Phys 514 - Assignment 4
Solutions

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1. In class I defined the covariant derivative of a vector field. Prove that the coefficients of the covariant derivative transform as a tensor of type (1,1) under a coordinate transformation. Show that the partial derivatives of such a vector field do not transform as a tensor

Solution

We are looking to determine how the covariant derivative of a vector field transforms. Recall that

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{1}$$

We start by nothing that the partial derivative operator transforms like a dual vector, and the vector field transforms as a contravariant vector. The transformation can thus be written

$$\begin{aligned}\nabla_\mu V^\nu \to \nabla_{\mu'} V^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) + \Gamma^{\nu'}_{\mu'\lambda'}\left(\frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda\right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^{\nu'}_{\mu'\lambda'}\left(\frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda\right)\end{aligned}$$

Now, to check that $\nabla_\mu V^\nu$ transforms as a (1,1) tensor under coordinate transformations, we need to figure out how the connection, $\Gamma^{\nu'}_{\mu'\lambda'}$ transforms. Note that in Carroll, they DEMAND that the covariant derivative of a vector field transforms like a (1,1) tensor, whereas we are trying to show it. Thus, we cannot use the transformations result in equation (3.10) of Carroll for the connection, and must show it ourselves. Consider now the expression for the transformation of the connection

$$\begin{aligned}\Gamma^\nu_{\mu\lambda} \to \Gamma^{\nu'}_{\mu'\lambda'} &= \frac{1}{2} g^{\nu'\sigma'}(\partial_{\mu'} g_{\lambda'\sigma'} + \partial_{\lambda'} g_{\sigma'\mu'} - \partial_{\sigma'} g_{\mu'\lambda'}) \\ &= \frac{1}{2}\left(\frac{\partial x^{\nu'}}{\partial x^\tau}\frac{\partial x^{\rho'}}{\partial x^\sigma} g^{\tau\sigma}\right)\left(\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^\alpha}{\partial x^{\lambda'}}\frac{\partial x^\beta}{\partial x^{\rho'}} g_{\alpha\beta}\right) + \frac{\partial}{\partial x^{\lambda'}}\left(\frac{\partial x^\alpha}{\partial x^{\rho'}}\frac{\partial x^\beta}{\partial x^{\mu'}} g_{\alpha\beta}\right) - \frac{\partial}{\partial x^{\rho'}}\left(\frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\lambda'}}\right)\right)\end{aligned}$$

Now the this is quite clearly becoming an ordeal to expand, but we must press on. Using some shorthand, $\Lambda^\alpha_\beta = \frac{\partial x^\alpha}{\partial x^\beta}$, we can carry on with our expansion

$$\begin{aligned}\Gamma^{\nu'}_{\mu'\lambda'} = \frac{1}{2}\Lambda^{\nu'}_\tau \Lambda^{\rho'}_\sigma g^{\tau\sigma}(&\frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^{\lambda'}}\Lambda^\beta_{\rho'} g_{\alpha\beta} + \Lambda^\alpha_{\lambda'}\frac{\partial^2 x^\beta}{\partial x^{\mu'}\partial x^{\rho'}} g_{\alpha\beta} + \Lambda^\alpha_{\lambda'}\Lambda^\beta_{\rho'}\frac{\partial g_{\alpha\beta}}{\partial x^{\mu'}} \\ &+ \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\rho'}}\Lambda^\beta_{\mu'} g_{\alpha\beta} + \Lambda^\alpha_{\rho'}\frac{\partial x^\beta}{\partial x^{\lambda'}\partial x^{\mu'}} g_{\alpha\beta} + \Lambda^\alpha_{\rho'}\Lambda^\beta_{\mu'}\frac{\partial g_{\alpha\beta}}{\partial x^{\lambda'}} \\ &- \frac{\partial^2 x^\alpha}{\partial x^{\rho'}\partial x^{\mu'}}\Lambda^\beta_{\lambda'} g_{\alpha\beta} - \Lambda^\alpha_{\mu'}\frac{\partial^2 x^\beta}{\partial x^{\rho'}\partial x^{\lambda'}} g_{\alpha\beta} - \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{\partial g_{\alpha\beta}}{\partial x^{\rho'}})\end{aligned}$$

