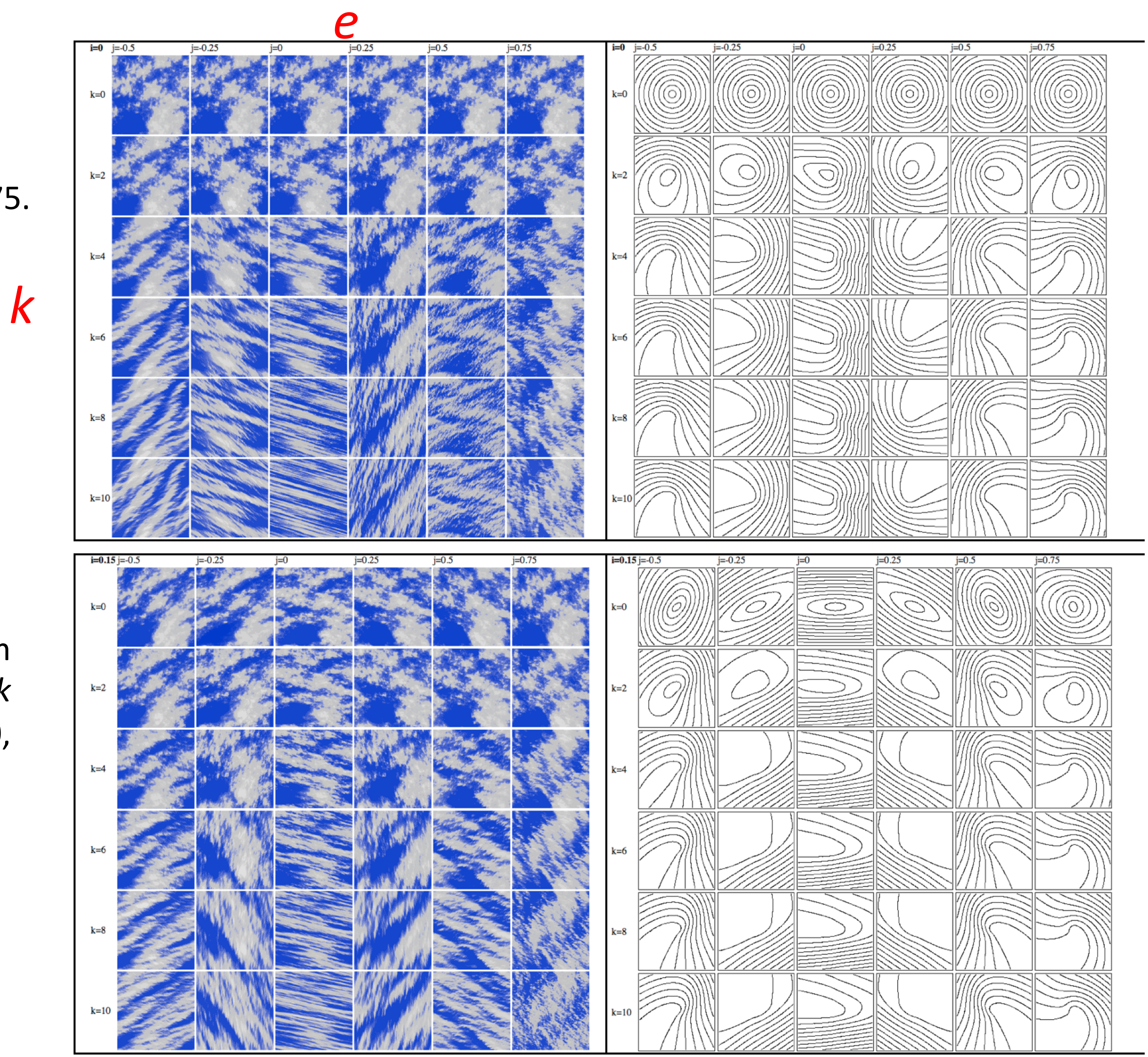


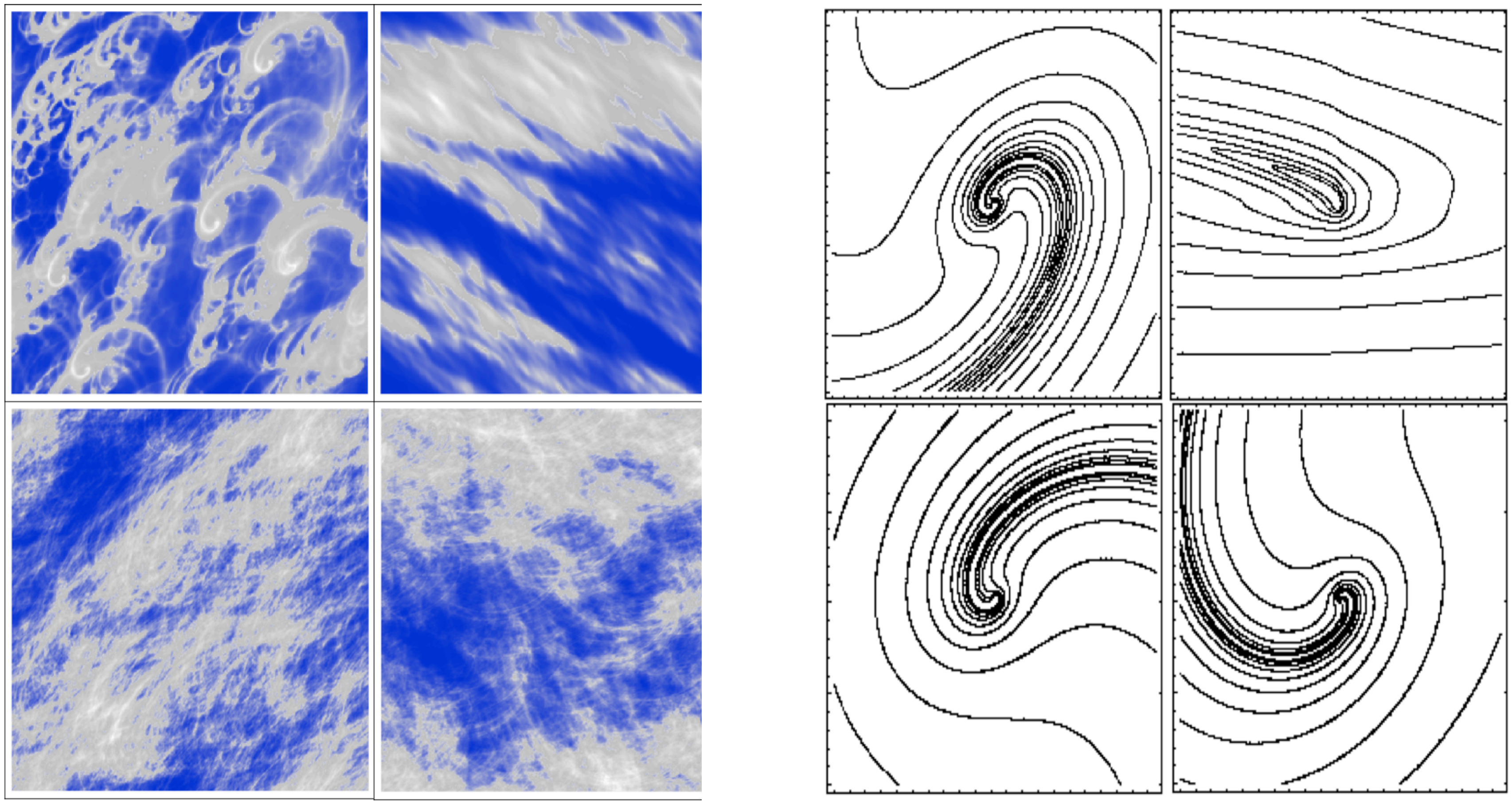
r = 0

e left to right is: -0.5, -0.25, …0.75.

r = 0.15

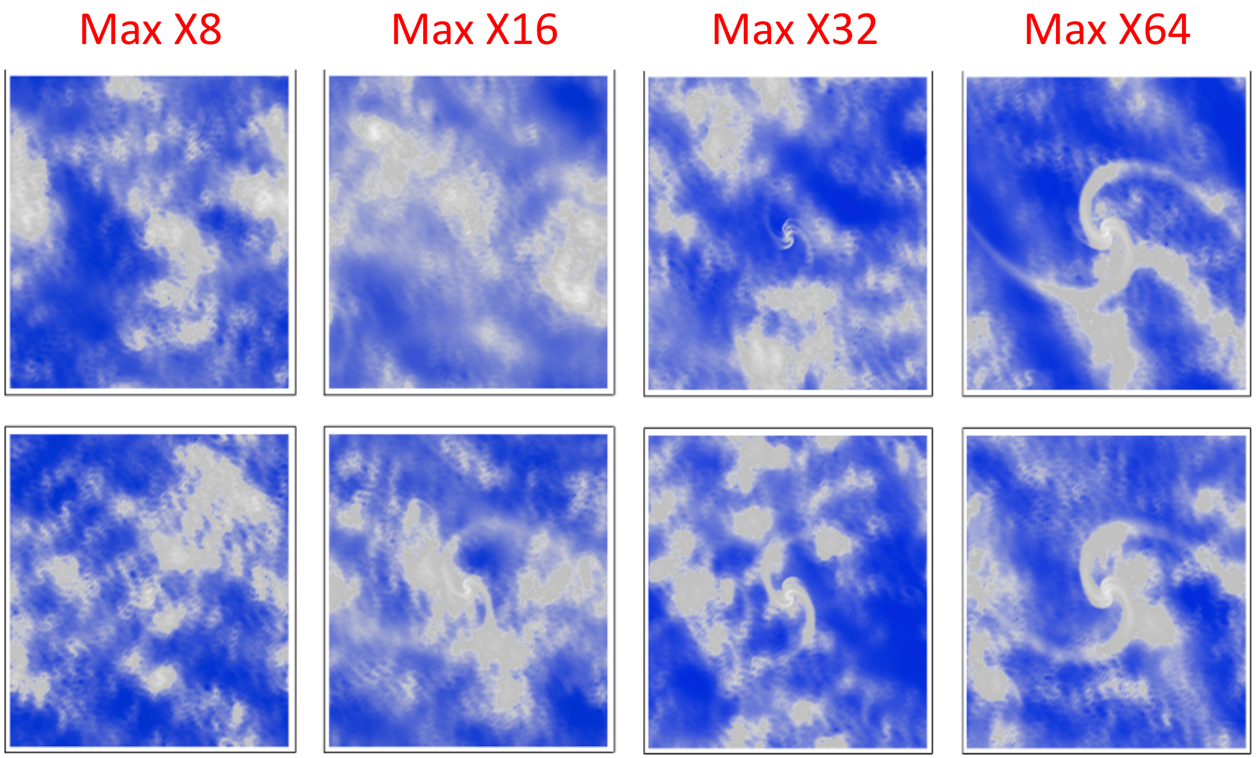
In all rows, from top to bottom, *k* is increased (0, 2, 4,..10),

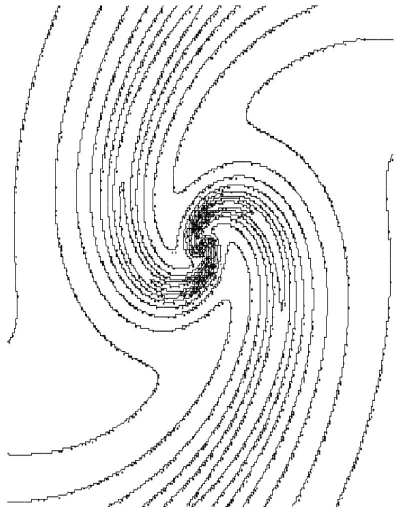




Examples of 2D simulations on 512x512 pixel grids with $\alpha = 1.8$, $C_1 = 0.1$, $H = 0.333$, d =1, f =0. Upper left: $c = 0.8$, $e = 2$, l_s =512, x = 1.3 ($2^k = r_{max}/r_{min} \approx 54$), upper right: $c = -2/7$, $e = 0.1$, $l_s = 32$, $2^k \approx 5$, lower left: $c = 0.3$, $e = 1.2$, $l_s = 32$, $2^k \approx 800$, lower right: $c = 0.3$, $e = 1.2$, $l_s = 1$, $2^k \approx 800$.

