Introduction to QCD and Jet I

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Overview of the Lectures

- Lecture 1 - Introduction to QCD and Jet
  - QCD basics
  - Sterman-Weinberg Jet in e^+e^- annihilation
  - Collinear Factorization and DGLAP equation
  - Basic ideas of k_t factorization
- Lecture 2 - k_t factorization and Dijet Processes in pA collisions
  - k_t Factorization and BFKL equation
  - Non-linear small-x evolution equations.
  - Dijet processes in pA collisions (RHIC and LHC related physics)
- Lecture 3 - k_t factorization and Higher Order Calculations in pA collisions
- No much specific exercise. 1. filling gaps of derivation; 2. Reading materials.

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Outline

1. Introduction to QCD and Jet
   - QCD Basics
   - Sterman-Weinberg Jets
   - Collinear Factorization and DGLAP equation
   - Transverse Momentum Dependent (TMD or k_t) Factorization

References:

- R.D. Field, Applications of perturbative QCD A lot of detailed examples.
- R. K. Ellis, W. J. Stirling and B. R. Webber, QCD and Collider Physics
- CTEQ, Handbook of Perturbative QCD
- CTEQ website.
- John Collins, The Foundation of Perturbative QCD Includes a lot new development.
- Yu. L. Dokshitzer, V. A. Khoze, A. H. Mueller and S. I. Troyan, Basics of Perturbative QCD More advanced discussion on the small-x physics.
- S. Donnachie, G. Dosch, P. Landshoff and O. Nachtmann, Pomeron Physics and QCD
- V. Barone and E. Predazzi, High-Energy Particle Diffraction

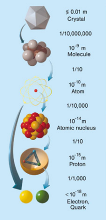
QCD

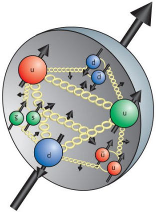
QCD Lagrangian

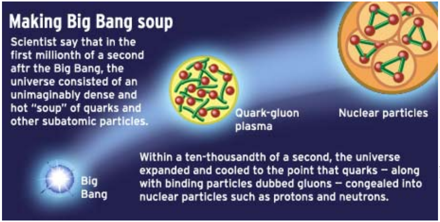
$$L = \bar{\psi}(i\gamma \cdot \partial - m_q)\psi - \frac{1}{4} F^{\mu\nu a} F_{\mu\nu a} - g_s \bar{\psi}\gamma \cdot A\psi$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$.

- Non-Abelian gauge field theory. Lagrangian is invariant under SU(3) gauge transformation.
- Basic elements:
  - Quark Ψ^i with 3 colors, 6 flavors and spin $1/2$.
  - Gluon $A^{a\mu}$ with 8 colors and spin 1.





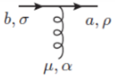


QCD Feynman Rules

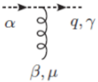
$$\left[-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2+i0}\right]\frac{i}{p^2+i0}$$

$$\frac{i(\not{p}+m_f)_{\rho\sigma}}{p^2-m_f^2+i0}$$

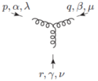
$$\frac{i}{p^2+i0}$$



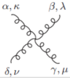
$$-ig\mu^\epsilon (t^\alpha)_{ab}\gamma^\mu_{\rho\sigma}$$



$$-g\mu^\epsilon f_{\alpha\beta\gamma} q^\mu$$



$$-g\mu^\epsilon f_{\alpha\beta\gamma}\left[(p-q)^\nu g^{\lambda\mu} + (q-r)^\lambda g^{\mu\nu} + (r-p)^\mu g^{\nu\lambda}\right]$$



$$\begin{aligned}
&-ig^2\mu^{2\epsilon} f_{\epsilon\alpha\beta} f_{\epsilon\gamma\delta}(g^{\kappa\mu}g^{\lambda\nu} - g^{\kappa\nu}g^{\lambda\mu}) \\
&-ig^2\mu^{2\epsilon} f_{\epsilon\alpha\gamma} f_{\epsilon\beta\delta}(g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\nu}g^{\lambda\mu}) \\
&-ig^2\mu^{2\epsilon} f_{\epsilon\alpha\delta} f_{\epsilon\beta\gamma}(g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\mu}g^{\lambda\nu})
\end{aligned}$$

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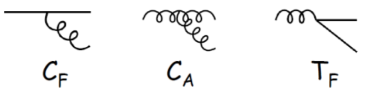
Color Structure

Fundamental representation: T^a_{ij} and Adjoint representation: $t^a_{bc} = -if_{abc}$

The effective color charge:

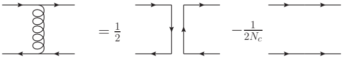
- $\left[T^a, T^b\right] = if^{abc}T^c$
- $\mathrm{Tr}\left(T^aT^b\right) = T_F\delta^{ab}$
- $T^aT^a = C_F \times 1$
- $f^{abc}f^{abd} = C_A\delta^{cd}$



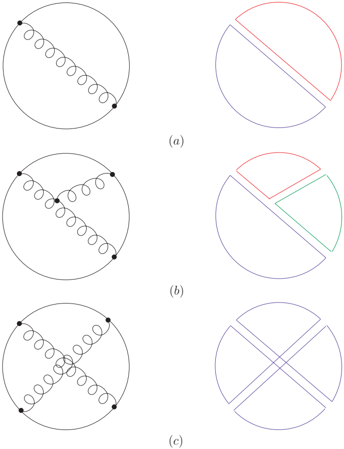
Symbol	SU(n)	SU(3)
T_F	$\frac{1}{2}$	$\frac{1}{2}$
C_F	$\frac{n^2-1}{2n}$	$\frac{4}{3}$
C_A	n	3