Now recall the two useful identities, $g_{\mu\nu}g^{\nu\sigma} = \delta^\mu_\sigma$, and $\Lambda^\mu_\nu \Lambda^\nu_\sigma = \frac{\partial x^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial x^\sigma} = \delta^\mu_\sigma$. Expanding the brackets and using these identities a few times yields

$$\begin{aligned}\Gamma^{\nu'}_{\mu'\lambda'} = \frac{1}{2}(&\Lambda^{\nu'}_\alpha \frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^{\lambda'}} + \Lambda^{\nu'}_\tau g^{\tau\sigma}\Lambda^\alpha_{\lambda'}\frac{\partial^2 x^\beta}{\partial x^{\mu'}\partial x^\sigma}g_{\alpha\beta} + \Lambda^{\nu'}_\tau g^{\tau\sigma}\Lambda^\beta_{\mu'}\frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^\sigma}g_{\alpha\beta} \\ &+ \Lambda^{\nu'}_\beta \frac{\partial^2 x^\beta}{\partial x^{\lambda'}\partial x^{\mu'}} - \Lambda^{\nu'}_\tau g^{\tau\sigma}\Lambda^\beta_{\lambda'}\frac{\partial^2 x^\alpha}{\partial x^\sigma \partial x^{\mu'}}g_{\alpha\beta} - \Lambda^{\nu'}_\tau g^{\tau\sigma}\Lambda^\alpha_{\mu'}\frac{\partial^2 x^\beta}{\partial x^\sigma\partial x^{\lambda'}}g_{\alpha\beta}) \\ &+ \frac{1}{2}\Lambda^{\nu'}_\tau g^{\tau\sigma}(\Lambda^\alpha_{\lambda'}\frac{\partial g_{\alpha\sigma}}{\partial x^{\mu'}} + \Lambda^\beta_{\mu'}\frac{\partial g_{\sigma\beta}}{\partial x^{\lambda'}} - \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma})\end{aligned}$$

Reordering the terms gives us

$$\begin{aligned}\Gamma^{\nu'}_{\mu'\lambda'} = &\frac{1}{2}\left(\Lambda^{\nu'}_\alpha \frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^{\lambda'}} + \Lambda^{\nu'}_\beta \frac{\partial^2 x^\beta}{\partial x^{\lambda'}\partial x^{\mu'}}\right) \\ &+ \Lambda^{\nu'}_\tau g^{\tau\sigma}g_{\alpha\beta}\left(\Lambda^\alpha_{\lambda'}\frac{\partial^2 x^\beta}{\partial x^{\mu'}\partial x^\sigma} + \Lambda^\beta_{\mu'}\frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^\sigma} - \Lambda^\beta_{\lambda'}\frac{\partial^2 x^\alpha}{\partial x^\sigma\partial x^{\mu'}} - \Lambda^\alpha_{\mu'}\frac{\partial^2 x^\beta}{\partial x^\sigma\partial x^{\lambda'}}\right) \\ &+ \frac{1}{2}\Lambda^{\nu'}_\tau g^{\tau\sigma}\left(\Lambda^\alpha_{\lambda'}\Lambda^\beta_{\mu'}\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \Lambda^\beta_{\mu'}\Lambda^\alpha_{\lambda'}\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} - \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)\end{aligned}$$

Using the fact that $g_{\alpha\beta} = g_{\beta\alpha}$ (symmetric metric), and the fact that we can rename and exchange α and β since they are dummy indices, lets examine this line by line.

In the first line, calling $\beta \to \alpha$ for the second term shows that the two terms are equal.

For the second line, note that

$$\begin{aligned}g_{\alpha\beta}\Lambda^\alpha_{\lambda'}\frac{\partial^2 x^\beta}{\partial x^{\mu'}\partial x^\sigma} &= g_{\beta\alpha}\Lambda^\beta_{\lambda'}\frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^\sigma} \\ &= g_{\alpha\beta}\Lambda^\beta_{\lambda'}\frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^\sigma}\end{aligned}$$

Which cancels with the third term in the second line. The second and fourth terms cancel in the same way, so this second term disappears.

The third line can be rearranged as thus

$$\begin{aligned}\Lambda^\alpha_{\lambda'}\Lambda^\beta_{\mu'}\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \Lambda^\beta_{\mu'}\Lambda^\alpha_{\lambda'}\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} - \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma} &= \Lambda^\beta_{\lambda'}\Lambda^\alpha_{\mu'}\frac{\partial g_{\beta\sigma}}{\partial x^\alpha} + \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{\partial g_{\sigma\alpha}}{\partial x^\beta} - \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \\ &= \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\left(\frac{\partial g_{\beta\sigma}}{\partial x^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)\end{aligned}$$

Thus we can write our whole transformation as

$$\Gamma^{\nu'}_{\mu'\lambda'} = \Lambda^{\nu'}_\alpha \frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^{\lambda'}} + \Lambda^{\nu'}_\tau \Lambda^\alpha_{\mu'}\Lambda^\beta_{\lambda'}\frac{1}{2}g^{\tau\sigma}\left(\frac{\partial g_{\beta\sigma}}{\partial x^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)$$

