UV Divergences in Maximal and Half-Maximal Supergravities

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Adventures in Superspace Workshop
in Honor of Marc Grisaru

McGill University
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G. Bossard, P.S. Howe & K.S.S. 0901.4661, 0908.3883, 1009.0743
G. Bossard, P.S. Howe, K.S.S. & P. Vanhove 1105.6087
G. Bossard, P.S. Howe & K.S.S., 1212.0841
G. Bossard, P.S. Howe & K.S.S., in preparation

Marc and the origins of Superspace nonrenormalization theorems

- Marc was a pioneer in the application of S-matrix amplitude methods to the analysis of supergravity infinities. Grisaru, Pendleton & Van Nieuwenhuizen 1977
- He also pioneered the application of the background field method in the analysis of infinities. Abbott, Grisaru & Schaefer 1983, Grisaru, van Nieuwenhuizen & Wu 1975
- These techniques were put to use in a dramatic way in the calculation of 4-loop beta functions for supersymmetric sigma models. Grisaru, van de Ven & Zanon 1986
- He also introduced the key nonrenormalization theorem for extended supersymmetry. Grisaru & Siegel 1982

- Key tools in proving non-renormalization theorems are superspace formulations and the background field .

- For example, the Wess-Zumino model in N=1, D=4 supersymmetry is formulated in terms of a chiral superfield $\phi(x,\theta,\bar{\theta})$: $\bar{D}\phi = 0$; $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$.

- In the background field method, one splits the superfield into "background" and "quantum" parts,

$$\phi = \varphi + Q$$

background   quantum

- The chiral constraint on $Q(x,\theta,\bar{\theta})$ can be solved by introducing a "prepotential": $Q = \bar{D}^2 X$ $\quad (\bar{D}^3 \equiv 0)$

- Although the Wess-Zumino action requires chiral superspace integrals $I = \int d^4x d^4\theta \bar{\phi}\phi + Re \int d^4x d^2\theta \phi^3$ when they are written in terms of the total field ϕ , the parts involving the quantum field Q appearing inside loop diagrams can be re-written as $\int d^4x d^4\theta = \int d^4x d^2\theta d^2\bar{\theta}$ full-superspace integrals using the "integration = differentiation" property of Berezin integrals.

- Upon expanding into background and quantum parts, one finds, *e.g.*, that the chiral interaction terms can be re-written as full superspace integrals:

$$\int d^4x d^2\theta Q^2 \varphi = \int d^4x d^4\theta X \bar{D}^2 X \varphi$$

- Thus all counterterms written using the background field φ must be writable as full-superspace integrals.

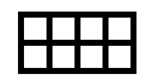
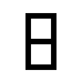
Grisaru, Siegel & Rocek 1979

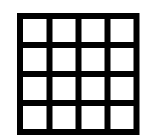
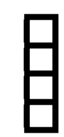
- The degree of "off-shell" supersymmetry is the maximal supersymmetry for which the algebra can close without use of the equations of motion.
- Knowing the extent of this off-shell supersymmetry is tricky, and may involve formulations (*e.g.* harmonic superspace) with infinite numbers of auxiliary fields. Galperin, Ivanov, Kalitsin, Ogievetsky & Sokatchev
- For maximal N=4 Super Yang-Mills and maximal N=8 supergravity, the fraction of off-shell realizable supersymmetry is known to be at least *half* the full supersymmetry of the theory, but the maximum realizable fraction in harmonic superspace is not currently known. Assuming that the maximal fraction is 1/2 lead originally to the expectation that the first allowable counterterms would have "1/2 BPS" structure.