Fierz identity and Large N_c limit

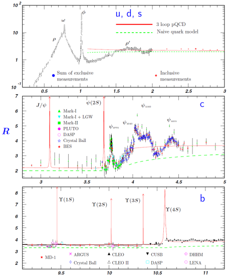
- Fierz identity: $T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta_{kl}$



- Large N_c limit: $3 \gg 1$



Evidence for colors



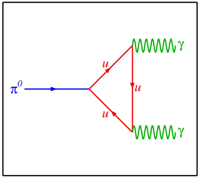
- The ratio between the $e^+e^- \to$ hadrons total cross section and the $e^+e^- \to \mu^+\mu^-$ cross section.

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{u,d,s,\ldots} e_i^2 \left[1 + \frac{\alpha_s(Q^2)}{\pi}\right]$$

  - $N_c \sum_{u,d,s} e_i^2 = 2$
  - $N_c \sum_{u,d,s,c} e_i^2 = \frac{10}{3}$
  - $N_c \sum_{u,d,s,c,b} e_i^2 = \frac{11}{3}$.

Triangle anomaly:

π^0 Decay



- The decay rate is given by the quark triangle loop:

$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = N_c^2 \left(e_u^2 - e_d^2\right)^2 \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} = 7.7\text{eV}$$

  - $f_\pi = 92.4\text{MeV}$ is $\pi^- \to \mu^-\nu$ decay constant.
  - The data give $\Gamma\left(\pi^0 \to \gamma\gamma\right) = 7.7 \pm 0.6\text{eV}$.
  - Nonrenormalization of the anomaly. [Adler, Bardeen, 69]

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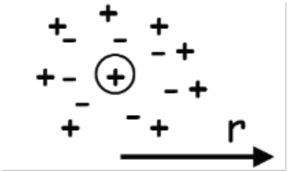
QCD beta function and running coupling

[Gross, Wilczek and Politzer, 73]

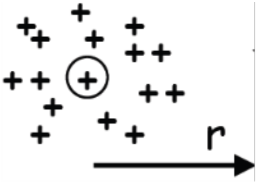
- The QCD running coupling

$$\alpha_s(Q) = \frac{2\pi}{\left(\frac{11}{6}N_c - \frac{2}{3}T_F n_f\right) \ln Q^2/\Lambda^2}$$

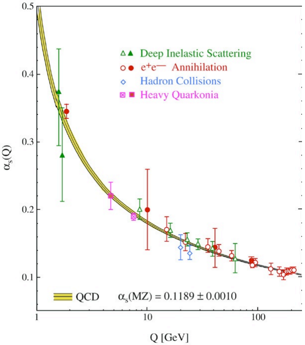
- QED has only fermion loop contributions, thus its coupling runs in opposite direction.



QED like contribution



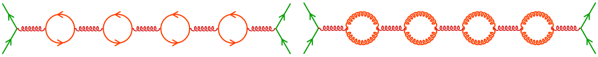
gluon contribution



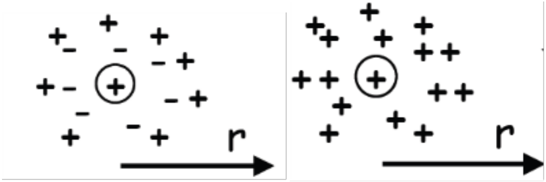
QCD beta function and running coupling

The QCD running coupling

$$\alpha_s(Q) = \frac{2\pi}{\left(\frac{11}{6}N_c - \frac{2}{3}T_F n_f\right)\ln Q^2/\Lambda^2}$$



Screening   Anti-Screening



Quark loop QED like contribution   Non-Abelian gluon contribution

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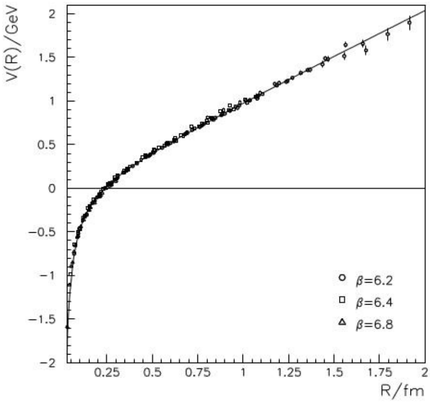
Brief History of QCD beta function

- 1954 Yang and Mills introduced the non-Abelian gauge thoery.
- 1965 Vanyashin and Terentyev calculated the beta function for a massive charged vector field theory.
- 1971 ’t Hooft computed the one-loop beta function for SU(3) gauge theory, but his advisor (Veltman) told him it wasn’t interesting.
- 1972 Gell-Mann proposed that strong interaction is described by SU(3) gauge theory, namely QCD.
- 1973 Gross and Wilczek, and independently Politzer, computed the 1-loop beta-function for QCD.
- 1999 ’t Hooft and Veltman received the 1999 Nobel Prize for proving the renormalizability of QCD.
- 2004 Gross, Wilczek and Politzer received the Nobel Prize.