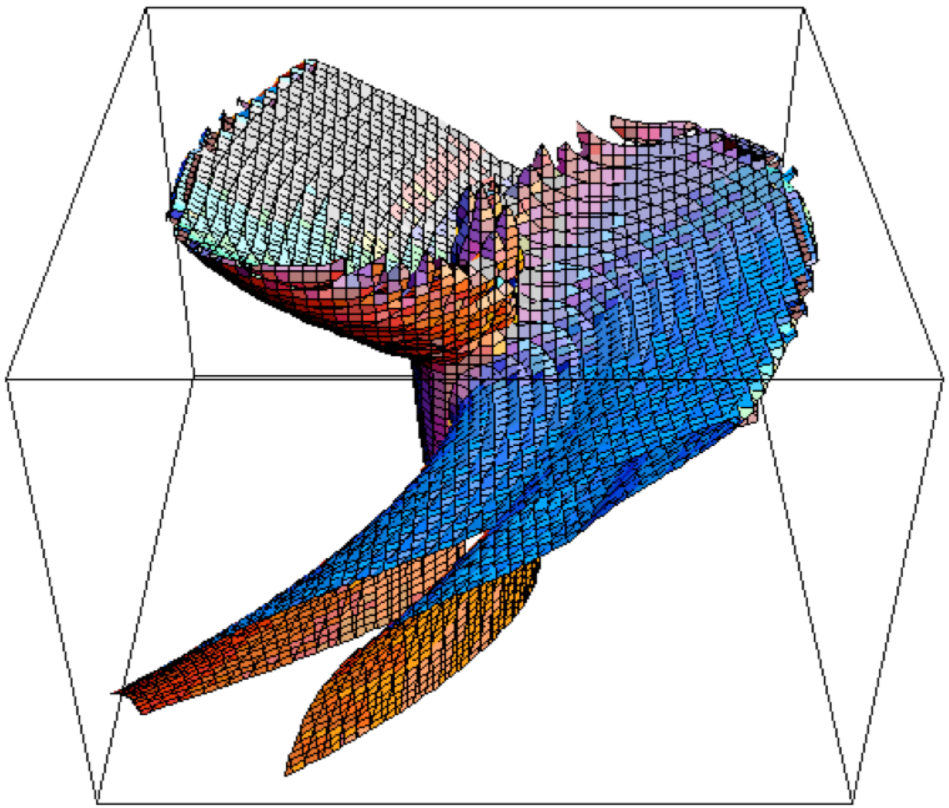
Order emerging from chaos



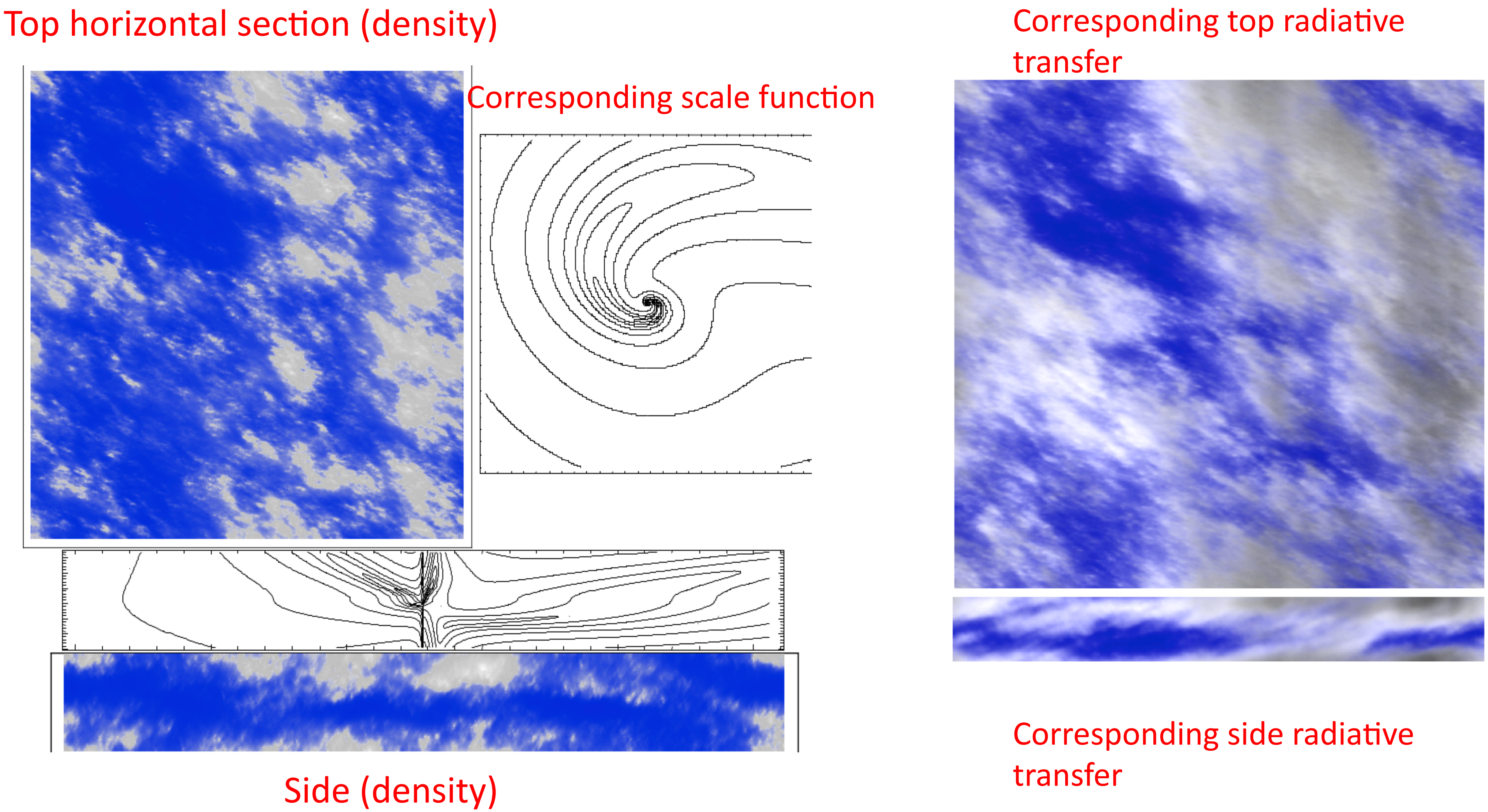


Each row shows a realization of a random multifractal process with a single value of of the subgenerator $\gamma(\underline{r})$ at the centre of a 512X512 grid replaced by the maximum of $\gamma(\underline{r})$ over the field boosted by factors of N increasing by 2 from left to right (from 8 to 64) in order to simulate very rare events (α = 1.8, C_1 = 0.1, H = 0.333). The scaling is anisotropic with complex eigenvalues of G, the scale function is shown at right.

Simulations in three dimensions, rendering with simulated radiative transfer

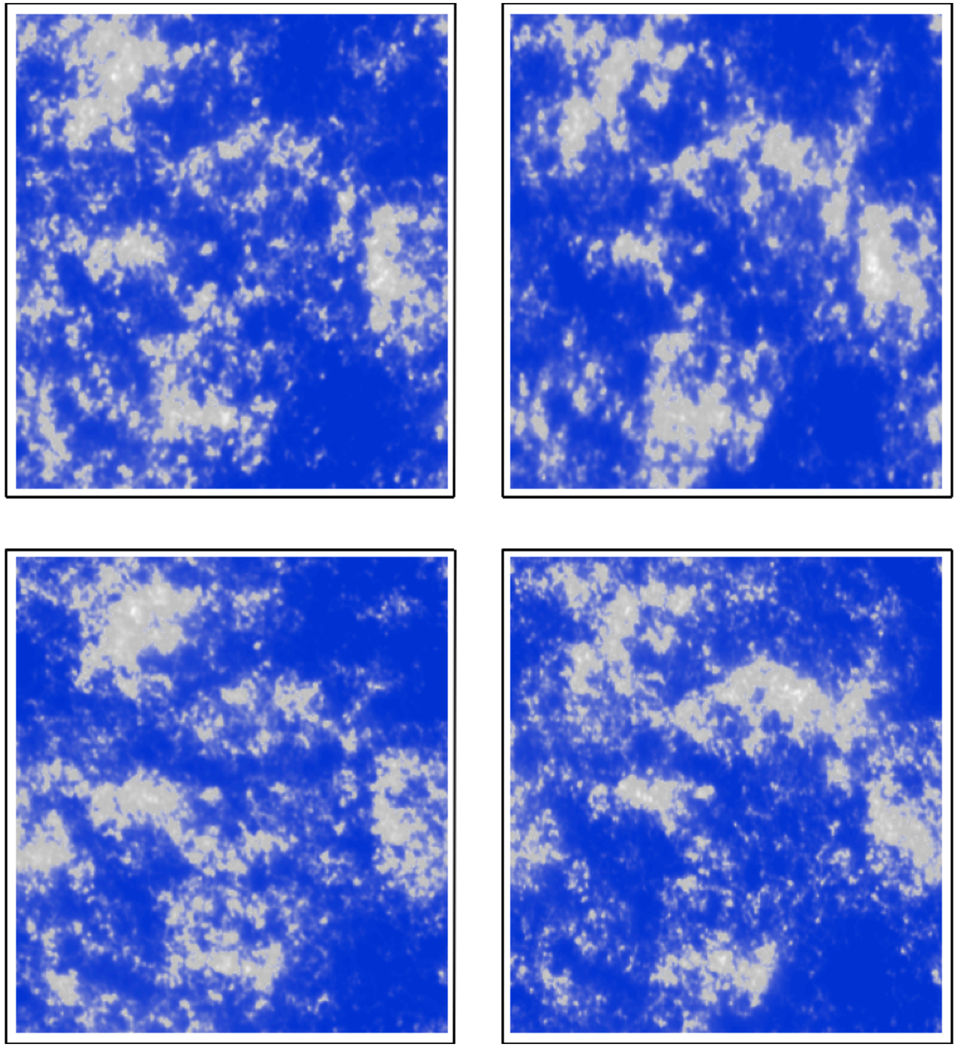


This is a contour of the scale function corresponding to a single scale; this is a strongly rotationally dominant case with n = 2, $x_q = x_f$ = 1.4, d =1, c = 0.5, e = 1, f = 0, H_z = 0.8, l_s =64,



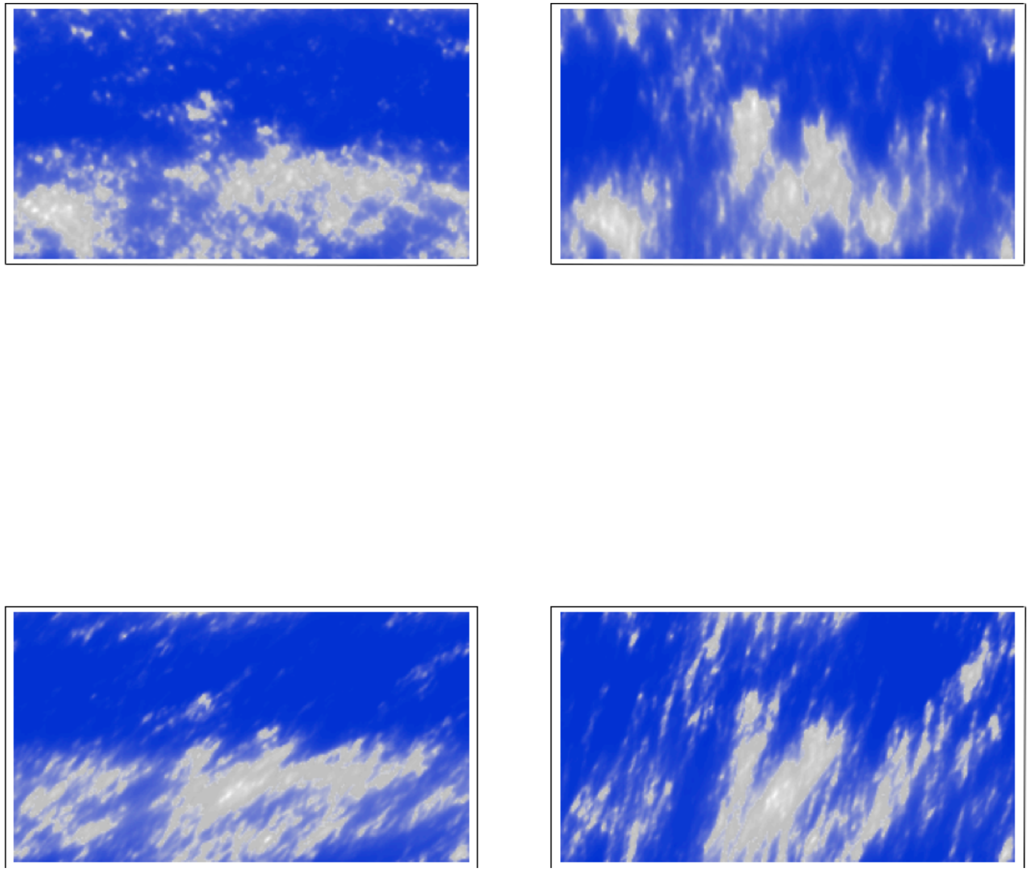
An example with a = 1.8, C_1 = 0.1, H = 0.333, on a 512x512x64 grid (the latter is the thickness). The parameters are n_q =1, n_f = 2, x_q = 0.3, x_f = 0.8, c = 0.2, e = 0.5, f = 0.2 (rotation dominant), H_z = 0.555 with l_s = 128, l_{sz} = 32. The upper left is the liquid water density field, top horizontal section, to the right is the corresponding central hrizontal cross section of the scale function. The bottom row shows one of the sides (512x64 pixels) with corresponding central part of the vertical cross section.