Deser, Kay & K.S.S. 1977 (N=1 case)

- The 3-loop R^4 maximal supergravity candidate counterterm has a structure under linearized supersymmetry that is very similar to that of an F^4 N=4 super Yang-Mills invariant. Both of these are 1/2 BPS invariants, involving integration over just half the corresponding full superspaces:

Howe, K.S.S. & Townsend 1981
Kallosh 1981

$$\Delta I_{SYM} = \int (d^4\theta d^4\bar{\theta})_{105} \operatorname{tr}(\phi^4)_{105}$$  ***105*** ϕ_{ij}  ***6*** of SU(4)

$$\Delta I_{SG} = \int (d^8\theta d^8\bar{\theta})_{232848} (W^4)_{232848}$$  ***232848*** W_{ijkl}  ***70*** of SU(8)

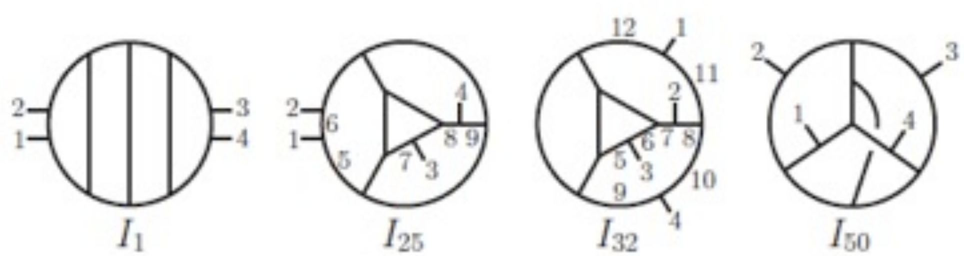
- Versions of these supergravity and SYM counterterms indeed do occur at one loop in D=8. This implies that, at least in that spacetime dimension, the full nonlinear structure of such counterterms exists and is consistent with all symmetries.

Unitarity-based calculations

Bern, Carrasco, Dixon, Johansson, Roiban et al. 2007 ... 2011

- The calculational front has now made substantial progress since the late 1990s.

- This has led to unanticipated and surprising cancellations at the 3- and 4-loop orders, yielding new lowest possible orders for the super Yang-Mills and supergravity divergence onset.



plus 46 more topologies

Max. SYM first divergences, current lowest possible orders (for integral spacetime dimensions).

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	6?	∞
BPS degree	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	$\partial^2 F^4$	finite

Blue: known divergences

Max. supergravity first divergences, current lowest possible orders (for integral spacetime dimensions).

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	6?	5?
BPS degree	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{4}$
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^{12} R^4$	$\partial^4 R^4$

Algebraic Renormalization and Ectoplasm

Dixon; Howe, Lindstrom & White; Piguet & Sorella; Hennaux; Stora;
Baulieu & Bossard; Voronov 1992; Gates, Grisaru, Knut-Whelau, & Siegel 1998
Berkovits and Howe 2008; Bossard, Howe & K.S.S. 2009

- The construction of supersymmetric invariants is isomorphic to the construction of cohomologically nontrivial closed forms in superspace: $I = \int_{M_0} \sigma^* \mathcal{L}_D$ is invariant (where σ^* is a pull-back to a section of the projection map down to the purely bosonic "body" subspace M_0) if $\mathcal{L}_D$ is a closed form in superspace, and it is nonvanishing only if $\mathcal{L}_D$ is nontrivial.
- Using the BRST formalism, one can handle all gauge symmetries including space-time diffeomorphisms by the nilpotent BRST operator s. The invariance condition for $\mathcal{L}_D$ is $s\mathcal{L}_D + d_0\mathcal{L}_{D-1} = 0$, where d_0 is the usual bosonic exterior derivative. Since $s^2 = 0$ and s anticommutes with d_0, one obtains using Poincaré's lemma $s\mathcal{L}_{D-1} + d_0\mathcal{L}_{D-2} = 0$, etc.

Cohomological non-renormalization theorem

- Counterterm cohomology then allows one to derive non-renormalization theorems: the cocycle structure of a candidate counterterm and its associated operators must match that of the classical action.
  - In maximal SYM, this leads to a non-renormalization theorem ruling out the F^4 counterterm that was otherwise expected at L=4 in D=5.
  - Similar non-renormalization theorems exist in supergravity, but their study is complicated by local supersymmetry and the density character of counterterm integrands.

Duality invariance constraints

- Maximal supergravity has a series of duality symmetries which extend the automatic GL(11-D) symmetry obtained upon dimensional reduction down from D=11. The classic example is E_7 in the N=8, D=4 theory, with the 70=133-63 scalars taking their values in an $E_7/SU(8)$ coset target space.