Confinement



- Non-perturbative QCD
- Linear potential ⇒ constant force.
- Intuitively, confinement is due to the force-carrying gluons having color charge, as compared to photon which does not carry electric charge.
- Color singlet hadrons : no free quarks and gluons in nature

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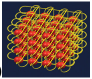
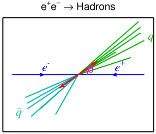
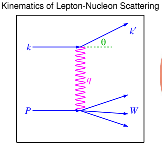
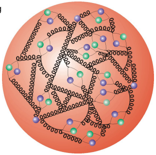
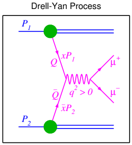
How to test QCD ?

- Non-perturbative part:
  - Hadron mass (Lattice QCD) 
  - Parton distributions (No free partons in the initial state)
  - Fragmentation function (No free quarks and gluons in the final state)
- Perturbative QCD: needs to have Factorization to separate the short distances (perturbative) physics from the long distance (non perturbative) physics.
  - e^+e^- annihilation.
  - Deep inelastic scattering.
  - Hadron-hadron collisions, such as Drell-Yan processes.





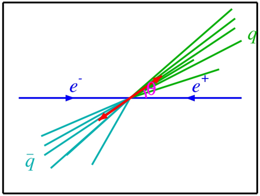


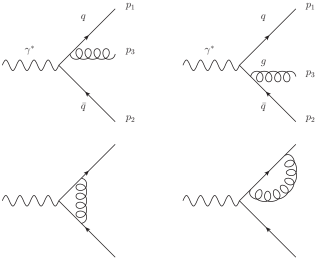


- Collinear factorization demonstrates that collinear parton distribution and fragmentation function are universal.
- k_t factorization is more complicated.

e^+e^- annihilation

$e^+e^- \rightarrow$ Hadrons





- Born diagram () gives $\sigma_0 = \frac{4\pi}{3}\frac{\alpha_{em}^2}{q^2} N_c \sum_q e_q^2$
- NLO: real contribution (3 body final state)

$$\frac{d\sigma_3}{dx_1 dx_2} = C_F \frac{\alpha_s}{2\pi}\sigma_0 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

$$\text{with} \quad \frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left[\frac{1}{(1-x_1)} + \frac{1}{(1-x_2)}\right]$$

- Energy conservation $\Rightarrow x_1 + x_2 + x_3 = 2$.
- $(p_1 + p_3)^2 = 2p_1 \cdot p_3 = (Q - p_2)^2 = Q^2(1 - x_2)$
- $x_2 \rightarrow 1 \Rightarrow \vec{p}_3 \,||\, \vec{p}_1 \Rightarrow$ Collinear Divergence (Similarly $x_1 \rightarrow 1$)
- $x_3 \rightarrow 0 \Rightarrow$ Soft Divergence.

Dimensional Regularization

To generate a finite contribution to the total cross section, use the standard procedure dimensional regularization:

- Analytically continue in the number of dimensions from $d = 4$ to $d = 4 - 2\epsilon$.
- Convert the soft and collinear divergence into poles in ϵ.
- To keep g_s dimensionless, substitue $g_s \to g_s \mu^\epsilon$ with renormalization scale μ.

At the end of the day, one finds

$$\begin{aligned}
\sigma_r &= \sigma_0 \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right] \\
\sigma_v &= \sigma_0 \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]
\end{aligned}$$

and the sum $\sigma = \sigma_0 \left(1 + \frac{\alpha_s(\mu)}{\pi}\right)$.

- Cancellation between real and virtual for total cross section. Bloch-Nordsieck theorem
- For more exclusive observables, the cancellation is not always complete. One needs to do subtractions of $\frac{1}{\epsilon} + \ln 4\pi - \gamma_E$ ($\overline{\text{MS}}$ scheme).
- Sterman-Weinberg Jets.

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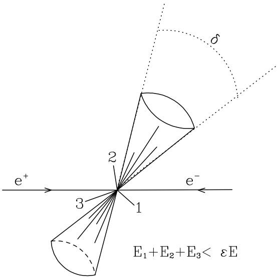
Sterman-Weinberg Jets

Definition: We define, an event contributes if we can find two cones of opening angle δ that contain all of the energy of the event, excluding at most a fraction ϵ of the total, as the production of a pair of Sterman Weinberg jets.



- Jets in experiments are defined as a collimated distribution of hadrons with total energy E within the jet cone size $R \equiv \sqrt{\delta\phi^2 + \delta\eta^2}$.
- Jets in QCD theory are defined as a collimated distribution of partons. Need to assume the parton-hadron duality.
- Jet finding algorithm: (k_t, cone and anti-k_t)See other lecture. [M. Cacciari, G. P. Salam and G. Soyez, 08]

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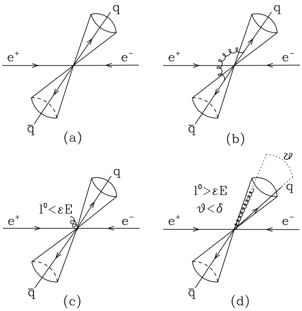
$e^+e^- \to \gamma^* \to$ jets



- a. The Born contribution: σ_0
- b. The virtual contribution: $-\sigma_0 C_F \frac{\alpha_s}{2\pi} \int_0^E \frac{dl}{l} \int_0^\pi \frac{4 d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$
- c. The soft real contribution: $\sigma_0 C_F \frac{\alpha_s}{2\pi} \int_0^{\epsilon E} \frac{dl}{l} \int_0^\pi \frac{4 d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$
- d. The hard real contribution: $\sigma_0 C_F \frac{\alpha_s}{2\pi} \int_{\epsilon E}^E \frac{dl}{l} \left[\int_0^\delta + \int_{\pi-\delta}^\pi\right] \frac{4 d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$
- sum $= \sigma_0 \left[1 - C_F \frac{\alpha_s}{2\pi} \int_{\epsilon E}^E \frac{dl}{l} \int_\delta^{\pi-\delta} \frac{4 d\cos\theta}{1-\cos^2\theta}\right] = \sigma_0 \left[1 - \frac{4 C_F \alpha_s}{\pi} \ln\epsilon \ln\delta\right]$