Cloud tops (densities)

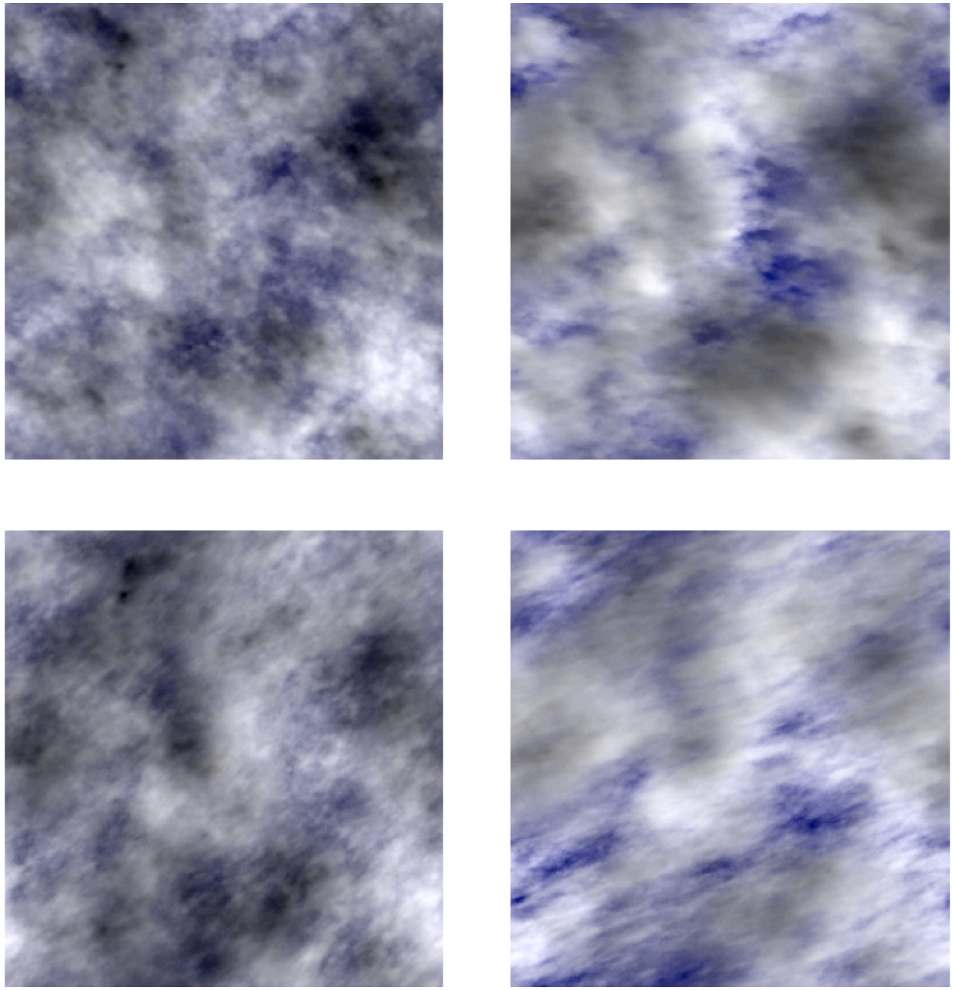


This shows the top layers of three dimensional cloud liquid water density simulations (false colours) all have $d = 1$, $c = 0.05$, $e = 0.02$, $f = 0$, $H_z = 0.555$, $\alpha = 1.8$, $C_1 = 0.1$, $H = 0.333$ and are simulated on a 256x256x128 point grid ($a^2 > 0$; stratification dominant in the horizontal). The simulations in the top row have $l_s = 8$ pixels, (left column), 64 pixels (right column), k=0, k=32 (bottom row). Note that in these simulations, the $l_s = 8$, 64 applies to both vertical and horizontal cross-sections (i.e. $l_s = l_{sz}$). Show an example with IR scattering?

Sides, same clouds (densities)

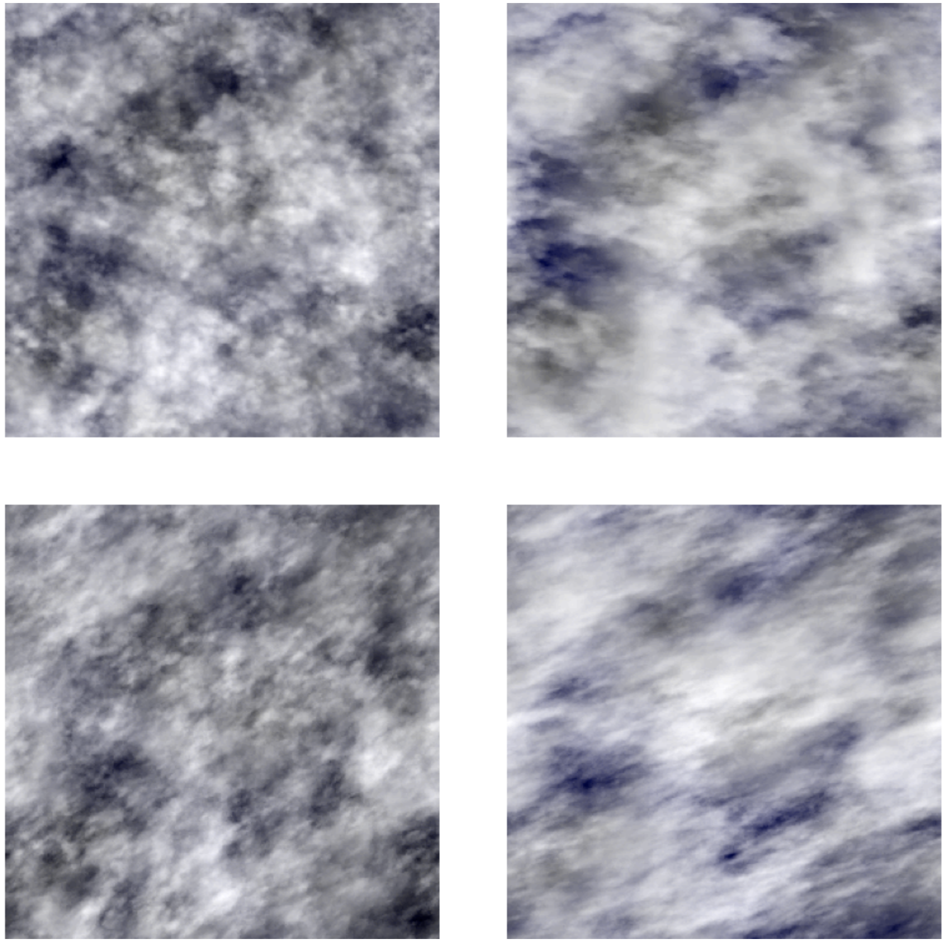


Same clouds radiative transfer, top view



The top view with single scattering radiative transfer; incident solar radiation at 45° from the right, mean vertical optical thickness = 50

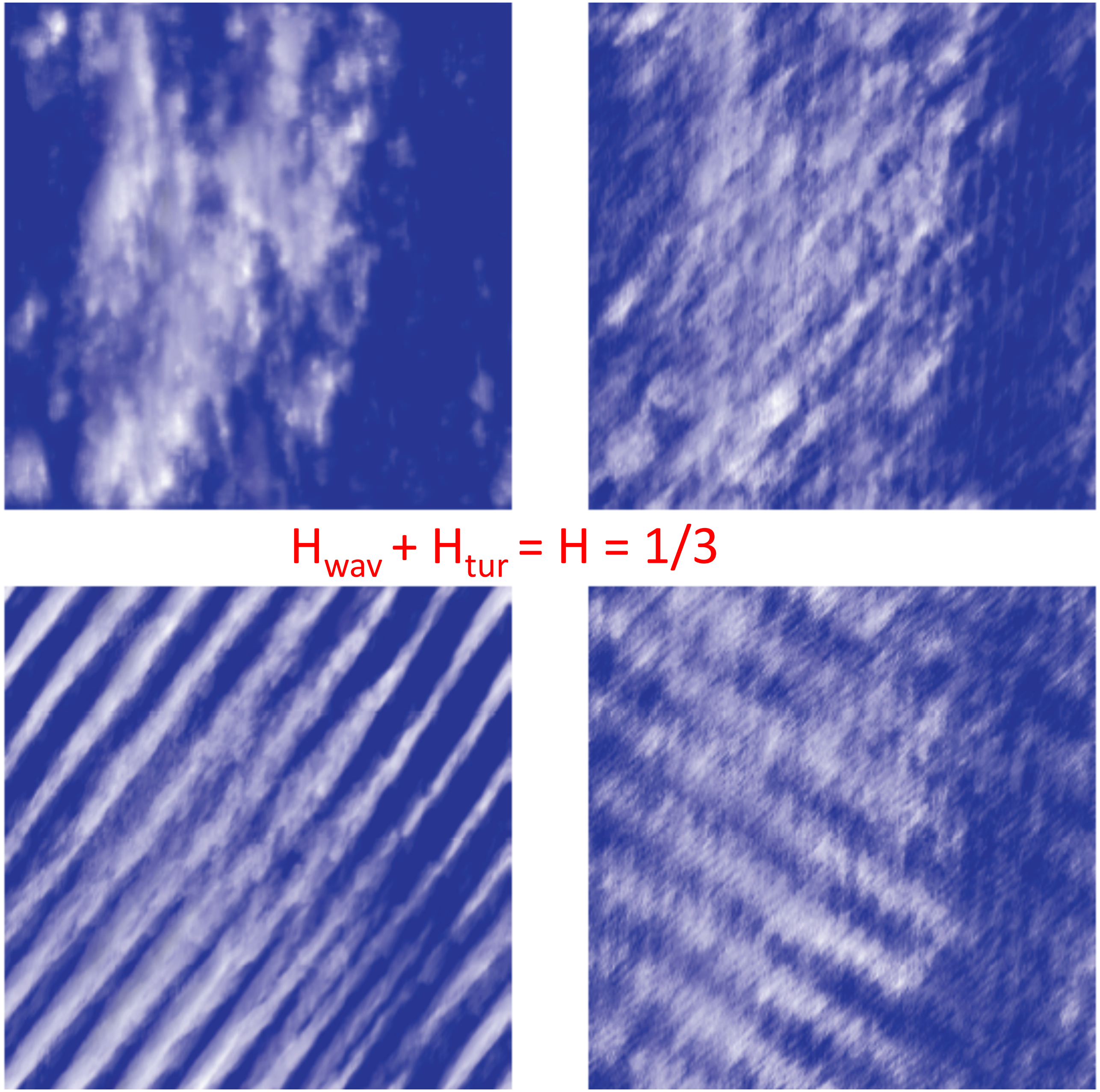
Same clouds radiative transfer, bottom view



The same except viewed from the bottom.

