

# Phys 514 - Midterm Solutions

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1. (5 pts) Consider a variant of the Pound-Rebka experiment. An experimenter on a tower of height  $h$  above the surface of the earth lets a ball of mass  $m$  fall to the ground from rest. At the surface, the total energy is "magically" transformed into energy of a single photon which travels back up to the experimenter, where the energy is transformed back to rest mass. Show that if not for the gravitational redshift, you could construct a perpetual motion machine (a machine from which you could extract an arbitrary amount of energy). What happens if you take gravitational redshift into account?

## Solution

The easiest way to solve this is to consider what happens to the rest mass after each full cycle, with and without redshifting. First, without redshifting. We assume a static gravitational field throughout.

Initially, the object at the top has a total energy

$$E_{top} = m_0c^2 + m_0gh$$

Where  $m_0$  corresponds to the mass of the object being dropped at the start of the first cycle. We wish to see how this mass evolves. When the mass reaches the bottom, conservation of energy tells us that

$$\begin{aligned} E_{bottom} &= E_{top} = m_0c^2 + m_0gh \\ &= h\nu = E_\gamma \end{aligned}$$

Where in the second line we have equated the energy to that of a photon. Without redshifting, the frequency of the photon doesn't change between the bottom and the top. Thus, when the photon reaches the top and changes back into an object with rest mass energy equal to  $h\nu$ , we get a new total energy at the top of

$$E'_{top} = m_1c^2 + m_1gh$$

Where  $m_1$  is the new mass of the object formed by this

$$\begin{aligned} m_1c^2 &= h\nu = m_0(c^2 + gh) \\ m_1 &= m_0 \left( 1 + \frac{gh}{c^2} \right) \end{aligned}$$

Thus, the mass of the object has increased by a factor of  $gh/c^2$ , giving the system more energy than it initially started with! This is the signature of a perpetual motion machine, as our new energy at the top is

$$E'_{top} = m_0c^2 + 2m_0gh + m_0\frac{(gh)^2}{c^2} > E_{top}$$

Lets look at the case with gravitational redshift. The energy at the top and the bottom is still the same, as

$$\begin{aligned} E_{top,red} &= m_0c^2 + m_0gh \\ E_{bottom,red} &= m_0c^2 + m_0gh \\ &= h\nu_0 = E_\gamma \end{aligned}$$

Where we have distinguished  $\nu_0$  as the frequency of the photon at the bottom of the tower. Gravitational redshift affects frequencies by

$$\nu_f = \frac{\nu_0}{1 + \frac{gh}{c^2}}$$

So including this redshifting gives us a photon energy at the top of the tower of

$$h\nu_f = \frac{h\nu_0}{1 + \frac{gh}{c^2}} = m_1c^2$$

We know what  $h\nu_0$  was from before, so we can determine the new mass of our created object at the top of the tower

$$\begin{aligned} m_1c^2 &= m_0(c^2 + gh)\frac{c^2}{c^2 + gh} = m_0c^2 \\ m_1 &= m_0 \end{aligned}$$

The energy at the top is now

$$E'_{top,red} = m_1c^2 + m_1gh = m_0c^2 + m_0gh = E_{top,red}$$

Clearly, no energy was gained in this process, thus we cannot extract an arbitrary amount of energy by running this cycle as many times as we want.

2. (5 pts) Consider the surface of the earth (radius  $R_E$ ) and write down the metric of the surface in some convenient system of coordinates.

a) Determine the distance between two points both of latitude 45 deg and with longitudes 0 and 12 hours, respectively (the range of longitudes is from 0 to 24 hours).

b) Do the same calculation if the first point is at latitude 45 deg and longitude 0 hrs, and the second point lies on the equator at longitude 6 hrs.

N.B. No calculus is required to solve this problem!

### Solution

a) The metric on a sphere with (constant) radius  $R_E$  is

$$ds^2 = R_E^2 d\theta^2 + R_E^2 \sin^2(\theta) d\phi^2$$

The path length is simply  $s$ . There is a trick to this section. The path of constant latitude is not the shortest distance between the two points. In fact, the shortest distance will always be given by the intersect of a plane containing the origin, the start, and the endpoint with the sphere. This is called the great circle distance.

For the case of starting and ending points on opposite sides of the sphere (separated by a longitude of 12 hrs), the shortest path is straight over the north pole.