Infrared Safety

- We have encountered two kinds of divergences: collinear divergence and soft divergence.
- Both of them are of the Infrared divergence type.That is to say, they both involve long distance.
  - According to uncertainty principle, soft $\leftrightarrow$ long distance;
  - Also one needs an infinite time in order to specify accurately the particle momenta, and therefore their directions.
- For a suitable defined inclusive observable (e.g., $\sigma_{e^+e^-\to\text{hadrons}}$), there is a cancellation between the soft and collinear singularities occurring in the real and virtual contributions. Kinoshita-Lee-Nauenberg theorem
- Any new observables must have a definition which does not distinguish between

$$\text{parton} \leftrightarrow \text{parton + soft gluon}$$
$$\text{parton} \leftrightarrow \text{two collinear partons}$$

- Observables that respect the above constraint are called infrared safe observables. Infrared safety is a requirement that the observable is calculable in pQCD.
- Other infrared safe observables, for example, Thrust: $T = \max \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$ ...

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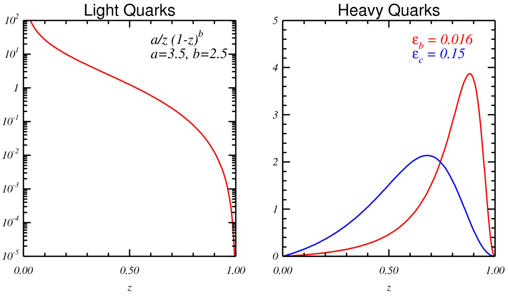
Fragmentation function

Factorization of single inclusive hadron production:

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma(e^+e^- \to h+X)}{\mathrm{d}x} = \sum_i \int_x^1 C_i\left(z, \alpha_s(\mu^2), s/\mu^2\right) D_{h/i}(x/z, \mu^2) + \mathcal{O}(1/s)$$



- $D_{h/i}(x/z, \mu^2)$ encodes the probability that the parton i fragments into a hadron h carrying a fraction z of the parton's momentum.
- Energy conservation $\Rightarrow$ $\sum_h \int_0^1 dz\, z\, D_i^h(z,\mu^2) = 1\,.$
- Heavy quark fragmentation function: $f(z) \propto \frac{1}{z\left(1-\frac{1}{z}-\frac{\epsilon_Q}{1-z}\right)^2}\,,$.

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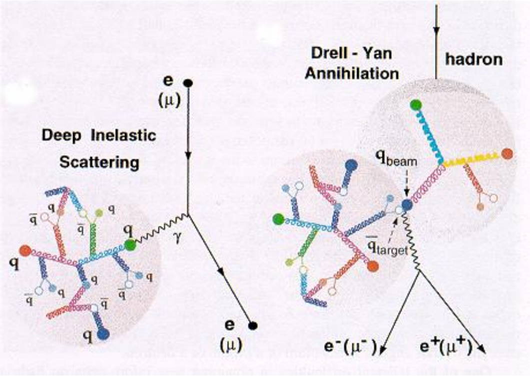
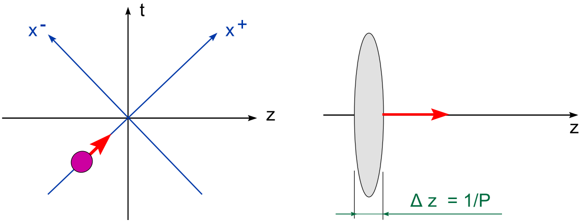
Deep inelastic scattering and Drell-Yan process



Light Cone coordinates and gauge

For a relativistic hadron moving in the $+z$ direction



- In this frame, the momenta are defined

$$P^+ = \frac{1}{\sqrt{2}}(P^0 + P^3) \quad \text{and} \quad P^- = \frac{1}{\sqrt{2}}(P^0 - P^3) \to 0$$

- $P^2 = 2P^+P^- - P_\perp^2$
- Light cone gauge for a gluon with momentum $k^\mu = (k^+, k^-, k_\perp)$, the polarization vector reads

$$k^\mu \epsilon_\mu = 0 \Rightarrow \quad \epsilon = (\epsilon^+ = 0, \epsilon^- = \frac{\epsilon_\perp \cdot k_\perp}{k^+}, \epsilon_\perp^\pm) \quad \text{with} \quad \epsilon_\perp^\pm = \frac{1}{\sqrt{2}}(1, \pm i)$$

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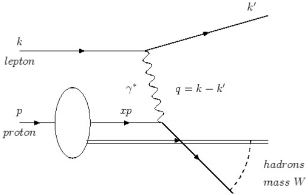
Deep inelastic scattering

Summary of DIS:



$$\frac{d\sigma}{dE'd\Omega} = \frac{\alpha_{\mathrm{em}^2}}{Q^4}\frac{E'}{E}L_{\mu\nu}W^{\mu\nu}$$

with $L_{\mu\nu}$ the leptonic tensor and $W^{\mu\nu}$ defined as

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)W_1 + \frac{1}{m_p^2}\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)W_2$$