Multifractals with wave character

$H_{wav} = 0$ $H_{wav} = 0.33$



$H_{wav} + H_{tur} = H = 1/3$

$H_{wav} = 0.52$ $H_{wav} = 0.38$

Localized in space, unlocalized in space-time (product of turbulent and wave-like scaling propagators).

propagator ↓

$$\tilde{I}(\underline{k},\omega) = \tilde{g}(\underline{k},\omega)\tilde{\varepsilon}(\underline{k},\omega)$$

$$\tilde{g}(\underline{k},\omega) = \left(-i\omega + \|\underline{k}\|\right)^{-H_{tur}}\left(\omega^2 V^{-2} - \|\underline{k}\|^2\right)^{-H_{wav}/2}$$

Fly by of anisotropic (multifractal, cascade) cloud

Rocks

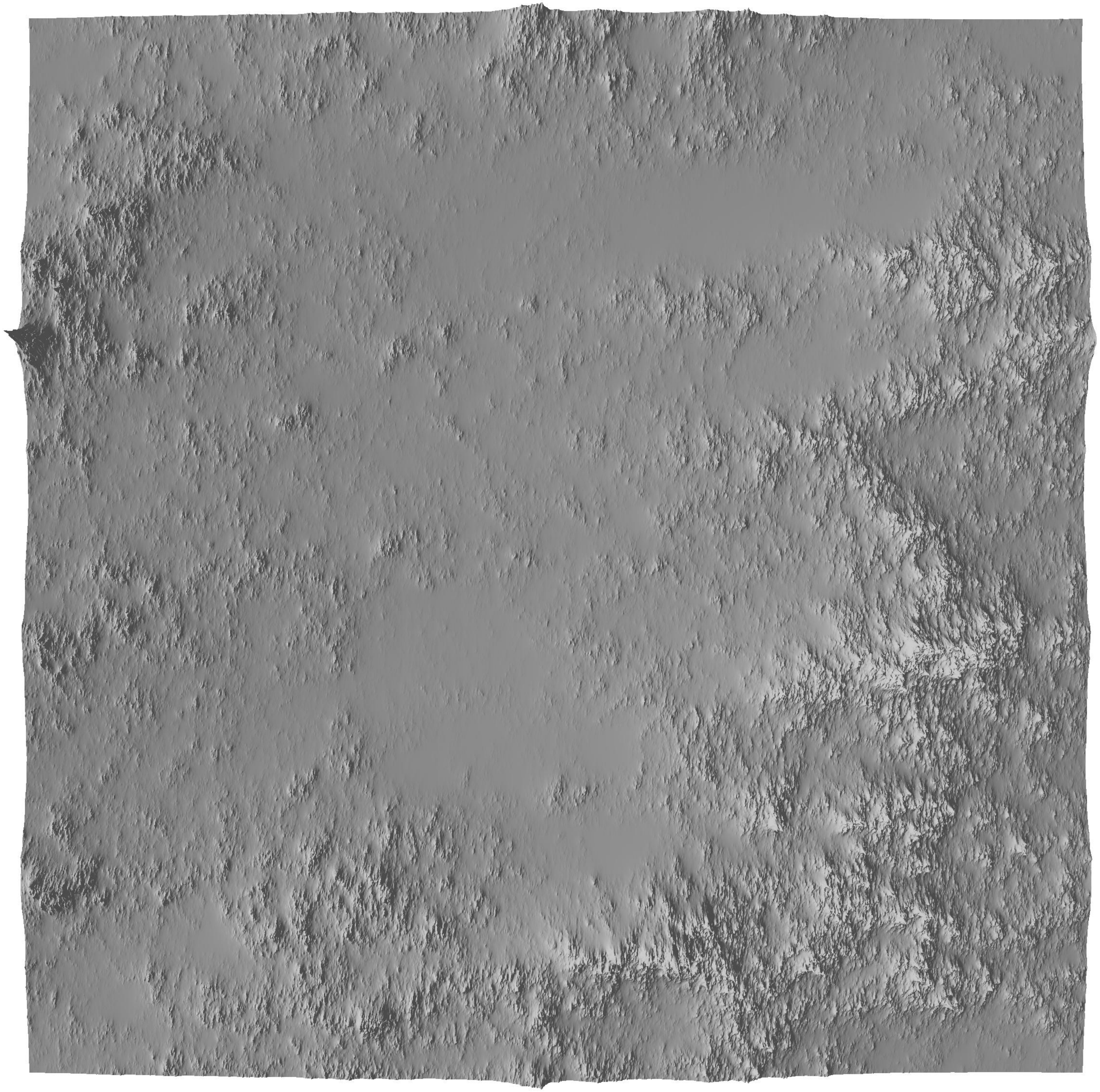
Flyby 1

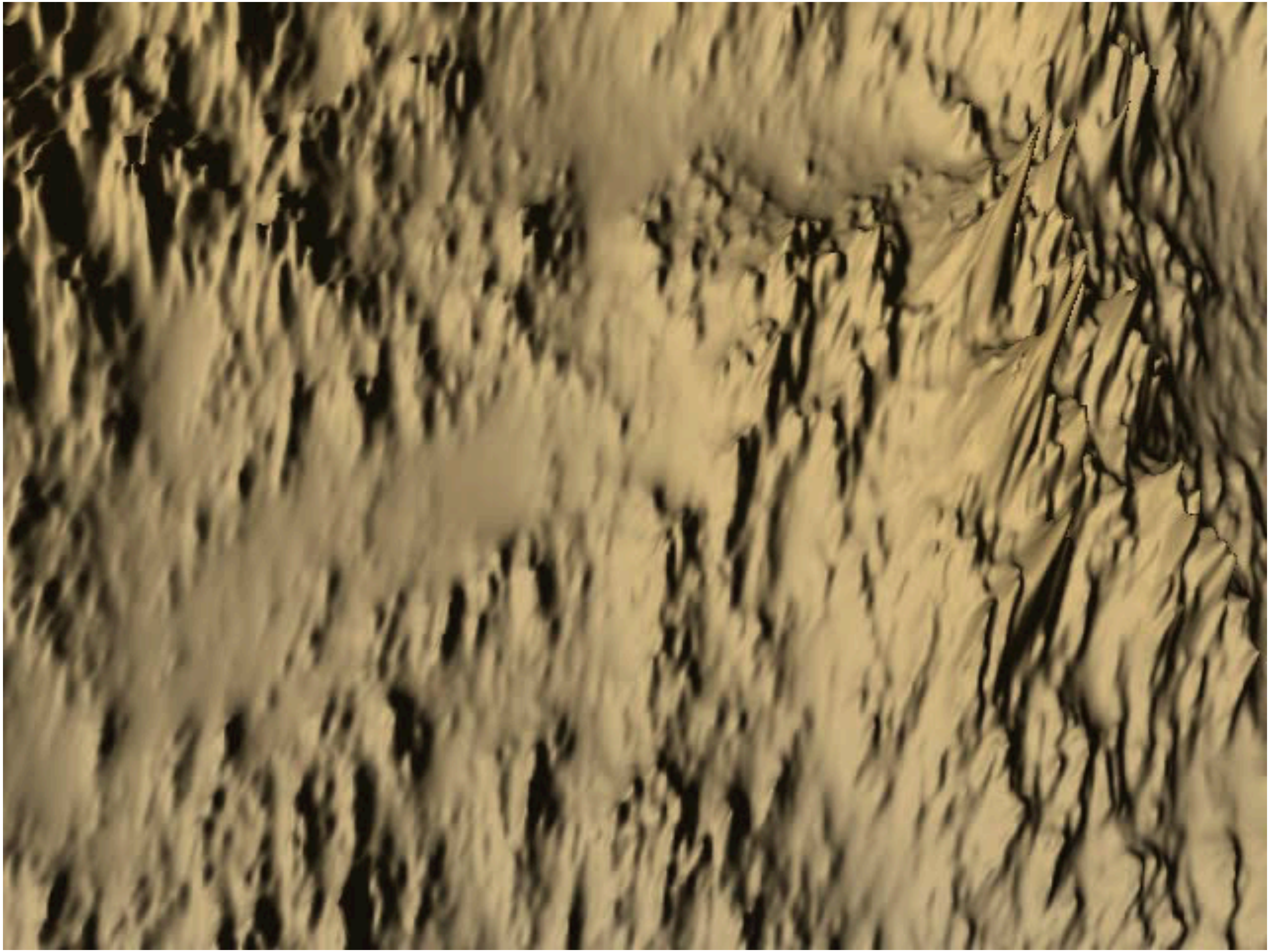
This 4096X4096 simulation is flown over

α=1.8, C_1=0.12, H=0.7

$$G = \begin{pmatrix} 0.65 & -0.1 \\ 0.1 & 1.35 \end{pmatrix}$$

l_s=64 pixels

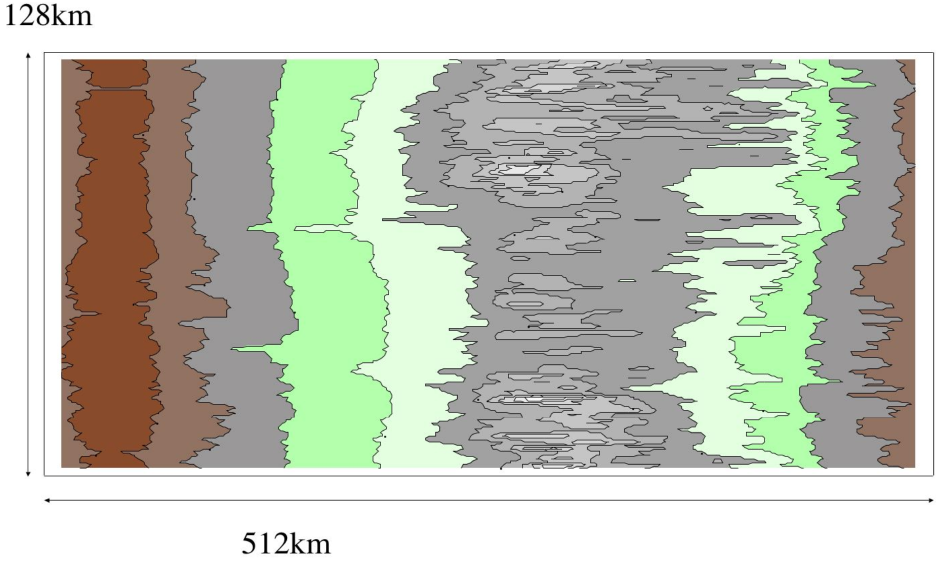




Stratified Multifractal Crust, Mantle rock density simulation

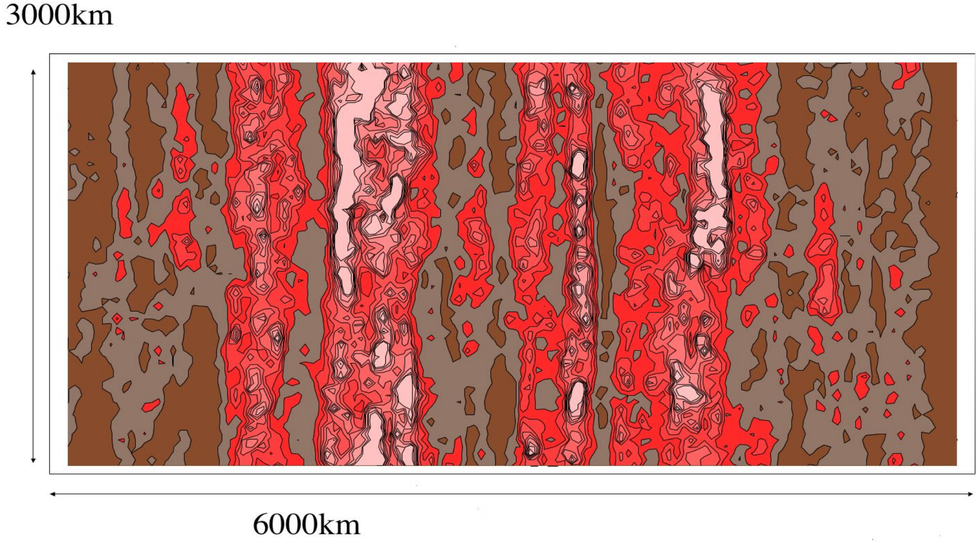
Vertical cross-sections $D_{el}=3$

Lithospheric rock density



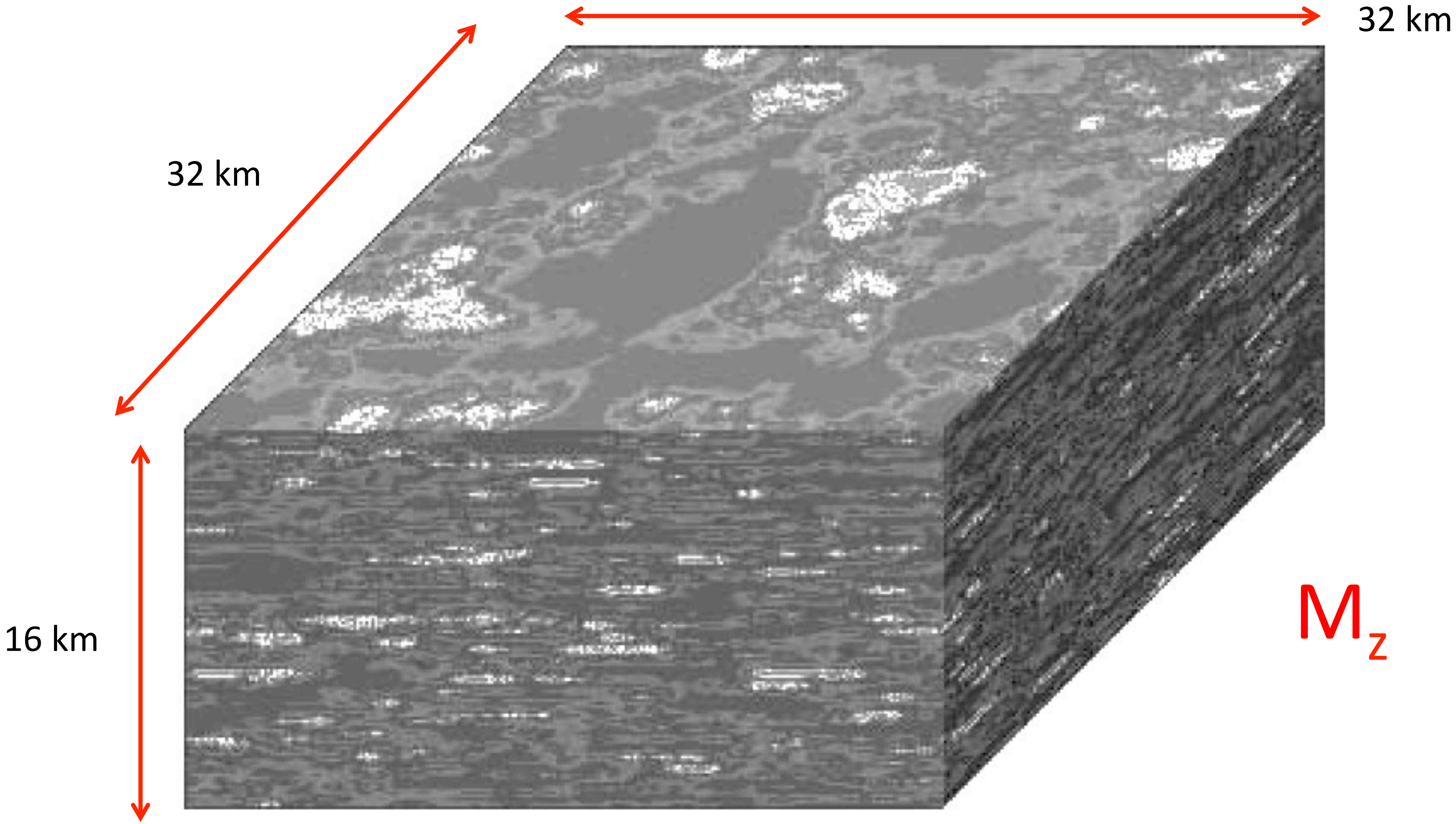
Sphero-scale l_s=256km, with 1 pixel = 1km.

Mantle density



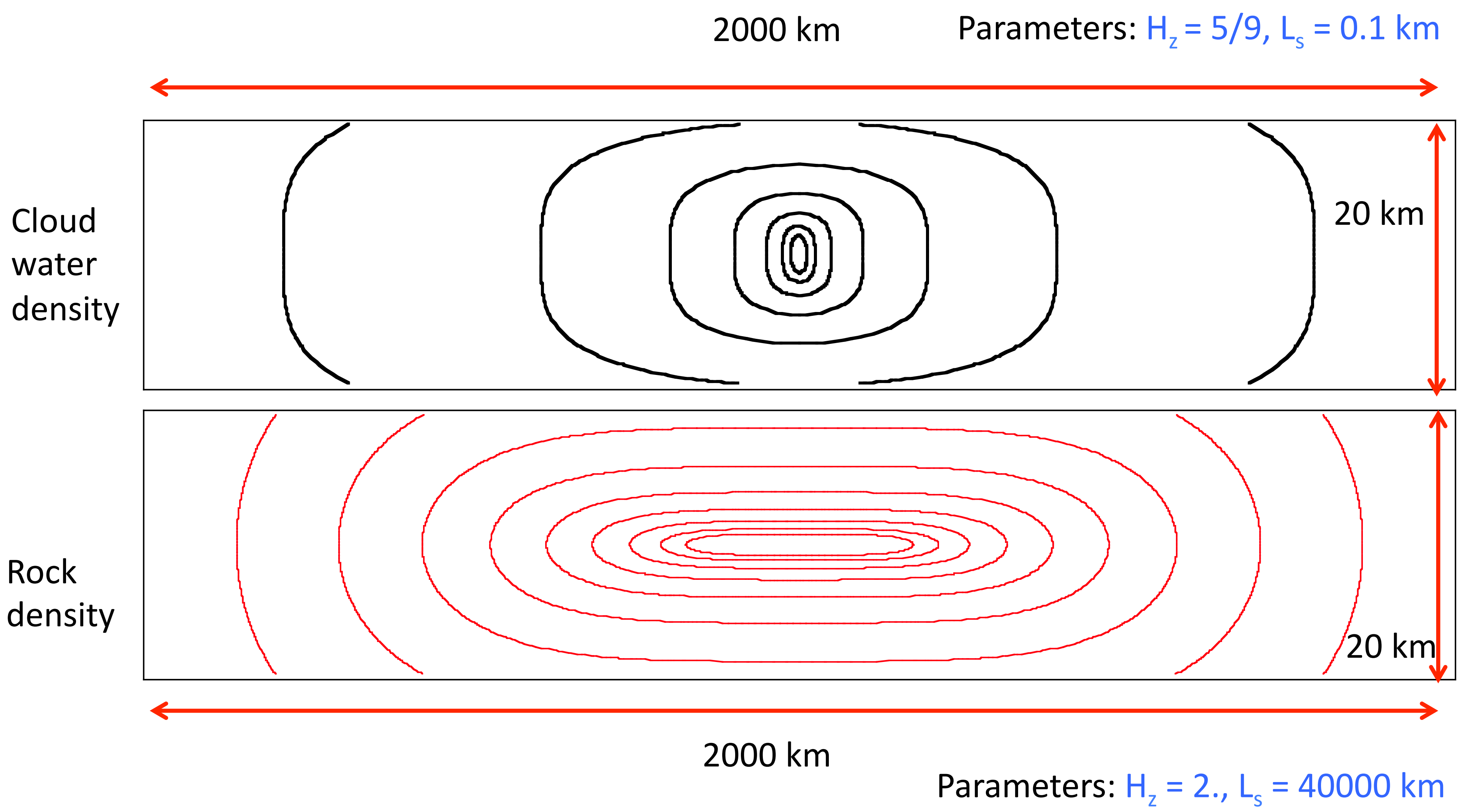