Where we recognize part of the last term as

$$\frac{1}{2}g^{\tau\sigma}\left(\frac{\partial g_{\beta\sigma}}{\partial x^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right) = \Gamma^\tau_{\alpha\beta}$$

Consider now the two forms of the expression

$$\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\nu'}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\lambda'}}\right) = \frac{\partial^2 x^{\nu'}}{\partial x^{\mu'}\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\lambda'}} + \frac{\partial x^{\nu'}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\lambda'}}$$

But also

$$\frac{\partial}{\partial x^{\mu'}}\left(\frac{\partial x^{\nu'}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\lambda'}}\right) = \frac{\partial}{\partial x^{\mu'}}\delta^{\nu'}_{\lambda'} = 0$$

So we can rearrange our first expression as

$$\Lambda^{\nu'}_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\lambda'}} = -\Lambda^{\alpha}_{\lambda'}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu'}\partial x^{\alpha}}$$

Thus we can write our transformation as

$$\Gamma^{\nu'}_{\mu'\lambda'} = \Lambda^{\nu'}_{\tau}\Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\lambda'}\Gamma^{\tau}_{\alpha\beta} - \Lambda^{\alpha}_{\lambda'}\Lambda^{\beta}_{\mu'}\frac{\partial^2 x^{\nu'}}{\partial x^{\beta}\partial x^{\alpha}}$$

Since τ, α, β are dummy indices, we can finally write our transformation rule for the connection coefficients as

$$\Gamma^{\nu}_{\mu\lambda} \to \Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}$$

Which verifies Carroll's assumed transformation rule for the coefficients. note that they don't transform as a proper tensor, which we didn't expect in the first place.

Now we can continue with our transformation of the covariant derivative. It is

$$\begin{aligned}
\nabla_{\mu}V^{\nu} \to \nabla_{\mu'}V^{\nu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}}V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial V^{\nu}}{\partial x^{\mu}} + \Gamma^{\nu'}_{\mu'\lambda'}\left(\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}V^{\lambda}\right) \\
&= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}}V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial V^{\nu}}{\partial x^{\mu}} + \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right)\left(\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}V^{\lambda}\right) \\
&= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}}V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}V^{\lambda} \\
&= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\left(\frac{\partial V^{\nu}}{\partial x^{\mu}} + \Gamma^{\nu}_{\mu\lambda}V^{\lambda}\right) \\
&= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\nabla_{\mu}V^{\nu}
\end{aligned}$$

So it transforms like a $(1,1)$ tensor, as we had hoped. Note in going from line 3 to 4 we swapped $\lambda \to \nu$ as it was a dummy index.

The partial derivatives do not transform like a proper tensor, since

$$
\begin{aligned}
\partial_\mu V^\nu \to \partial_{\mu'} V^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) \\
&= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\mu} \\
&\neq \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu
\end{aligned}
$$

Due to the presence of the first term in line 2.

2. In class I showed that the parallel transport along a closed curve spanned by two coordinate vector fields X and Y is given by the operator

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X \tag{2}$$

Show that this leads to, for general vector fields X and Y, the result

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{3}$$

Solution

Lets take our coordinate vector fields to be untilded quantities, and our generalized vector fields to be tilded quantities, where

$$X = f\tilde{X} \qquad\qquad Y = g\tilde{Y}$$

Here, f and g are non-trivial functions with respect to the coordinates. Now, we can write

$$\begin{aligned}
R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X &= \nabla_{f\tilde{X}} \nabla_{g\tilde{Y}} - \nabla_{g\tilde{Y}} \nabla_{f\tilde{X}} \\
&= f\nabla_{\tilde{X}}(g\nabla_{\tilde{Y}}) - g\nabla_{\tilde{Y}}(f\nabla_{\tilde{X}}) \\
&= f\nabla_{\tilde{X}}(g)\nabla_{\tilde{Y}} + fg\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - g\nabla_{\tilde{Y}}(f)\nabla_{\tilde{X}} - gf\nabla_{\tilde{Y}}\nabla_{\tilde{X}} \\
&= fg(\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}) + f\nabla_{\tilde{X}}(g)\nabla_{\tilde{Y}} - g\nabla_{\tilde{Y}}(f)\nabla_{\tilde{X}} \\
&= fg(\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}) + f\tilde{X}(g)\nabla_{\tilde{Y}} - g\tilde{Y}(f)\nabla_{\tilde{X}}
\end{aligned}$$