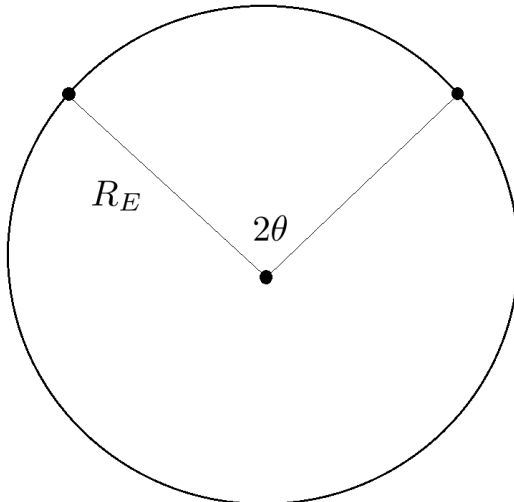


Figure 1: The shortest path length from two opposite ends of the sphere at constant latitude is the distance over the pole between the two points.

Referring to figure 1, this path length is easily seen to be the portion of the circle subtended by an angle  $2\theta$ . Here,  $\theta = \pi/4$ , so the length is

$$s = R_E \cdot \frac{\pi}{2}$$

Note that a calculation of the constant latitude path yields  $s_{\theta=\text{const}} = R_E \cdot \frac{\pi}{\sqrt{2}}$ , so going over the pole is indeed shorter.

b) For this part, it is perhaps easiest to refer to figure 2. Our method for finding this path will be to hold the ending point fixed (as it lies on the  $x$  axis), and rotating the starting point to lie on the  $z$  axis.

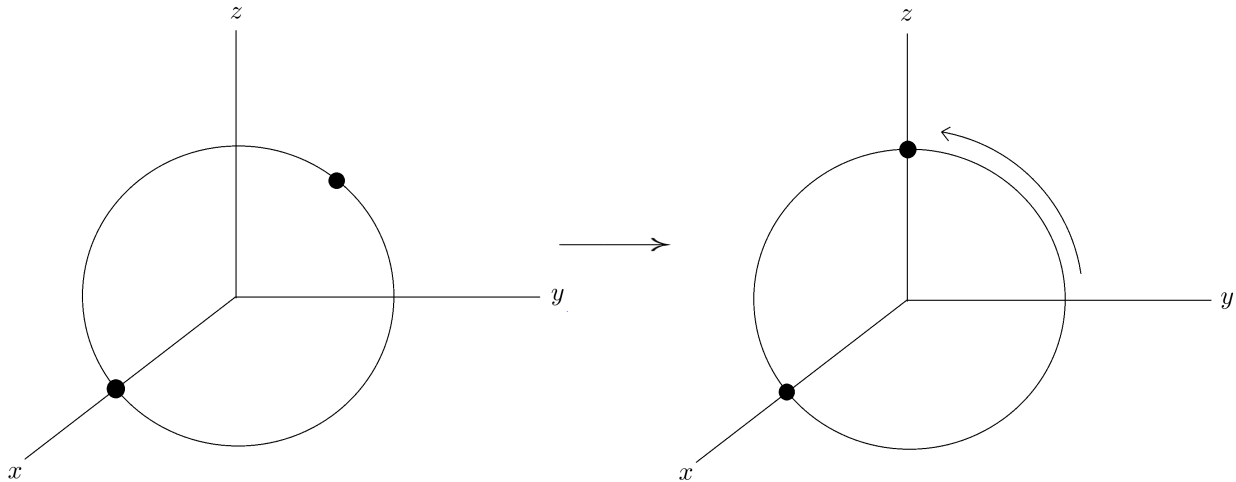


Figure 2: The distance between the points is easiest to see by rotating about the  $x$  axis so that the path starts on the  $z$  axis, and ends on the  $x$ .

With this rotation (it is not necessary to actually perform the rotation, as we start and end in the  $yz$  plane with the starting point), we can see that the angle between the two points is  $\pi/2$  (the angle between the  $x$  and  $z$  axis). Therefore, the arc length between the two points is simply

$$s = R_E \cdot \frac{\pi}{2}$$

The same as in part  $a$ .