Sphero scale = 1 pixel. Each pixel is 50 km, sphero-scale = 25km. Hot (low density) plumes shown as white/red (this is a model for either density or temperature fluctuations (the two being proportional; we assume constant expansion coefficient). These are for fluctuations with respect to the mean vertical profile

Simulated magnetization field for horizontally isotropic crustal magnetization



Parameters: are H_z =1.7, s = 4, H = 0.2, α =1.98, C_1 = 0.08, l_s = 2500 km,

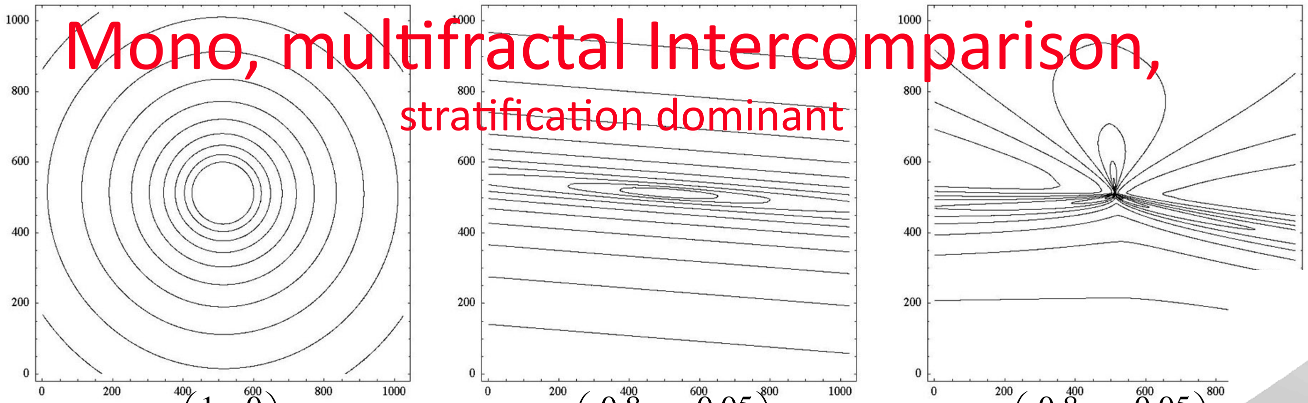
The unity of geosciences: clouds and rocks



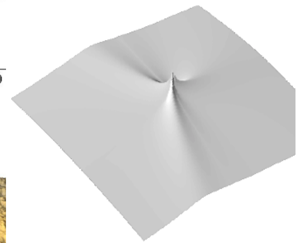
aspect ratios = 1/5

Mono, multifractal Intercomparison,
stratification dominant

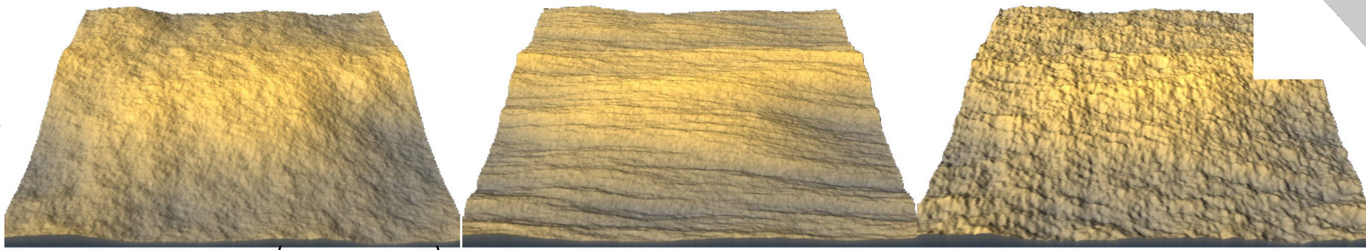
Contours of the scale functions



$$G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\qquad G=\begin{pmatrix}0.8 & -0.05\\ 0.05 & 1.2\end{pmatrix}\qquad G=\begin{pmatrix}0.8 & -0.05\\ 0.05 & 1.2\end{pmatrix}$$

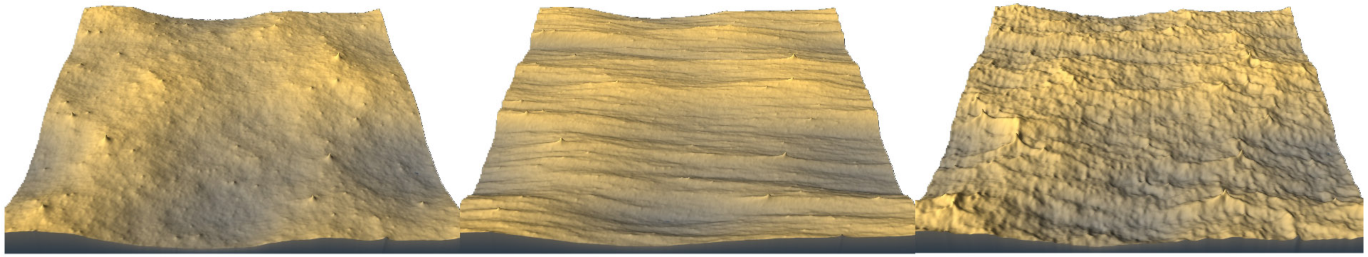


Fractional Brownian motion, H=0.7



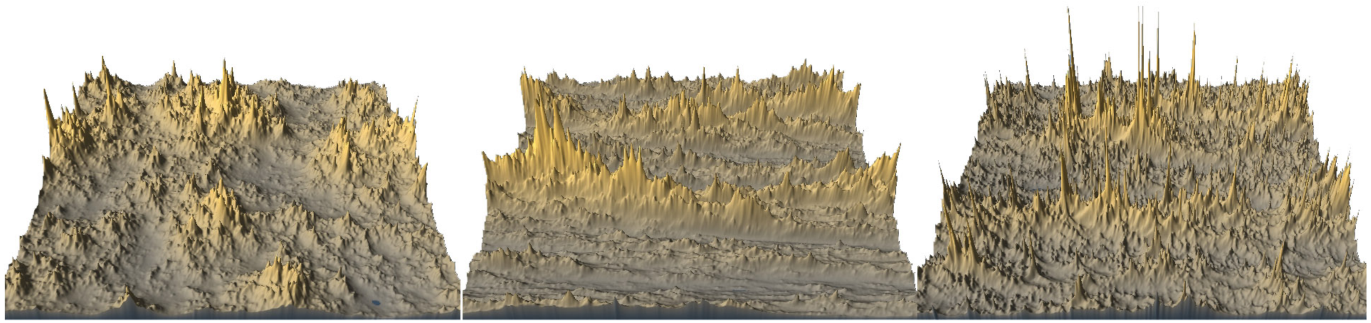
$$\left\langle \Delta h(\Delta r)^{q}\right\rangle \approx \Delta r^{qH-K(q)}\qquad K(q)=0$$