Where from line 1 to 2 we have used that $\nabla_{f\tilde{X}} = f\nabla_{\tilde{X}}$, and in getting to the final line we have used that $\nabla_X(g) = X^\mu \nabla_\mu g = X^\mu \partial_\mu g = X(g)$ since the covariant derivative reduces to a partial derivative on scalar functions. Now lets compute the commutator of the generalized vector fields

$$\begin{aligned}
[\tilde{X},\tilde{Y}] = [\frac{1}{f}X, \frac{1}{g}Y] &= \frac{1}{f}X\left(\frac{1}{g}Y\right) - \frac{1}{g}Y\left(\frac{1}{f}X\right) \\
&= f^{-1}X(g^{-1})Y + f^{-1}g^{-1}X(Y) - g^{-1}Y(f^{-1})X - f^{-1}g^{-1}Y(X) \\
&= \frac{-1}{fg^2}X(g)Y + \frac{1}{f^2g}Y(f)X + \frac{1}{fg}(X(Y) - Y(X)) \\
&= \frac{1}{f}\tilde{Y}(f)\tilde{X} - \frac{1}{g}\tilde{X}(g)\tilde{Y}
\end{aligned}$$

Where in going from line 2 to 3 we have used $X(g^{-1}) = X^\mu \partial_\mu (g^{-1}) = \frac{-1}{g^2}X(g)$, and in going to the last line the fact that for coordinate vector fields, the commutator vanishes. Now lets see what the covariant derivative would look like

$$\nabla_{[\tilde{X},\tilde{Y}]} = \nabla_{\frac{1}{f}\tilde{Y}(f)\tilde{X} - \frac{1}{g}\tilde{X}(g)\tilde{Y}} = \frac{1}{f}\tilde{Y}(f)\nabla_{\tilde{X}} - \frac{1}{g}\tilde{X}(g)\nabla_{\tilde{Y}}$$

Rearranging our last term in the $R(X,Y)$ equation gives us

$$\begin{aligned}
R(X,Y) &= fg\left(\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}} - \left(-\frac{1}{g}\tilde{X}(g)\nabla_{\tilde{Y}} + \frac{1}{f}\tilde{Y}(f)\nabla_{\tilde{X}}\right)\right) \\
&= fg\left(\nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}} - \nabla_{[\tilde{X},\tilde{Y}]}\right)
\end{aligned}$$

Since we also know that $R(X,Y) = fgR(\tilde{X},\tilde{Y})$, we can note that

$$R(\tilde{X},\tilde{Y}) = \nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}} - \nabla_{[\tilde{X},\tilde{Y}]}$$

3. Consider a two dimensional cone. Use polar coordinates with the origin taken to be the top of the cone (the singular point of this 'manifold' - it has manifold structure everywhere except at the tip of the cone). Write down the metric on the surface of the cone induced by the Euclidean metric of the three-dimensional space in which the cone lives. Find the geodesics between two points an angle $\delta\phi$ apart at the same radius.

Solution

The surface of a cone with apex at the origin (opening upwards) is

$$z = c\sqrt{x^2 + y^2} \tag{4}$$

Where c is a positive real constant. Since we are embedding this two dimensional surface in three dimensions, it is prudent to convert to spherical coordinates.

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{5}$$

For this construction, the polar angle ϕ ranges from 0 to 2π, and θ will remain fixed, at some value called θ_0, which will control the steepness of the cone in 3 dimensions. Indeed, in spherical coordinates a cone is defined by fixing θ to be constant. This angle is related to c by $\theta_0 = \cot^{-1}(c)$. Holding θ constant, our differentials become

$$\begin{aligned} dx &= \sin\theta_0\cos\phi dr - r\sin\theta_0\sin\phi d\phi \\ dy &= \sin\theta_0\sin\phi dr + r\sin\theta_0\cos\phi d\phi \\ dz &= \cos\theta_0 dr \end{aligned}$$

Since the surface of a cone can be defined by identifying two edges of a partial circle together, say at $\phi = 0$ and $\phi = \beta$ (so that $\phi = \phi + \beta$), we will look at the flat space metric. The two dimensional metric induced by the three dimensional flat space it lives in is simply