Introduce the dimensionless structure function:

$$F_1 \equiv W_1 \quad \text{and} \quad F_2 \equiv \frac{Q^2}{2m_p x}W_2$$

$$\Rightarrow \frac{d\sigma}{dxdy} = \frac{\alpha_{4\pi \mathrm{sem}^2}}{Q^4}\left[(1-y)F_2 + xy^2F_1\right] \quad \text{with} \quad y = \frac{P\cdot q}{P\cdot k}.$$

Quark Parton Model: Callan-Gross relation

$$F_2(x) = 2xF_1(x) = \sum_q e_q^2 x\left[f_q(x) + f_{\bar{q}}(x)\right].$$

Callan-Gross relation

Callan - Gross Relation

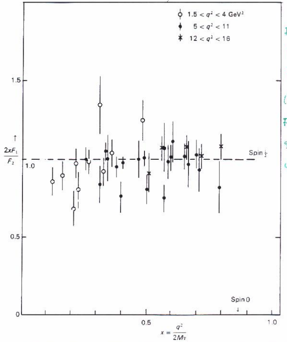
$$\left(\frac{d\sigma}{dq^2}\right)_{\text{Dirac}} = \frac{4\pi\alpha^2 z^2}{q^4}\left(\frac{E'}{E}\right)^2\left(\cos^2\frac{\theta}{2} + \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)$$

$$\left(\frac{d^2\sigma}{dq^2\,dx}\right)_{\text{inelastic}} = \frac{4\pi\alpha^2}{q^4}\frac{E'}{E}\left(F_2(x)\cos^2\frac{\theta}{2} + \frac{q^2}{2M^2x^2}2xF_1(x)\sin^2\frac{\theta}{2}\right)\frac{1}{x}$$

Electric — Magnetic

$$\frac{2xF_1(x)}{F_2(x)} = 1$$

For spin $\frac{1}{2}$ partons



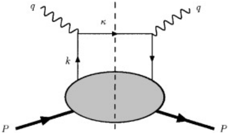
For spin 0 partons: $F_1 = 0$ (Remember: F_1 describes spin-spin interactions)

Figure 8.10 The ratio $2xF_1/F_2$ measured in SLAC electron-nucleon scattering experiments. For spin-$\frac{1}{2}$ partons, with $g = 2$, a ratio of unity is expected in the limit of large q^2—the Callan-Gross relation. (Data compiled from published SLAC data.)

- The relation ($F_L = F_2 - 2xF_1$) follows from the fact that a spin-$\frac{1}{2}$ quark cannot absorb a longitudinally polarized vector boson.
- In contrast, spin-0 quark cannot absorb transverse bosons and so would give $F_1 = 0$.

Parton Density

The probabilistic interpretation of the parton density.



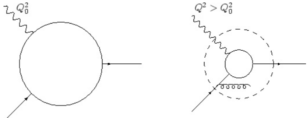
$$\Rightarrow \quad f_q(x) = \int \frac{\mathrm{d}\zeta^-}{4\pi} e^{ixP^+\zeta^-} \langle P \left| \bar{\psi}(0)\gamma^+\psi(0,\zeta^-) \right| P \rangle$$

Comments:

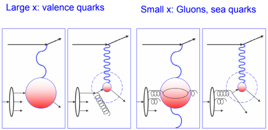
- Gauge link $\mathcal{L}$ is necessary to make the parton density gauge invariant.

$$\mathcal{L}(0,\zeta^-) = \mathcal{P}\exp\left(\int_0^{\zeta^-} \mathrm{d}s_\mu A^\mu\right)$$

- Choose light cone gauge $A^+ = 0$ and right path, one can eliminate the gauge link.
- Now we can interpret $f_q(x)$ as parton density in the light cone frame.
- Evolution of parton density: Change of resolution



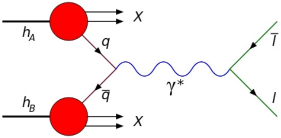
a) b)



- At low-x, dominant channels are different.

Drell-Yan process

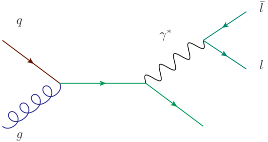
For lepton pair productions in hadron-hadron collisions:



the cross section is

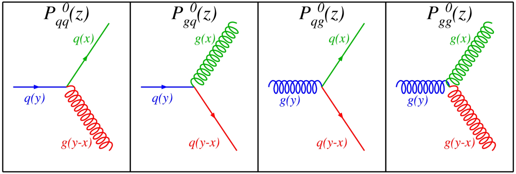
$$\frac{d\sigma}{dM^2 dY} = \sum_q x_1 f_q(x_1) x_2 f_{\bar{q}}(x_2) \frac{1}{3} e_q^2 \frac{4\pi\alpha^2}{3M^4} \quad \text{with} \quad Y = \frac{1}{2} \ln \frac{x_1}{x_2}.$$

- Collinear factorization proof shows that $f_q(x)$ involved in DIS and Drell-Yan process are the same.
- At low-x and high energy, the dominant channel is $qg \to q\gamma^*(l^+ l^-)$.