Fractional Levy motion, H=0.7, α =1.8



Multifractal FIF H=0.7, α =1.8, C_1=0.12

$$K(q)=\frac{C_1}{\alpha-1}\left(q^{\alpha}-q\right)$$

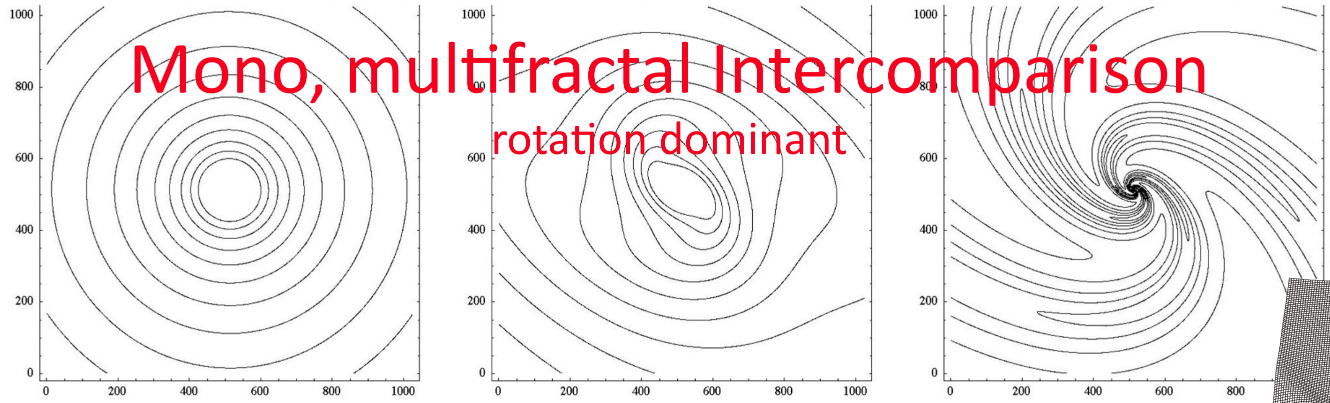


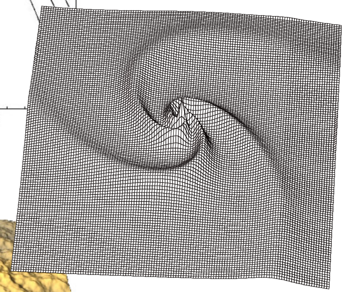
isotropic | Anisotropic no trivial anisotropy | Anisotropic with trivial anisotropy

Mono, multifractal Intercomparison

rotation dominant

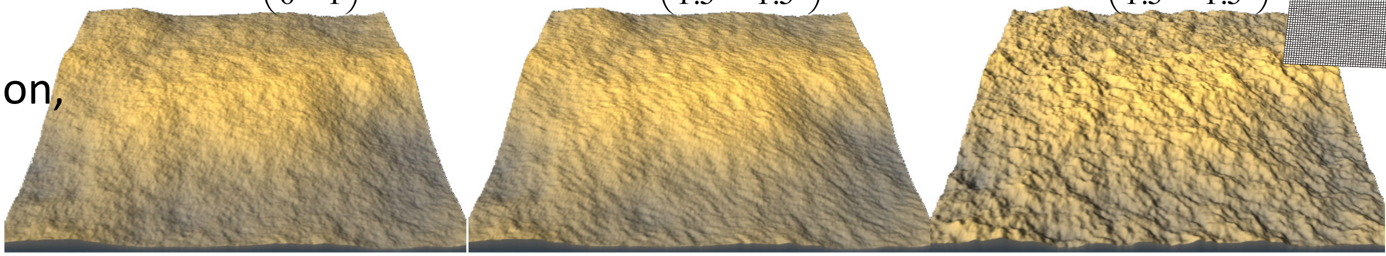
Contours of the scale functions



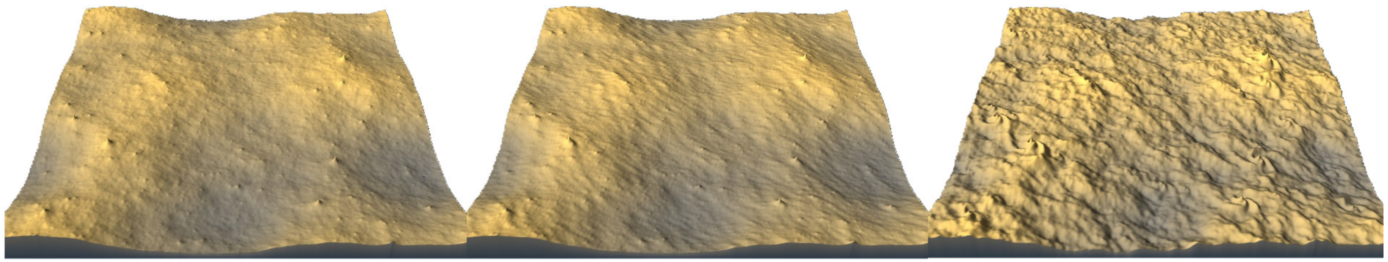


$$G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad G=\begin{pmatrix}0.5 & -1.5\\ 1.5 & 1.5\end{pmatrix} \qquad G=\begin{pmatrix}0.5 & -1.5\\ 1.5 & 1.5\end{pmatrix}$$