- The N=8, D=4 theory can be formulated in a manifestly E_7 covariant (but non-manifestly Lorentz covariant) formalism.
  Bossard, Hillman & Nicolai 2010
  Anomalies for SU(8), and hence E_7, cancel.
  Marcus 1985

- Combining the requirement of continuous duality invariance with the superspace cohomology requirements gives further powerful restrictions on counterterms.

Other approach to duality analysis from string amplitudes:
Broedel & Dixon 2010
Elvang & Kiermeier 2010;
Beisert, Elvang, Freedman, Kiermaier, Morales & Stieberger 2010

Supergravity Densities

- In a curved superspace, an invariant is constructed from the top (pure "body") component in a coordinate basis:

$$I = \frac{1}{D!}\int d^D x\, \varepsilon^{m_D \dots m_1}\, E_{m_D}{}^{A_D} \cdots E_{m_1}{}^{A_1}\, L_{A_1 \dots A_D}(x, \theta = 0)$$

- Referring this to a preferred "flat" basis and identifying $E_M{}^A$ components with vielbeins and gravitinos, one has, *e.g.* in D=4

$$I = \frac{1}{24}\int \big(e^a_\wedge e^b_\wedge e^c_\wedge e^d\, L_{abcd} + 4 e^a_\wedge e^b_\wedge e^c_\wedge \psi^{\underline{\alpha}} L_{abc\underline{\alpha}} + 6 e^a_\wedge e^b_\wedge \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}\, L_{ab\underline{\alpha}\,\underline{\beta}}$$
$$+ 4 e^a_\wedge \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}_\wedge \psi^{\underline{\gamma}} L_{a\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}} + \psi^{\underline{\alpha}}_\wedge \psi^{\underline{\beta}}_\wedge \psi^{\underline{\gamma}}_\wedge \psi^{\underline{\delta}} L_{\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}\,\underline{\delta}}\big)$$

  - Thus the "soul" components of the cocycle also contribute to the local supersymmetric covariantization.

- Since the gravitinos do not transform under the D=4 E_7 duality, the L_{ABCD} form components have to be *separately* duality invariant.

- At leading order, the $E_7/SU(8)$ coset generators of E_7 simply produce *constant shifts* in the 70 scalar fields. This leads to a much easier check of invariance than analyzing the full superspace cohomology problem.

Howe, K.S.S. & Townsend 1981

- Although the pure-body (4,0) component L_{abcd} of the R^4 counterterm has long been known to be shift-invariant at lowest order (since all 70 scalar fields are covered by derivatives), it is harder for the fermionic "soul" components to be so, since they are of lower dimension.

- Thus, one finds that the maxi-soul (0,4) $L_{\alpha\beta\gamma\delta}$ component is *not* invariant under constant shifts of the 70 scalars. Hence the D=4, N=8, 3-loop R^4 1/2 BPS counterterm is not E_7 duality invariant, so it is ruled out as an allowed counterterm. Bossard, Howe & K.S.S. 2010

L=7 and Vanishing Volume

Bossard, Howe, K.S.S. & Vanhove 2011

- The above type of analysis knocks out all the candidates in D=4, N=8 supergravity through L=6 loops. This leaves 7 loops (Δ=16) as the first order where a fully acceptable candidate might occur, with the volume of superspace as a prime candidate: $\int d^4x d^{32}\theta E(x,\theta)$.
- Explicitly integrating out the volume into component fields using the superspace constraints implying the classical field equations would be an ugly task.
  - However, using an on-shell implementation of harmonic superspace together with a superspace implementation of the normal-coordinate expansion, one can in fact evaluate it, but one then finds that the volume *vanishes*:

Kuzenko & Tartaglino-Mazzucchelli 2008

$$\int d^4x d^{32}\theta E(x,\theta) = 0 \ \text{ on-shell}$$

1/8 BPS E_7 invariant candidate notwithstanding

- Despite the vanishing of the full N=8 superspace volume, one can nonetheless use an on-shell harmonic superspace formalism to construct a different manifestly E_7 -invariant but 1/8 BPS candidate:

Bossard, Howe, K.S.S. & Vanhove 2011

$$I^8 := \int d\mu_{(8,1,1)} \, B_{\alpha\dot{\beta}} \, B^{\alpha\dot{\beta}} \qquad\qquad B_{\alpha\dot{\beta}} = \bar{\chi}_{\dot{\beta}}^{1ij} \chi_{\alpha\, 8ij}$$

- At the leading 4-point level, this invariant of generic $\partial^8 R^4$ structure can be written as a full superspace integral with respect to the linearized N=8 supersymmetry. It cannot, however, be rewritten as a non-BPS full-superspace integral with a duality-invariant integrand at the nonlinear level.