PENNSTATE

Splitting function



$$\mathcal{P}^0_{qq}(\xi) = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi),$$

$$\mathcal{P}^0_{gq}(\xi) = \frac{1}{\xi}\left[1+(1-\xi)^2\right],$$

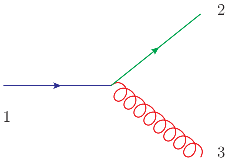
$$\mathcal{P}^0_{qg}(\xi) = \left[(1-\xi)^2+\xi^2\right],$$

$$\mathcal{P}^0_{gg}(\xi) = 2\left[\frac{\xi}{(1-\xi)_+} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right] + \left(\frac{11}{6} - \frac{2N_f T_R}{3N_c}\right)\delta(1-\xi).$$

- $\xi = z = \frac{x}{y}$.
- $\int_0^1 \frac{d\xi f(\xi)}{(1-\xi)_+} = \int_0^1 \frac{d\xi [f(\xi)-f(1)]}{1-\xi} \Rightarrow \int_0^1 \frac{d\xi}{(1-\xi)_+} = 0$

Derivation of $\mathcal{P}^0_{qq}(\xi)$

The real contribution:

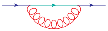


$$k_1 = (P^+, 0, 0_\perp) \quad ; \quad k_2 = (\xi P^+, \frac{k_\perp^2}{\xi P^+}, k_\perp)$$

$$k_3 = ((1-\xi)P^+, \frac{k_\perp^2}{(1-\xi)P^+}, -k_\perp) \quad \epsilon_3 = (0, -\frac{2k_\perp \cdot \epsilon_\perp^{(3)}}{(1-\xi)P^+}, \epsilon_\perp^{(3)})$$

$$|V_{q\to qg}|^2 = \frac{1}{2}\mathrm{Tr}\left(\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\right)\sum \epsilon_3^{*\mu}\epsilon_3^\nu = \frac{2k_\perp^2}{\xi(1-\xi)}\frac{1+\xi^2}{1-\xi}$$

$$\Rightarrow \qquad \mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{1-\xi} \quad (\xi < 1)$$

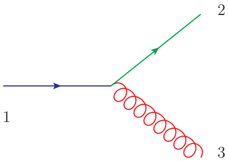
- Including the virtual graph  , use $\int_a^1 \frac{d\xi g(\xi)}{(1-\xi)_+} = \int_a^1 \frac{d\xi g(\xi)}{1-\xi} - g(1)\int_0^1 \frac{d\xi}{1-\xi}$

$$\frac{\alpha_s C_F}{2\pi}\left[\int_x^1 \frac{d\xi}{\xi} q(x/\xi)\frac{1+\xi^2}{1-\xi} - q(x)\int_0^1 d\xi \frac{1+\xi^2}{1-\xi}\right]$$

$$= \frac{\alpha_s C_F}{2\pi}\left[\int_x^1 \frac{d\xi}{\xi} q(x/\xi)\frac{1+\xi^2}{(1-\xi)_+} - q(x)\underbrace{\int_0^1 d\xi \frac{1+\xi^2}{(1-\xi)_+}}_{=-\frac{3}{2}}\right].$$

Derivation of $\mathcal{P}^0_{qq}(\xi)$

The real contribution:



$$k_1 = (P^+, 0, 0_\perp) \quad ; \quad k_2 = (\xi P^+, \frac{k_\perp^2}{\xi P^+}, k_\perp)$$

$$k_3 = ((1-\xi)P^+, \frac{k_\perp^2}{(1-\xi)P^+}, -k_\perp) \quad \epsilon_3 = (0, -\frac{2k_\perp \cdot \epsilon_\perp^{(3)}}{(1-\xi)P^+}, \epsilon_\perp^{(3)})$$

$$|V_{q\to qg}|^2 = \frac{1}{2}\mathrm{Tr}\left(\not{k}_2\gamma_\mu \not{k}_1\gamma_\nu\right)\sum \epsilon_3^{*\mu}\epsilon_3^\nu = \frac{2k_\perp^2}{\xi(1-\xi)}\frac{1+\xi^2}{1-\xi}$$

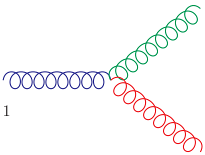
$$\Rightarrow \qquad \mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{1-\xi} \quad (\xi < 1)$$

- Regularize $\frac{1}{1-\xi}$ to $\frac{1}{(1-\xi)_+}$ by including the divergence from the virtual graph.
- Probability conservation:

$$P_{qq} + \mathrm{d}P_{qq} = \delta(1-\xi) + \frac{\alpha_s C_F}{2\pi}\mathcal{P}^0_{qq}(\xi)dt \quad \text{and} \quad \int_0^1 \mathrm{d}\xi \mathcal{P}_{qq}(\xi) = 0,$$

$$\Rightarrow \mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi) = \left(\frac{1+\xi^2}{1-\xi}\right)_+ .$$

Derivation of $\mathcal{P}^0_{gg}(\xi)$



$$k_1 = (P^+, 0, 0_\perp) \quad \epsilon_1 = (0, 0, \epsilon_\perp^{(1)}) \quad \text{with} \quad \epsilon_\perp^\pm = \frac{1}{\sqrt{2}}(1, \pm i)$$

$$k_2 = (\xi P^+, \frac{k_\perp^2}{\xi P^+}, k_\perp) \quad \epsilon_2 = (0, \frac{2k_\perp \cdot \epsilon_\perp^{(2)}}{\xi P^+}, \epsilon_\perp^{(2)})$$

$$k_3 = ((1-\xi)P^+, \frac{k_\perp^2}{(1-\xi)P^+}, -k_\perp) \quad \epsilon_3 = (0, -\frac{2k_\perp \cdot \epsilon_\perp^{(3)}}{(1-\xi)P^+}, \epsilon_\perp^{(3)})$$