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\sin^2\theta_0 d\phi^2 \tag{6}$$

by substitution of our computed differentials. For clarity, it can be useful to visualize the situation by considering an upward opening cone with θ_0 being the angle from the z axis to an edge of the cone in 3 dimensional space. We now wish to determine geodesics in this geometry, so we must first determine the nonzero Christoffel symbols. The nonzero elements of our induced metric are

$$g_{rr} = 1 \qquad g_{\phi\phi} = r^2\sin^2\theta_0 \tag{7}$$

Now, if we had chosen cylindrical coordinates to represent our cone ($x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$), we would get a slightly different metric (note now that ρ is the radius vector in the xy plane, $rho^2 = x^2 + y^2$). Using the equation for a cone, we note that we still have $z = c\rho = \rho\cot\theta_0$, and so our differentials become

$$\begin{aligned} dx &= \cos\phi d\rho - \rho\sin\phi d\phi \\ dy &= \sin\phi d\rho + \rho\cos\phi d\phi \\ dz &= \cot\theta_0 d\rho \end{aligned}$$

Combining these gives the line element

$$ds^2 = (1 + \cot^2\theta_0)d\rho^2 + \rho^2 d\phi^2 = \csc^2\theta_0 d\rho^2 + \rho^2 d\phi^2 \tag{8}$$

Thereby inducing metric elements as

$$g_{\rho\rho} = \csc^2\theta_0 \qquad g_{\phi\phi} = \rho^2 \tag{9}$$

We will continue working in our originally chosen spherical system, but the two coordinates are simply related by $\sin\theta_0 = \rho/r$

Recall the connection coefficient equation is

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) \tag{10}$$

A quick computation gives the nonzero connection coefficients as

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta_0 \qquad\qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$$

Recall now the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{11}$$

So we have the equations

$$\frac{d^2 r}{d\lambda^2} - r\sin^2\theta_0 \left(\frac{d\phi}{d\lambda}\right)^2 = 0$$
$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

We are interested in determining the orbit $r(\phi)$, since we are interested in a geodesic at constant r separated by $\delta\phi$. For this, we go to the geodesic equations and eliminate the λ variable. Looking at the first equation, we can write $\ddot{r} = \frac{d}{d\lambda}\left(\frac{dr}{d\phi}\frac{d\phi}{d\lambda}\right)$ to find a new form

$$\dot{\phi}^2 \frac{d^2 r}{d\phi^2} + \ddot{\phi}\frac{dr}{d\phi} - r\sin^2\theta_0 \dot{\phi}^2 = 0 \tag{12}$$

Where the overdots represent derivatives with respect to λ.

Now recall the other geodesic equation for ϕ

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0$$

Rewriting $\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda}$, we can rearrange this to be

$$\ddot{\phi} = -\frac{2}{r}\frac{dr}{d\phi}\dot{\phi}^2$$

Substituting this form into the first geodesic equation yields

$$\dot{\phi}\left(\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r\sin^2\theta_0\right) = 0 \tag{13}$$

Since we are interested in geodesics which will vary in ϕ ($\dot{\phi} \neq 0$), we merely need to solve the internal equation,

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r\sin^2\theta_0 = 0 \tag{14}$$

The solution is

$$r(\phi) = D\sec((\phi + E)\sin\theta_0)$$

Where D and E are integration constants as usual, which we will set by consider a specific geodesic. The geodesic to consider is one which starts and ends at the same radial distance, shifted by an angle $\delta\phi$. This condition is

$$r(\phi_0) = r(\phi_0 + \delta\phi) = r_0 \tag{15}$$

In order for this to occur, there must be a critical point in the interval between ϕ_0 and $\phi_0 + \delta\phi$ so the geodesic can loop back to the same distance, r_0. We can find

$$\frac{d}{d\phi}r(\phi) = D\sin\theta_0(\sec((\phi + E)\sin\theta_0)\tan((\phi + E)\sin\theta_0))$$

This hits a minimum when $(\phi_c + E)\sin\theta_0 = 0$, or $E = -\phi_c$ (assuming $\sin\theta_0 \neq 0$). We can now use the symmetry of our geodesic (must take half the distance to go out, and half the distance to come back into the original r_0 to state that $\phi_c = \phi_0 + \delta\phi/2$, so we can find $E = -\phi_0 - \delta\phi/2$. Using our initial condition, we can also set D by

$$\begin{aligned} r(\phi_0) = D\sec\left(\left(\phi_0 - \phi_0 - \frac{\delta\phi}{2}\right)\sin\theta_0\right) &= r_0 \\ D &= r_0\cos\left(\frac{\delta\phi\sin\theta_0}{2}\right) \end{aligned}$$

So the geodesics beginning and ending at a radius r_0 are of the form

$$\begin{aligned} r(\phi) &= r_0\cos\left(\frac{\delta\phi\sin\theta_0}{2}\right)\sec\left(\left(\phi - \phi_0 - \frac{\delta\phi}{2}\right)\sin\theta_0\right) \\ &= r_0\frac{\cos((\delta\phi/2)\sin\theta_0)}{\cos((\phi - \phi_0 - \delta\phi/2)\sin\theta_0} \end{aligned}$$

4. Consider the metric

$$ds^2 = dt^2 - a(t)^2[dx^2 + dy^2 + dz^2] \tag{16}$$

where $a(t)$ is an increasing function of time (and x, y, z are Euclidean spatial coordinates), which describes a homogenous and isotropic expanding universe.

a) Compute the Christoffel symbols (check your answers with those in the text).

b) Write down the equation of motion of a point particle in this metric in the absence of external forces and derive the time dependence of the physical velocity. Comment on the result.