- Non-BPS full-superspace and manifestly E_7-invariant candidates do exist in any case from 8 loops onwards.

Howe & Lindstrom 1981
Kallosh 1981

The N=4 Supergravity L=3 surprise

- Not everything is perfect in the understanding of supergravity divergences, however. A surprize has occurred in an unexpected sector: D=4, N=4 supergravity at L=3. The expected 3-loop R^4 divergence (Δ=8) does not occur in that theory. Bern, Davies, Dennen & Huang 2012
  - Yet, the L=7 candidate counterterm of N=8 supergravity has a natural analogue here as a 1/4 BPS (4,1,1) G-analytic invariant: $I^4 = \int d\mu_{(4,1,1)}\, B_{\alpha\dot\beta}\, B^{\alpha\dot\beta} \quad B_{\alpha\dot\beta} = \chi^1_\alpha \bar\chi_{\dot\beta 4}$
  - Expanding the content of this N=4 invariant at linearized level, one finds a leading R^4 structure undressed by the $SL(2,\mathbb{R})/U(1)$ complex scalar field: it is perfectly duality invariant, just like the 1/8 BPS candidate 7-loop N=8 counterterm. Bossard, Howe, K.S.S. & Vanhove 2011

Vanishing volumes and their consequences

- Another aspect of this story needs to be clarified. The vanishing of a superspace volume can open the door to another representation of candidate counterterms.
- Consider the cases where superspace volumes vanish on-shell:
  - The full superspace volumes of all D=4 pure supergravities vanish, for any extension N of supersymmetry.
  - In D=5, the volume of maximal (32 supercharge) supergravity does *not* vanish, but the volume of half-maximal (16 supercharge, i.e. N=2, D=5) supergravity *does*.

Half-maximal D=5, L=2

Bern, Davies, Dennen & Huang 2012

- Unitarity-based calculations in D=5 half-maximal supergravity show cancellation of R^4 divergences at the 2-loop level similar to those found in half-maximal D=4, L=3.

- This cancellation is equally surprising as in the N=4, D=4 case, because there is an available 1/4 BPS D=5 (4,1) G-analytic Sp(2)/(U(1)×Sp(1)) counterterm:

$$\int d\mu_{(4,1)}\Omega^{\alpha\beta}\Omega^{\gamma\delta}\left(\chi^1_\alpha\chi^1_\beta\chi^1_\gamma\chi^1_\delta\right)$$

  where $\Omega^{\alpha\beta}$ is the D=5 Lorentz Sp(1,1) symplectic matrix.

- Moreover, in D=5 there are no complications from anomalies to the "duality" shift symmetry for the single scalar ϕ of half-maximal D=5 supergravity, unlike the D=4, N=4 case.

- The vanishing volume of half-maximal D=5 supergravity invites another way to write a candidate Δ=8 counterterm in D=5. One can write simply

$$I^{4\prime} = \int d^{16}\theta E \Phi$$

where Φ is the D=5 field-strength superfield containing the scalar ϕ as its lowest component field.

- Also, this candidate is clearly invariant under the rather minimalistic D=5 duality symmetry Φ → Φ + constant, since $\int d^{16}\theta E = 0$.

- Moreover, this candidate turns out to be just a rewriting of the above (4,1) G-analytic manifestly duality invariant 1/4 BPS candidate counterterm.

- In this sense, the D=5 Δ=8 (4,1) R^4 counterterm is of *marginal* F/D type.

- The D=4 (4,1,1) G-analytic counterterm has the same marginal F/D character.