$$V_{g\to gg} = (k_1 + k_3)\cdot \epsilon_2 \epsilon_1 \cdot \epsilon_3 + (k_2 - k_3)\cdot \epsilon_1 \epsilon_2 \cdot \epsilon_3 - (k_1 + k_2)\cdot \epsilon_3 \epsilon_1 \cdot \epsilon_2$$

$$\Rightarrow \qquad |V_{g\to gg}|^2 = |V_{+++}|^2 + |V_{+--+}|^2 + |V_{++-}|^2 = 4k_\perp^2 \frac{[1-\xi(1-\xi)]^2}{\xi^2(1-\xi)^2}$$

$$\Rightarrow \qquad \mathcal{P}_{gg}(\xi) = 2\left[\frac{1-\xi}{\xi} + \frac{\xi}{1-\xi} + \xi(1-\xi)\right] \quad (\xi < 1)$$

- Regularize $\frac{1}{1-\xi}$ to $\frac{1}{(1-\xi)_+}$
- Momentum conservation:

$$\int_0^1 d\xi\, \xi\, [\mathcal{P}_{qq}(\xi) + \mathcal{P}_{gq}(\xi)] = 0 \qquad \int_0^1 d\xi\, \xi\, [2\mathcal{P}_{qg}(\xi) + \mathcal{P}_{gg}(\xi)] = 0,$$

$\Rightarrow$ the terms which is proportional to $\delta(1-\xi)$.

- HW: derive other splitting functions.

DGLAP equation

In the leading logarithmic approximation with $t = \ln \mu^2$, the parton distribution and fragmentation functions follow the DGLAP[Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-1977] evolution equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q(x,\mu) \\ g(x,\mu) \end{bmatrix} = \frac{\alpha(\mu)}{2\pi}\int_x^1 \frac{d\xi}{\xi}\begin{bmatrix} C_F P_{qq}(\xi) & T_R P_{qg}(\xi) \\ C_F P_{gq}(\xi) & N_c P_{gg}(\xi) \end{bmatrix}\begin{bmatrix} q(x/\xi,\mu) \\ g(x/\xi,\mu) \end{bmatrix},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} D_{h/q}(z,\mu) \\ D_{h/g}(z,\mu) \end{bmatrix} = \frac{\alpha(\mu)}{2\pi}\int_z^1 \frac{d\xi}{\xi}\begin{bmatrix} C_F P_{qq}(\xi) & C_F P_{gq}(\xi) \\ T_R P_{qg}(\xi) & N_c P_{gg}(\xi) \end{bmatrix}\begin{bmatrix} D_{h/q}(z/\xi,\mu) \\ D_{h/g}(z/\xi,\mu) \end{bmatrix},$$

Comments:

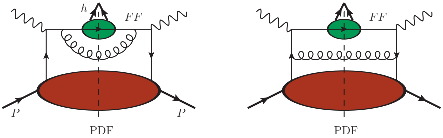
- In the double asymptotic limit, $Q^2 \to \infty$ and $x \to 0$, the gluon distribution can be solved analytically and cast into

$$xg(x,\mu^2) \simeq exp\left(2\sqrt{\frac{\alpha_s N_c}{\pi}\ln\frac{1}{x}\ln\frac{\mu^2}{\mu_0^2}}\right) \quad \text{Fixed coupling}$$

$$xg(x,\mu^2) \simeq exp\left(2\sqrt{\frac{N_c}{\pi b}\ln\frac{1}{x}\ln\frac{\ln\mu^2/\Lambda^2}{\ln\mu_0^2/\Lambda^2}}\right) \quad \text{Running coupling}$$

- The full DGLAP equation can be solved numerically.

Collinear Factorization at NLO



Use $\overline{\text{MS}}$ scheme ($\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} + \ln 4\pi - \gamma_E$) and dimensional regularization, DGLAP equation reads

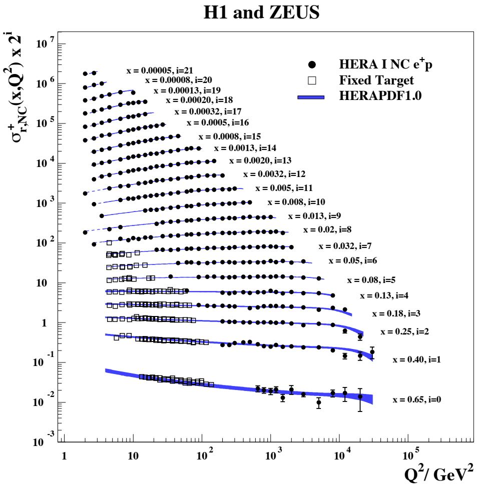
$$\begin{bmatrix} q(x,\mu) \\ g(x,\mu) \end{bmatrix} = \begin{bmatrix} q^{(0)}(x) \\ g^{(0)}(x) \end{bmatrix} - \frac{1}{\hat{\epsilon}} \frac{\alpha(\mu)}{2\pi} \int_x^1 \frac{d\xi}{\xi} \begin{bmatrix} C_F P_{qq}(\xi) & T_R P_{qg}(\xi) \\ C_F P_{gq}(\xi) & N_c P_{gg}(\xi) \end{bmatrix} \begin{bmatrix} q(x/\xi) \\ g(x/\xi) \end{bmatrix},$$