3. (15 pts) Consider the flow on a two dimensional plane which for  $y < 0$  is uniform in the  $y$  direction. Between  $y = 0$  and  $y = 1$  the flow is forced to converge towards the  $y$ -axis until the density has doubled. The convergence is smooth. For  $y > 1$  the flow is once again uniform along in  $y$  direction, maintaining the larger density. Consider the coordinate vector fields  $\partial_x$  and  $\partial_y$ .

a) Sketch the flow lines.

b) Write down a vector field  $X$  which generates this flow.

c) What are the covariant derivatives of  $\partial_x$  and  $\partial_y$  with respect to  $X$ ? Give a geometrical justification of your answer.

d) What are the Lie derivatives of  $\partial_x$  and  $\partial_y$  with respect to  $X$ ? Give a geometrical justification of your answer.

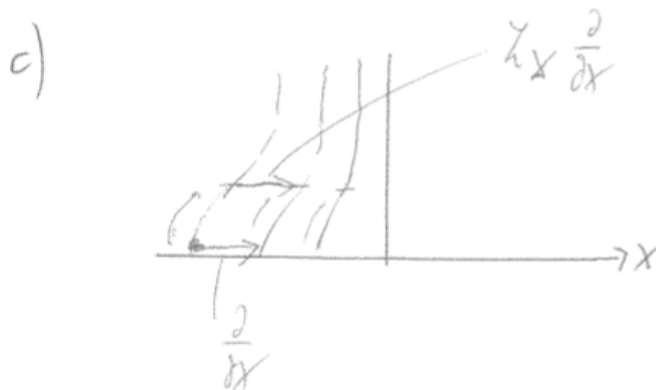
**Solution**



b)

$$(x_s(t), y_s(t)) = (x_e f(t+s), t+s)$$

$$X = \frac{d}{ds} \Big|_{s=0} (x_s(t), y_s(t)) = \left( x_e \frac{df}{dt} \Big|_t, 1 \right)$$



$$\mathcal{L}_X \left( \frac{\partial}{\partial x} \right) \approx \frac{\partial}{\partial x}$$

$$\left( \hbar x \frac{\partial}{\partial x} \right)^n = \left( X \frac{\partial}{\partial x} - \frac{\partial}{\partial x} X \right)^n = X^n \frac{\partial^n}{\partial x^n} - \frac{\partial^n}{\partial x^n} X^n$$

$$\text{with } \gamma = \frac{\partial}{\partial x}$$

$$\Rightarrow \hbar x \frac{\partial}{\partial x} = \left( - \frac{\partial F}{\partial t} \Big|_t, 0 \right)$$

$$\left( \hbar x \frac{\partial}{\partial y} \right) = \left( -x \frac{\partial F}{\partial t} \Big|_t, 0 \right)$$

by similar analysis

4. (15 pts) For the 2-dimensional metric

$$ds^2 = r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

Use the tetrad basis to compute the non-vanishing Riemann tensor elements and the Ricci scalar.

**Solution**

As usual with tetrad questions, we first define our tetrad basis. It will be

$$e^\theta = rd\theta \quad e^\phi = r \sin(\theta)d\phi$$

Where we note that  $r$  is a constant ( $r = 1$  for the unit circle). To find the spin connection we have to take the differential of our basis. Doing so yields

$$\begin{aligned} de^\theta &= 0 \\ de^\phi &= r \cos(\theta)d\theta \wedge d\phi \end{aligned}$$

Now to find the spin connection, we use the formula

$$de^a = e^b \wedge \omega_b^a$$

Writing our two equations, and noting that  $\omega_a^a = 0$  by antisymmetry, we have

$$\begin{aligned} de^\theta = 0 &= r \sin(\theta)d\phi \wedge \omega_\phi^\theta \\ de^\phi &= r \cos(\theta)d\theta \wedge d\phi = rd\theta \wedge \omega_\theta^\phi \end{aligned}$$

The first line doesn't help us at all. The second line allows us to identify the only spin connection element that is nonzero. That is,  $\omega_\theta^\phi = \cos(\theta)d\theta$  (by antisymmetry,  $\omega_\phi^\theta = -\cos(\theta)d\phi$ ). Now, we need to take the differential in order to determine our Riemann tensor in the tetrad basis. The differentials are

$$\begin{aligned} d\omega_\theta^\phi &= -\sin(\theta)d\theta \wedge d\phi \\ d\omega_\phi^\theta &= \sin(\theta)d\theta \wedge d\phi \end{aligned}$$