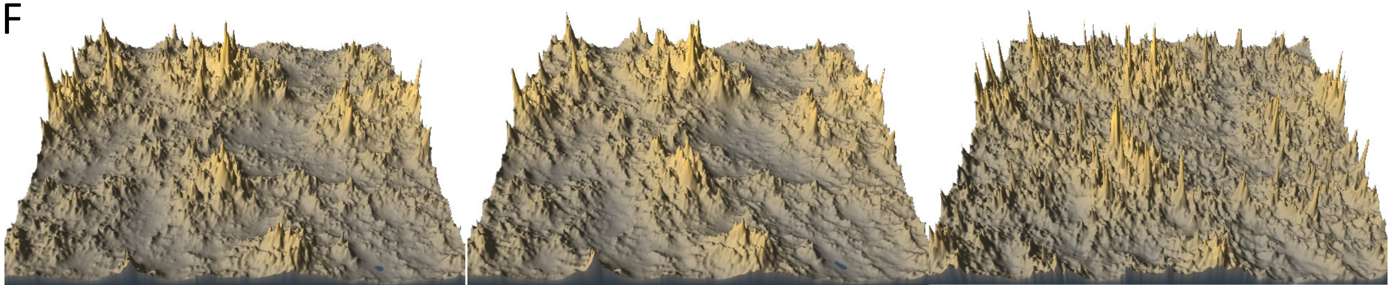
Fractional Brownian motion, H=0.7



Fractional Levy motion, H=0.7, α=1.8

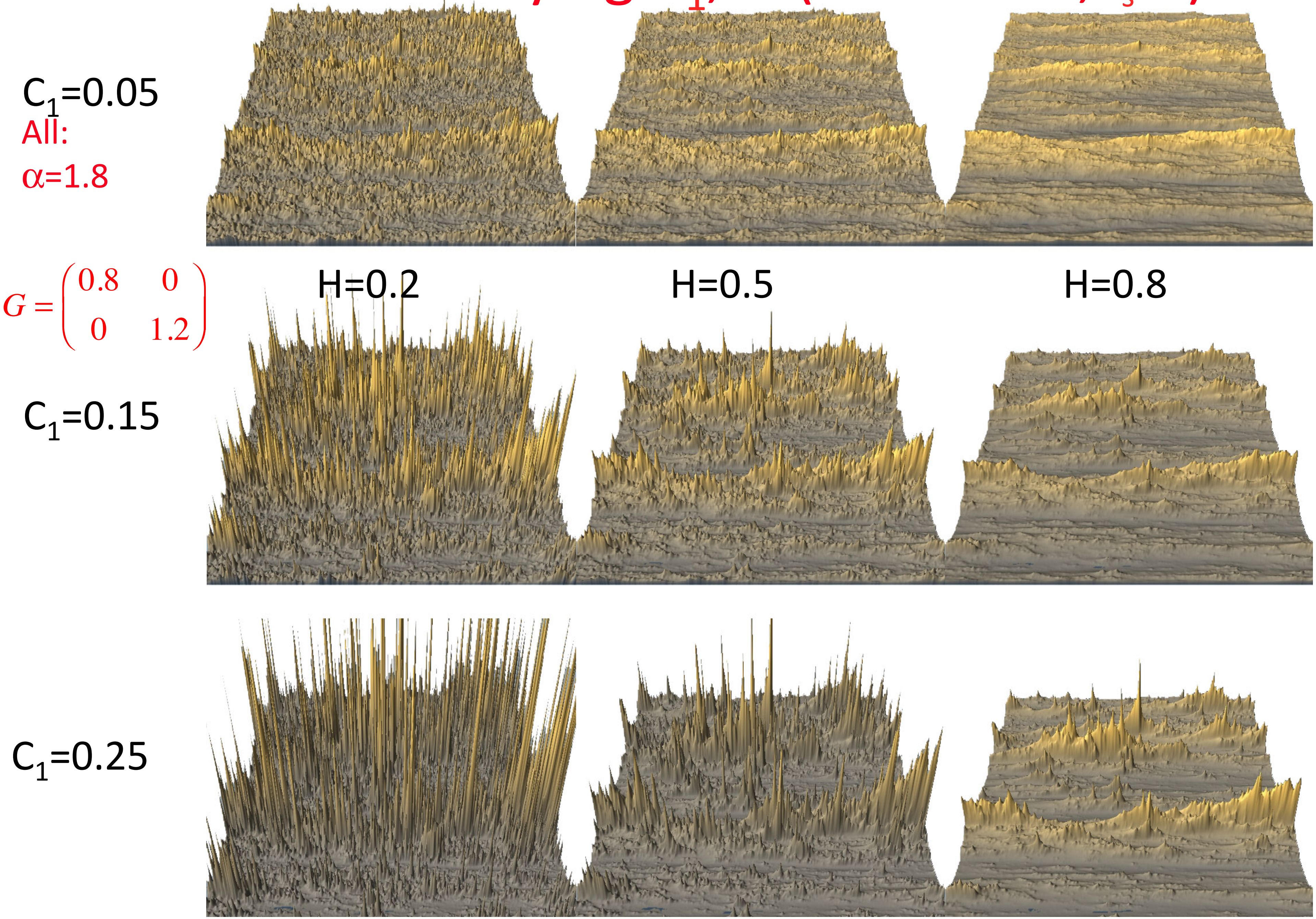


Multifractal, FIF H=0.7, α =1.8, C_1=0.12

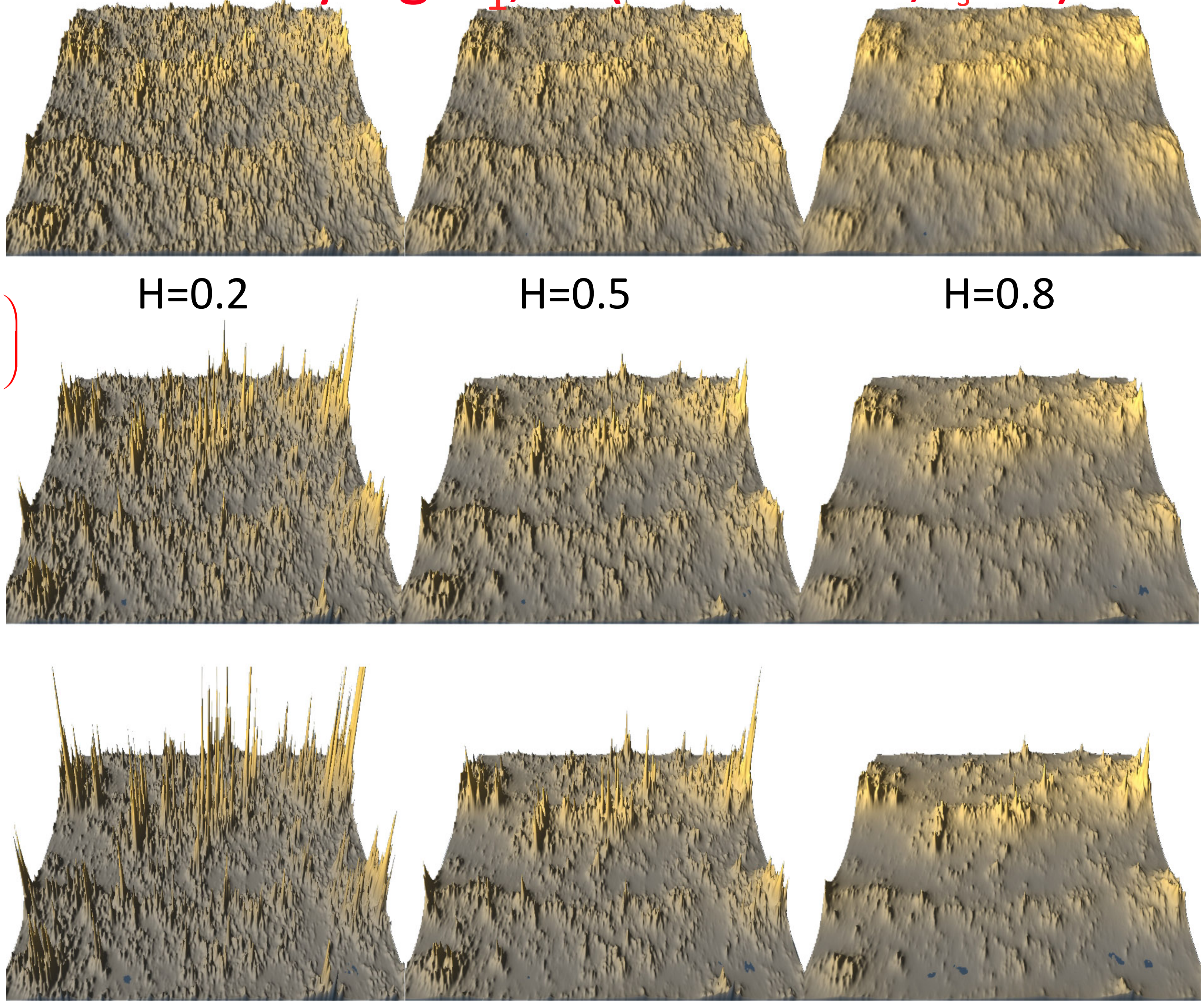


isotropic | Anisotropic no trivial anisotropy | Anisotropic with trivial anisotropy

Effect of varying C_1, H (self-affine, l_s=1)



Effect of varying C_1, H (self-affine, l_s=64)



C_1=0.05

All:
α=1.8

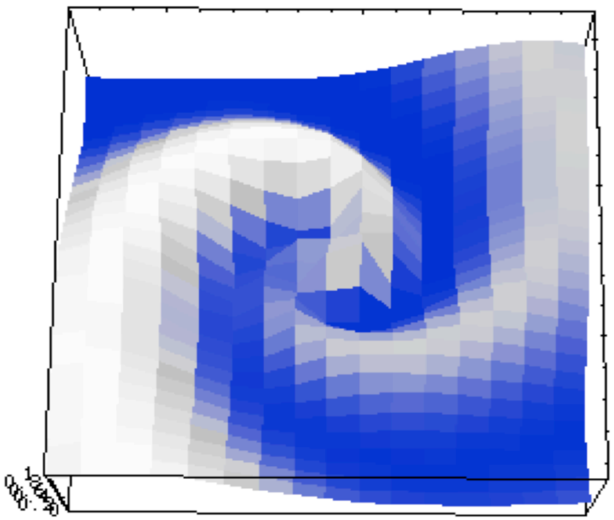
H=0.2 H=0.5 H=0.8

$$G = \begin{pmatrix} 0.8 & 0 \\ 0 & 1.2 \end{pmatrix}$$

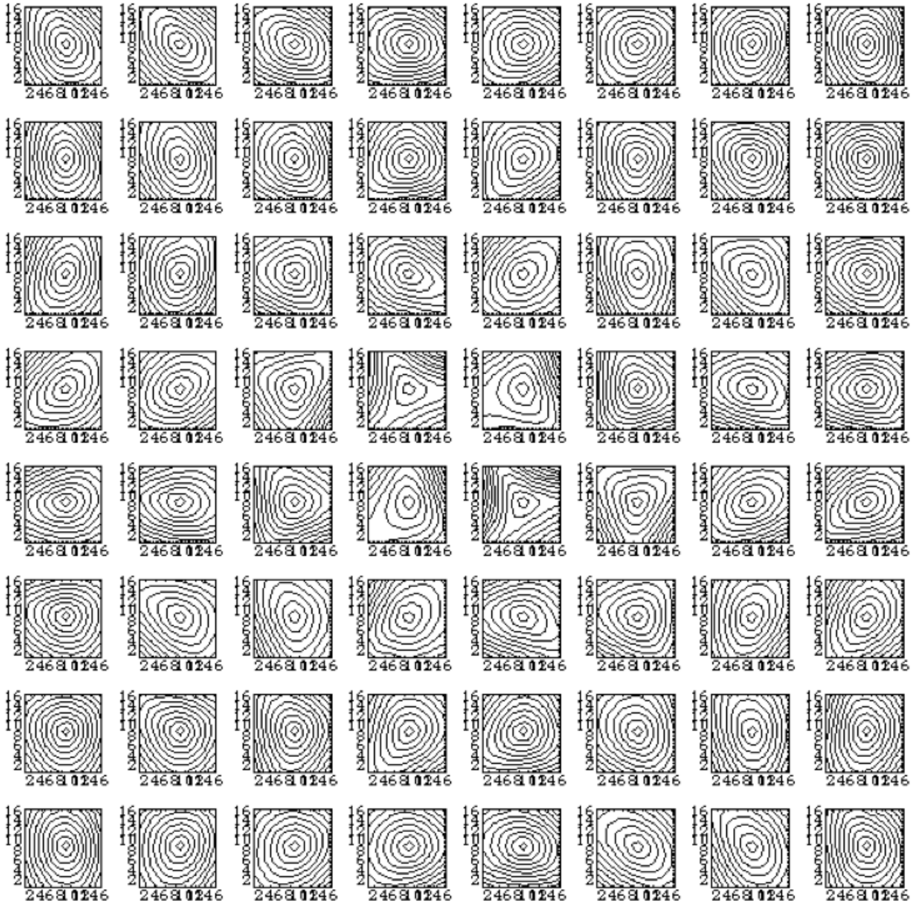
C_1=0.15

C_1=0.25

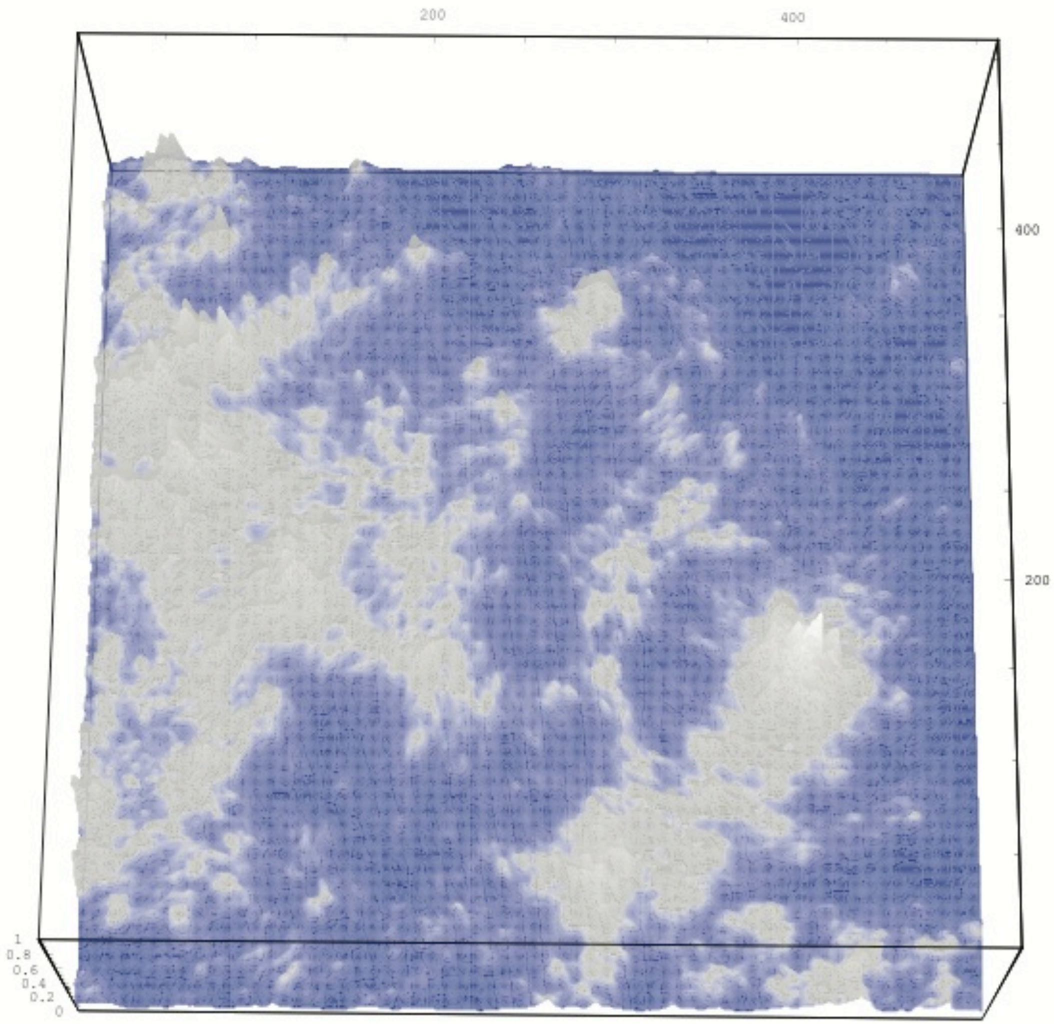
Examples of Nonlinear GSI



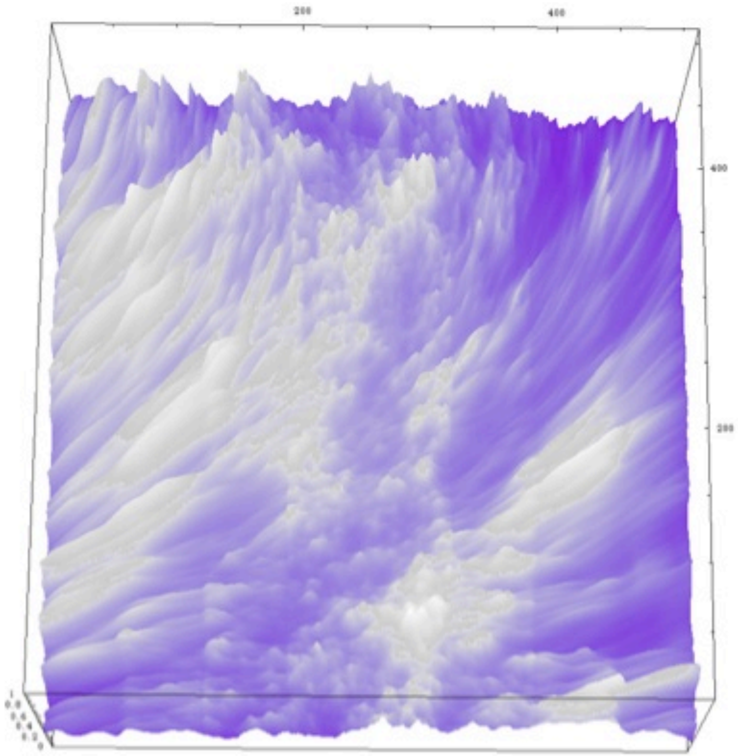
The (spiral shaped) scale scalar *h* function used to obtain *g*. The false colours indicate the relative values (as does the height).



The set of local scale functions displayed according to their relative positions obtained from the spiral shaped scalar h function at left using a combination of linear GSI with a quadratic transformation of variables to obtain functions accurate to cubic order in scale.