and

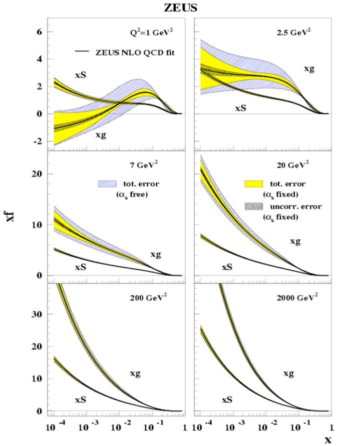
$$\begin{bmatrix} D_{h/q}(z,\mu) \\ D_{h/g}(z,\mu) \end{bmatrix} = \begin{bmatrix} D^{(0)}_{h/q}(z) \\ D^{(0)}_{h/g}(z) \end{bmatrix} - \frac{1}{\hat{\epsilon}} \frac{\alpha(\mu)}{2\pi} \int_z^1 \frac{d\xi}{\xi} \begin{bmatrix} C_F P_{qq}(\xi) & C_F P_{gq}(\xi) \\ T_R P_{qg}(\xi) & N_c P_{gg}(\xi) \end{bmatrix} \begin{bmatrix} D_{h/q}(z/\xi) \\ D_{h/g}(z/\xi) \end{bmatrix}.$$

- Soft divergence cancels between real and virtual diagrams;
- Gluon collinear to the initial state quark ⇒ parton distribution function; Gluon collinear to the final state quark ⇒ fragmentation function. KLN theorem does not apply.
- Other kinematical region of the radiated gluon contributes to the NLO ($\mathcal{O}(\alpha_s)$ correction) hard factor.

DGLAP evolution

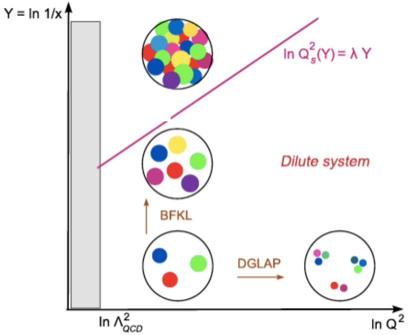


DGLAP evolution



- NLO DGLAP fit yields negative gluon distribution at low Q^2 and low x.
- Does this mean there is no gluons in that region? No

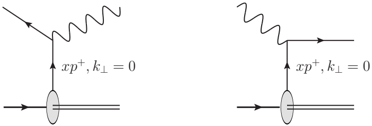
Phase diagram in QCD



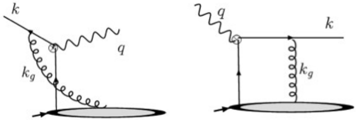
- Low Q^2 and low x region $\Rightarrow$ saturation region.
- Use BFKL equation and BK equation instead of DGLAP equation.
- BK equation is the non-linear small-x evolution equation which describes the saturation physics.

Collinear Factorization vs $k_\perp$ Factorization

Collinear Factorization



$k_\perp$ Factorization(Spin physics and saturation physics)



- The incoming partons carry no $k_\perp$ in the Collinear Factorization.
- In general, there is intrinsic $k_\perp$. It can be negligible for partons in protons, but should be taken into account for the case of nucleus target with large number of nucleons ($A \to \infty$).
- $k_\perp$ Factorization: High energy evolution with $k_\perp$ fixed.
- Initial and final state interactions yield different gauge links. (Process dependent)
- In collinear factorization, gauge links all disappear in the light cone gauge, and PDFs are universal.
- Other approaches, such as nuclear modification and higher twist approach. (See last year's lecture.)

k_t dependent parton distributions

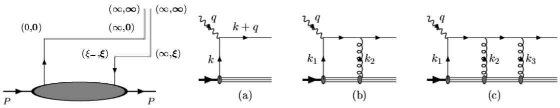
The unintegrated quark distribution

$$f_q(x,k_\perp) = \int \frac{d\xi^- d^2\xi_\perp}{4\pi(2\pi)^2} e^{ixP^+\xi^- + i\xi_\perp \cdot k_\perp} \langle P \left| \bar{\psi}(0)\mathcal{L}^\dagger(0)\gamma^+\mathcal{L}(\xi^-,\xi_\perp)\psi(\xi_\perp,\xi^-) \right| P\rangle$$

as compared to the integrated quark distribution

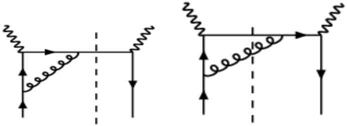
$$f_q(x) = \int \frac{d\xi^-}{4\pi} e^{ixP^+\xi^-} \langle P \left| \bar{\psi}(0)\gamma^+\mathcal{L}(\xi^-)\psi(0,\xi^-) \right| P\rangle$$

- The dependence of $\xi_\perp$ in the definition.
- Gauge invariant definition.
- Light-cone gauge together with proper boundary condition ⇒ parton density interpretation.
- The gauge links come from the resummation of multiple gluon interactions.
- Gauge links may vary among different processes.



TMD factorization

One-loop factorization:



For gluon with momentum k

- k is collinear to initial quark $\Rightarrow$ parton distribution function;
- k is collinear to the final state quark $\Rightarrow$ fragmentation function.
- k is soft divergence (sometimes called rapidity divergence) $\Rightarrow$ Wilson lines (Soft factor) or small-x evolution for gluon distribution.
- Other kinematical region of the radiated gluon contributes to the NLO ($\mathcal{O}(\alpha_s)$ correction) hard factor.
- See new development in Collins' book.