The Riemann tensor is defined as  $R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c$ , which with only one independent spin connection, gives us

$$\begin{aligned} R_{\phi'}^{\theta'} &= \sin(\theta)d\theta \wedge d\phi \\ R_{\theta'}^{\phi'} &= -\sin(\theta)d\theta \wedge d\phi \end{aligned}$$

Where we have used primes to distinguish between the tetrad and coordinate basis. We now switch back to the Riemann tensor in coordinate basis by the formula

$$R_{\sigma\mu\nu}^{\rho} = e_a^{\rho} e_{\sigma}^b R_{b\mu\nu}^a$$

Where

$$e_{\sigma}^b = \text{diag}(r, r \sin(\theta)) \quad e_a^{\rho} = \text{diag}\left(\frac{1}{r}, \frac{1}{r \sin(\theta)}\right)$$

So our Riemann tensor elements are

$$\begin{aligned} R_{\phi\mu\nu}^{\theta} &= e_{\theta'}^{\theta} e_{\phi}^{\phi'} R_{\phi'\mu\nu}^{\theta'} \\ &= \left(\frac{1}{r}\right) (r \sin(\theta)) (\sin(\theta) d\theta \wedge d\phi) \\ R_{\phi\theta\phi}^{\theta} &= \sin^2(\theta) \end{aligned}$$

And

$$\begin{aligned} R_{\theta\mu\nu}^{\phi} &= e_{\phi'}^{\phi} e_{\theta}^{\theta'} R_{\theta'\mu\nu}^{\phi'} \\ &= \left(\frac{1}{r \sin(\theta)}\right) (r) (-\sin(\theta) d\theta \wedge d\phi) \\ R_{\theta\theta\phi}^{\phi} &= -1 \end{aligned}$$

There are then a total of four nonzero components of the Riemann tensor. The Ricci tensor is defined as  $R_{\mu\nu} = R_{\mu\lambda\nu}^{\lambda}$ , so we have

$$\begin{aligned} R_{\theta\theta} &= R_{\theta\phi\theta}^{\phi} = 1 \\ R_{\phi\phi} &= R_{\phi\theta\phi}^{\theta} = \sin^2(\theta) \end{aligned}$$

Finally, the Ricci scalar is  $R = g^{\mu\nu} R_{\mu\nu}$ , so we get

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{r^2}$$

Which is just 2 for the unit sphere, just like usual!



5. (10 pts) You are on a planet located at a distance  $r_1 = 100r_s$  from a black hole with Schwarzschild radius  $r_s$ . Your parents travel to another planet located at a distance of  $r_2 = 9/8r_s$  from the same black hole and spend ten years there (according to their clocks). How much have you aged when your parents return? You can neglect the time it takes to travel to and from the resort. First, explain in words why the parents age more or less. Also, derive a formula which gives you the difference in ageing for general  $r_1$  and  $r_2$ .

### Solution

The Schwarzschild metric is

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Where  $r_s = 2GM$ . For a stationary test particle,  $dr = d\Omega = 0$ , and so we can derive the usual gravitational time dilation formula

$$t = \frac{\tau}{\sqrt{1 - \frac{r_s}{r}}}$$

Where  $\tau$  is the proper time in the gravitational potential, and  $t$  the time measured by an observer at infinity. We can work through this in two steps now. First, if your parents spend 10 years at  $r_s = 9/8r_s$ , an observer at infinity would find this time to be

$$t_\infty = \frac{10}{\sqrt{1 - 8/9}} \text{ Yrs} = 30 \text{ Yrs}$$

Now, we can consider the inverse operation. If an observer at infinity experiences 30 years worth of time, how much time is that for an observer at a distance  $r = 100r_s$ ?

$$\tau_{100r_s} = t_\infty \sqrt{1 - \frac{r_s}{100r_s}} = 3\sqrt{99} \text{ Yrs} \approx 29.8 \text{ Yrs}$$

Putting this together we can come up with the general formula for comparing the time between observers in two different gravitational potentials

$$\tau_{r_1} = \sqrt{1 - r_s/r_1} \frac{1}{\sqrt{1 - r_s/r_2}} \tau_{r_2} = \left( \frac{1 - r_s/r_1}{1 - r_s/r_2} \right)^{1/2} \tau_{r_2}$$