Solution

a) The nonzero elements of the metric are given by

$$g_{tt} = 1 \qquad g_{xx} = g_{yy} = g_{zz} = -(a(t))^2 \tag{17}$$

The connection symbols are computed in the usual way, using

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{18}$$

Thus you can show that the only nonzero connection coefficients are given by

$$\Gamma^0_{ij} = \dot{a}a\delta_{ij} \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j \tag{19}$$

b) To get the equation of motion for a point particle we must consider the geodesic equation. This is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{20}$$

Where τ is the proper time. Expanding this we find

$$\begin{aligned} \frac{d^2t}{d\tau^2} + \dot{a}a\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^J}{d\tau} &= 0 \\ \frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} &= 0 \end{aligned}$$

Recall the four velocity is $u^\mu = dx^\mu/d\tau$, so we can write this in a slightly simpler form

$$\begin{aligned} \frac{du^0}{d\tau} + \dot{a}a\sum_{i=1}^{3}(u^i)^2 &= 0 \\ \frac{du^i}{d\tau} + 2\frac{\dot{a}}{a}u^0u^i &= 0 \end{aligned}$$

The total physical velocity is thus

$$u = \sqrt{-u_iu^i} = \sqrt{-g_{ij}u^iu^j} = a(t)\sqrt{\delta_{ij}u^iu^j} \tag{21}$$

So we can write

$$u^2 = (a(t))^2\sum_{i=1}^{3}(u^i)^2 \tag{22}$$

Which we will use momentarily. Also recall that $u^\mu u_\mu = 1$, so we have

$$1 = u_\mu u^\mu = u_0 u^0 + u_i u^i = (u^0)^2 - u^2 = 1 \tag{23}$$

Differentiation yields the relationship

$$u^0 du^0 = u du \tag{24}$$

Substituting our expression for u^2 into the geodesic equations yields

$$\begin{aligned} \frac{du^0}{d\tau} + \frac{\dot{a}}{a} u^2 &= 0 \\ \frac{du^i}{d\tau} + 2\frac{\dot{a}}{a} u^0 u^i &= 0 \end{aligned}$$

Now realize the useful substitution

$$\frac{du^0}{d\tau} = \frac{du^0}{dt}\frac{dt}{d\tau} = u^0 \frac{du^0}{dt} = u\dot{u} \tag{25}$$

We are now in a position to solve our first differential equation. With this substitution, we have

$$\dot{u} + \frac{\dot{a}}{a} u = 0 \qquad (u \neq 0) \tag{26}$$

This expression is easily solved

$$\begin{aligned} \frac{du}{dt}\frac{1}{u} &= -\frac{da}{dt}\frac{1}{a} \\ \int \frac{du}{u} &= -\int \frac{da'}{a'} \\ \ln(u) &= -\ln(a/a_0) \\ u(t) &= \frac{a_0}{a(t)} \end{aligned}$$

Where a_0 is a real and positive constant of integration. We note that this implies the physical velocity of a particle on a manifold specified by the above metric will be decreasing with time, since $a(t) > a_0$.

5. Compute the Riemann tensor elements of the metric in the previous problem

Solution

The Riemann tensor is defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{27}$$

and so our first step should be to look at the nonzero connection coefficients defined in the previous solution. They were

$$\Gamma^{0}_{ij} = \dot{a}a\delta_{ij} \qquad \Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \frac{\dot{a}}{a}\delta^{i}_{j} \tag{28}$$

Where a is a function of time. We are going to want to make use of some symmetries of the Riemann tensor, but these are most manifest in the following 'all lower' form, so we will actually look to compute

$$R_{\alpha\sigma\mu\nu} = g_{\rho\alpha}R^{\rho}_{\sigma\mu\nu} = g_{\rho\alpha}(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}) \tag{29}$$