- The D=4, N=4 theory has as lowest-dimension physical component a complex scalar field τ taking its values in the Kähler space $SL(2,\mathbb{R})/U(1)$. In terms of τ, the Kähler potential is

$$K[\tau] = -\ln(\mathrm{Im}[\tau])$$

  and the N=4, Δ=8 (4,1,1) counterterm can equally well be written

$$\int d^{16}\theta E K[\tau]$$

- As in the D=5 case, although this full-superspace integral is duality invariant, its *integrand* is not duality invariant. The integrand varies as follows:

$$\delta\left(E\ln(\mathrm{Im}[\tau])\right) = 2hE + fE(\tau + \bar{\tau})$$

Superspace nonrenormalization theorems: refinement of the duality invariance requirement

- The marginal F/D structure of the Δ=8 counterterm candidates in half-maximal D=4 and D=5 supergravities requires a more careful treatment of the Ward identies for duality.
- If one makes the *assumption* that there exist off-shell full 16-supercharge superfield formulations for the half-maximal theories, then one can derive a stronger requirement for duality invariance: not only must the integrated counterterm be invariant, but also the counter-Lagrangian superfield integrand must itself be duality invariant.
- Proof of this refined theorem requires introduction of the notion of a chain of superspace co-forms arising from the duality variation of the Lagrangian density. In order for the duality Ward identities to be satisfied, the whole chain of co-forms must be renormalised consistently as a single cohomology class.

Nonrenormalization analogy: the N = (2,2) sigma model

Grisaru, van de Ven & Zanon 1986

- An analogy to the duality invariance requirement for counter-Lagrangian integrands can be found for N=(2,2) D=2 non-linear sigma models, based on chiral superfields T_a.
- The sigma-model action is given by the full superspace integral $I_\sigma = \int d^2x d^4\theta K(T_a, \bar{T}_b)$ where the Lagrangian integrand $K(T_a, \bar{T}_b)$ is the target-space Kähler potential.
- Although the classical superspace Lagrangian integrand is not itself a globally defined scalar, the N=(2,2) nonrenormalization theorem requires all counterterms $I_\sigma^{\rm ct} = \int d^2x d^4\theta S(T_a, \bar{T}_b)$ to have integrands $S(T_a, \bar{T}_b)$ that *are* globally defined scalars.

Howe, Papadopoulos & K.S.S. 1986

- Sigma models with isometries thus require counter-Lagrangian integrands that are isometrically invariant.

Off-shell half-maximal supergravity

- From the point of view of field-theoretic nonrenormalization theorems, the key question is whether there exists an off-shell linearly realised formulation of half-maximal supergravity. If so, then the nonrenormalization theorem would require a full-superspace $\int d^{16}\theta$ integral with a duality-invariant integrand, thus ruling out the F/D marginal D=4 and D=5 R^4 counterterms.
- Unfortunately, the answer to this question is not currently known. But there is a closely related off-shell formulation for linearized D=10, N=1 supergravity, with a finite number of component fields:

Howe, Nicolai & Van Proeyen 1982

$$\mathcal{L}_{10} = \frac{1}{2} V_{abc} \Delta_{abc,def} V_{def} - V_{abc} \bar{D} \Gamma_{abc} D S \qquad \Delta : \quad D^{16}, \ \partial D^{14}, \ \text{etc.}$$

- Upon dimensional reduction to D=4, the N=1, D=10 theory yields D=4, N=4 supergravity plus 6 N=4 super-Maxwell multiplets. So one has something close to the required formalism. Pure N=4 SG undoubtedly would require a harmonic superspace formulation.

Current outlook

- So far, things are under control for maximal supergravity from a purely field-theoretic analysis: what is prohibited does not occur, and what is not prohibited has occured, as far as one can see.

- As far as one knows, the first acceptable D=4 counterterm for maximal supergravity still occurs at L=7 loops ($\Delta = 16$); if not that, then they clearly exist at L=8 loops ($\Delta = 18$) and beyond.

Howe & Lindstrom 1981
Kallosh 1981

- The current divergence expectations for maximal supergravity are consequently:

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	6	7
BPS degree	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{8}$
Gen. form	$\partial^{12}R^4$	$\partial^{10}R^4$	R^4	∂^4R^4	∂^6R^4	$\partial^{12}R^4$	∂^8R^4

Blue: known divergences
Green: anticipated divergences