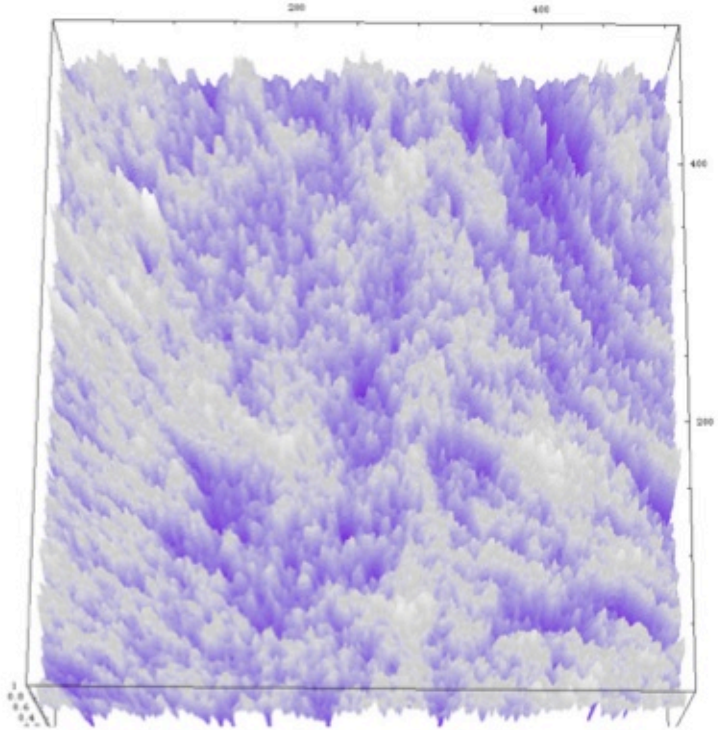


A nonlinear GSI multifractal simulation based on the spiral scale function indicated in the previous slide. The sphero-scale was held constant at 8 pixels, C_1 = 0.1, a =1.8, H=0. It can be seen that the spiral modulates the texture (determined primarily by the linear G approximation).

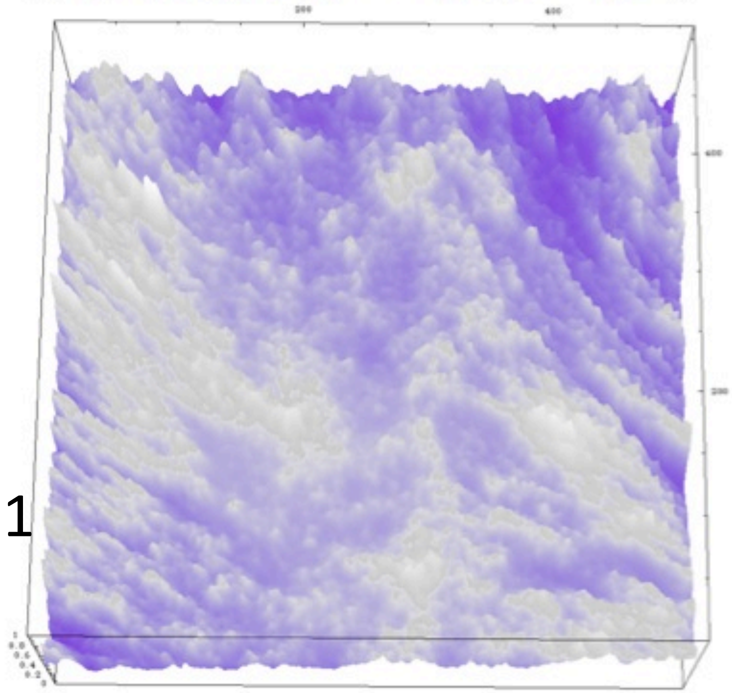


This shows a multifractal simulation of quadratic GSI (with *g* given by the cubic *h*, eq. 107) with a=1.8, C_1 = 0.1, *H* =0.33 and sphero-scale =256 pixels (the simulation is 512x512 pixels). The effect of the varying *G* is quite subtle.

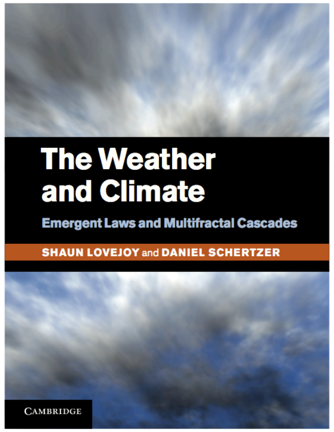
Same Left but for $l_s = 1$, $H = 0$.



Same but for $l_s =1$



Conclusions



1. Wide range scaling, multifractals in space, time and space time = unity of geophysics at the level of processes.

2. Cascades are the generic multifractal process.

3. Fractionally Integrated Flux (FIF) model for observables.

4. Universality classes make them manageable (H, C_1, α).

5. Wide ranges are possible due to anisotropic scaling: Generalized Scale Invariance:

Linear GSI: scale dependent anisotropy
Nonlinear GSI: scale and position dependent anisotropy

6. Phenomenological Fallacy.

Anisotropic singularities, Generalized Scale Invariance

Schertzer and Lovejoy 1987

Self-similar processses → Anisotropic scaling processes

noise → $\phi * \left|\underline{x}\right|^{-(D-H)}$ → $\phi * \left\|\underline{x}\right\|^{-(D_{el}-H)}$

Distance function (norm) → scale function

Scale function equation:

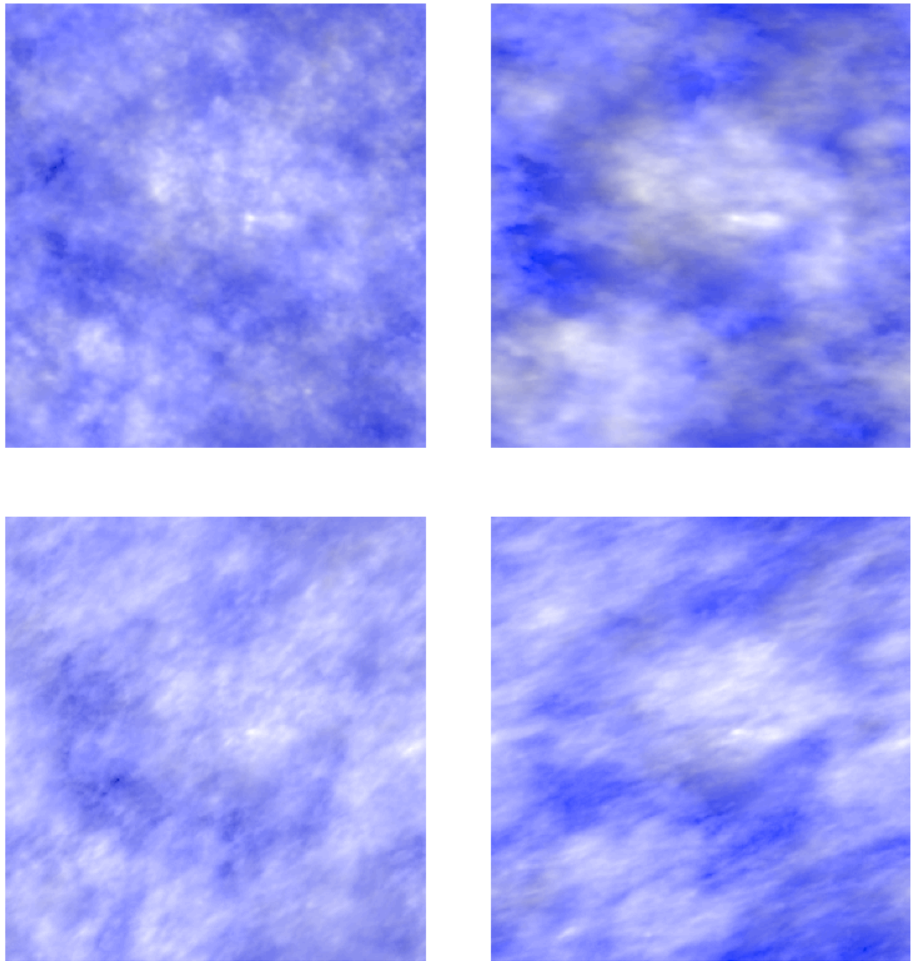
$$\left\|T_{\lambda}\underline{x}\right\| = \lambda^{-1}\left\|\underline{x}\right\|; \quad T_{\lambda} = \lambda^{-G}; \quad D_{el} = TraceG$$

Reduced scale vector

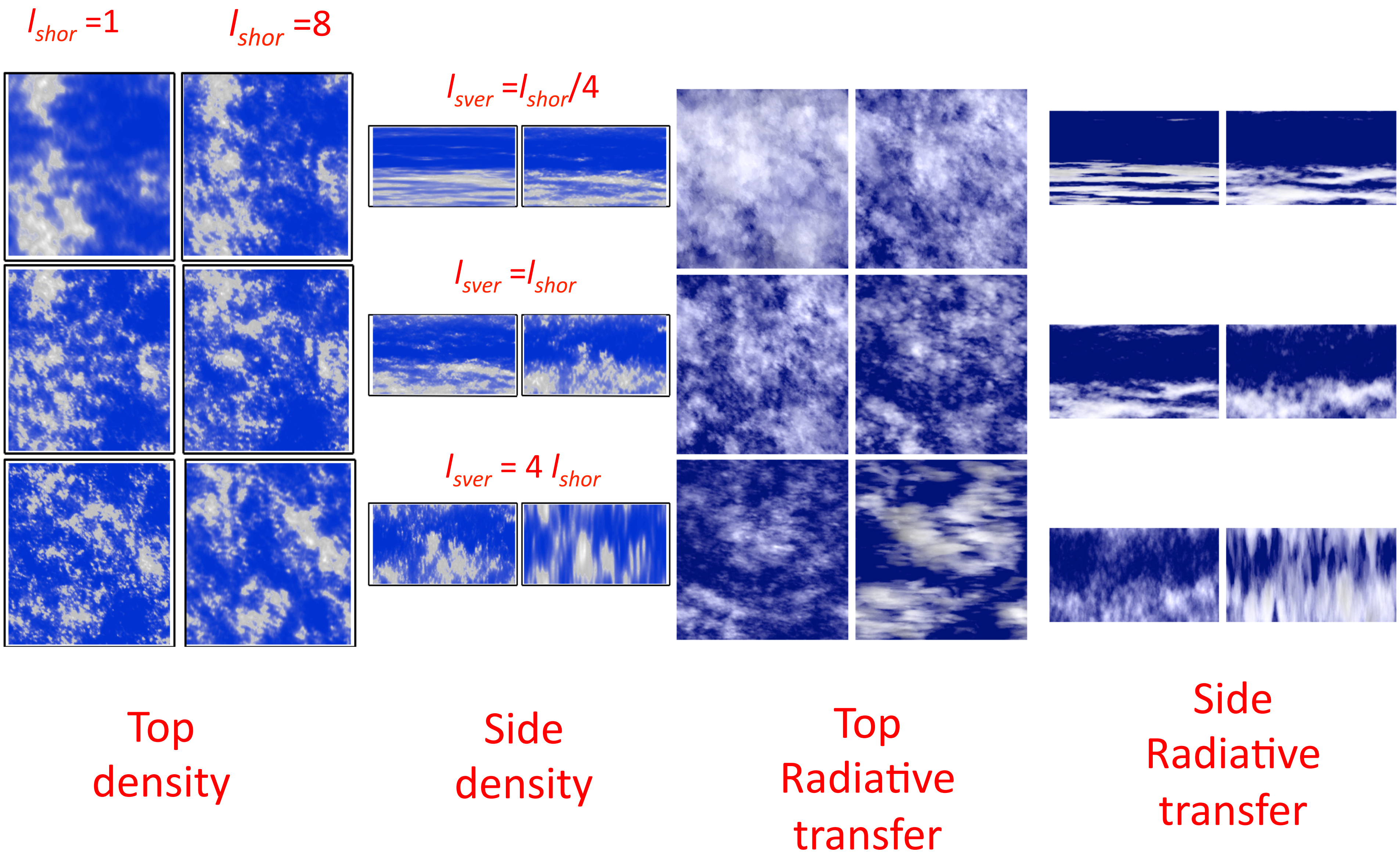
generator

Elliptical dimension

Same clouds Infra red emission, top view



The same as the previous except for a false colour rendition of a thermal infra red field (assuming a constant extinction coefficient and a linear vertical temperature profile).



$\alpha = 1.8$, $C_1 = 0.1$, $H = 0.333$, $d = 1$, $c = 0.5$, $e = 2$, $f = 0$, $H_z = 0.555$