We note that since our metric is diagonal ($g_{00} = 1$, $g_{ii} = -a^2$ where $i = 1, 2, 3$), nonzero components will be restricted to $\rho = \alpha$. The symmetries we will use to exploit elements of the tensor will be

$$\begin{aligned}
R_{\alpha\sigma\mu\nu} &= R_{\mu\nu\alpha\sigma} && (i)\\
R_{\alpha\sigma\mu\nu} &= -R_{\alpha\sigma\nu\mu} && (ii)\\
R_{\alpha\sigma\mu\nu} &= -R_{\sigma\alpha\mu\nu} && (iii)\\
R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} &= 0 && (iv)
\end{aligned}$$

These are derived and discussed in Carroll. Now with this in mind, it can be nice to consider two sets of lower indexed tensors because of the symmetry of our metric. We will consider finding the nonzero elements corresponding to $R_{0\sigma\mu\nu}$ and $R_{i\sigma\mu\nu}$ in a systematic way, since determining the nonzero elements of both will give us all the nonzero elements of the full Riemann tensor. Lets start with $R_{0\sigma\mu\nu}$. This can be written as

$$R_{0\sigma\mu\nu} = \partial_\mu \Gamma^{0}_{\nu\sigma} - \partial_\nu \Gamma^{0}_{\mu\sigma} + \Gamma^{0}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{0}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

The most straightforward way to check if the tensor components are nonzero is to consider when each term individually is nonzero. Lets start with the first term, $\partial_\mu \Gamma^{0}_{\nu\sigma}$. This term is nonzero when $\mu = 0$ (since our connection coefficients depend on time only) and when $\nu = i$ and $\sigma = j$. This gives us a tensor element of

$$\begin{aligned}
R_{0j0i} &= \partial_0 \Gamma^{0}_{ij} - \partial_i \Gamma^{0}_{0j} + \Gamma^{0}_{0\lambda}\Gamma^{\lambda}_{ij} - \Gamma^{0}_{i\lambda}\Gamma^{\lambda}_{0j}\\
&= \frac{d}{dt}(a\dot{a}\delta_{ij}) - 0 + 0 - \Gamma^{0}_{ik}\Gamma^{k}_{0j}\\
&= a\ddot{a}\delta_{ij} + \dot{a}^2\delta_{ij} - (a\dot{a}\delta_{ik})\left(\frac{\dot{a}}{a}\delta^{k}_{j}\right)\\
&= a\ddot{a}\delta_{ij}
\end{aligned}$$

So we have determined the components for R_{0j0i}! We can now utilize symmetry arguments to get a few other components, lets make up a short list of what we have now found

$$\begin{aligned}
R_{0j0i} &= a\ddot{a}\delta_{ij} & R_{j00i} &= -a\ddot{a}\delta_{ij}\\
R_{0ji0} &= -a\ddot{a}\delta_{ij} & R_{00ij} &= 0
\end{aligned}$$

Where the bottom right expression comes from identity (iv), like

$$R_{0j0i} + R_{00ij} + R_{0ij0} = 0 \tag{30}$$

Now lets move on to what it takes to keep the second term nonzero. We have $\partial_\nu \Gamma^0_{\mu\sigma}$, which means we need $\nu = 0$, $\mu = i$, and $\sigma = j$. This means we are looking to compute R_{0ji0}, but wait, we have already determined what the answer is for this component! Its simply $R_{0ji0} = -a\ddot{a}\delta_{ij}$ from our symmetry arguments above. So we don't have to explicitly compute it.

Lets move onto the third term. This is $\Gamma^0_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} = \Gamma^0_{\mu 0}\Gamma^0_{\nu\sigma} + \Gamma^0_{\mu k}\Gamma^k_{\nu\sigma}$. The first term here is always 0 because of the connection coefficients, and the second term is nonzero for $\mu = i$, $\nu = 0$ OR j, $\sigma = j$ OR 0 (respectively). We have a couple options to check here. For our first scenario, we are looking to compute $R_{0\sigma\mu\nu} \to R_{0ji0}$, which we just determined from above to be equal to $-a\ddot{a}\delta_{ij}$, so no explicit calculation is necessary. The second set of options puts us in $R_{0\sigma\mu\nu} \to R_{00ij}$, and we have this from our symmetry arguments already, as simply being 0.

Next we check the fourth and last term in our expression. This is $\Gamma^0_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} = \Gamma^0_{\nu 0}\Gamma^0_{\mu\sigma} + \Gamma^0_{\nu k}\Gamma^k_{\mu\sigma}$. Like before, the first term here is always 0, so we needn't worry about it. The second term is nonzero when $\nu = i$, $\mu = 0$ OR j, and $\sigma = j$ OR 0 respectively. This means we are computing first $R_{0\sigma\mu\nu} \to R_{0j0i}$, which we know from above is $a\ddot{a}\delta_{ij}$. The second case puts us at $R_{0\sigma\mu\nu} \to R_{00ji}$ which has been determined to be 0. Somewhat miraculously, we only had to perform one explicit calculation to get all the Riemann tensor coefficients for $R_{0\sigma\mu\nu}$. Now we must move onto the other case.

We are now cast with the duty of checking for nonzero components of the expression

$$R_{k\sigma\mu\nu} = -a^2(\partial_\mu \Gamma^k_{\nu\sigma} - \partial_\nu \Gamma^k_{\mu\sigma} + \Gamma^k_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^k_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})$$

We will do this in the same way as before, by checking each term individually. Start again with the first term, $\partial_\mu \Gamma^k_{\nu\sigma}$. This is nonzero for $\mu = 0$, $\nu = 0$ OR i, and $\sigma = j$ OR 0. We are left with two possible cases, first R_{kj00} which is clearly the same as R_{ij00}, and we know this is 0 from above. Second, check R_{k00i} which is just $-a\ddot{a}\delta_{ik}$ from above.

Now lets check the second term, $\partial_\nu \Gamma^k_{\mu\sigma}$. This is nonzero for $\nu = 0$, $\mu = 0$ OR i, and $\sigma = j$ OR 0 respectively. This yields the terms $R_{k\sigma\mu\nu} \to R_{kj00}$ which is 0, and $R_{k\sigma\mu\nu} \to R_{k0i0}$ which is once again $a\ddot{a}\delta_{ik}$.

Next term is $\Gamma^k_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} = \Gamma^k_{\mu 0}\Gamma^0_{\nu\sigma} + \Gamma^k_{\mu m}\Gamma^m_{\nu\sigma}$ where m is another index running from $1-3$. Things are a little more interesting here, as we now have three possible options. If $\mu = 0$, $\nu = 0$ OR i, $\sigma = j$ OR 0 respectively, the first term is 0 but the second term is nonzero. These two choices will yield $R_{k\sigma\mu\nu} \to R_{kj00} = 0$ and $R_{k\sigma\mu\nu} \to R_{k00i} = -a\ddot{a}\delta_{ik}$. The other case is the interesting one, where if $\mu = i$, $\nu = j$, and $\sigma = n$ the first term is nonzero. This yields

$$\begin{aligned}
R_{knij} &= -a^2(\partial_i \Gamma^k_{jn} - \partial_j \Gamma^k_{in} + \Gamma^k_{i\lambda}\Gamma^\lambda_{jn} - \Gamma^k_{j\lambda}\Gamma^\lambda_{in}) \\
&= -a^2(0 - 0 + \Gamma^k_{i0}\Gamma^0_{jn} + \Gamma^k_{im}\Gamma^m_{jn} - \Gamma^k_{j0}\Gamma^0_{in} - \Gamma^k_{jm}\Gamma^m_{in}) \\
&= -a^2(\left(\frac{\dot{a}}{a}\delta_{ik}\right)(a\dot{a}\delta_{jn}) + 0 - \left(\frac{\dot{a}}{a}\delta_{jk}\right)(a\dot{a}\delta_{in}) - 0) \\
&= a^2\dot{a}^2(\delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn})
\end{aligned}$$

So $R_{jiij} = a^2\dot{a}^2$ and $R_{ijij} = -a^2\dot{a}^2$, so the antisymmetric and symmetric properties of this one are manifest.

Finally, the last term is $\Gamma^k_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} = \Gamma^k_{\nu 0}\Gamma^0_{\mu\sigma} + \Gamma^k_{\nu m}\Gamma^m_{\mu\sigma}$. We have three options once again, as before. Keeping the second term alive requires $\nu = 0$, $\mu = 0$ OR i, and $\sigma = j$ OR 0. This means we are computing

$R_{k\sigma\mu\nu} \to R_{kj00} = 0$ and $R_{k\sigma\mu\nu} \to R_{k0i0} = a\ddot{a}\delta_{ik}$. To keep the first term alive requires $\nu = i$, $\mu = j$, and $\sigma = n$. This boils down to computing R_{knji}, which we know from above to be $R_{knji} = -a^2\dot{a}^2(\delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn})$. So $R_{jiji} = -a^2\dot{a}^2$ and $R_{ijji} = a^2\dot{a}^2$ which is consistent with what we saw prior. Our new updated list of Riemann lower tensor coefficients is then

$$
\begin{aligned}
R_{0j0i} &= a\ddot{a}\delta_{ij} \qquad & R_{j00i} &= -a\ddot{a}\delta_{ij} \\
R_{0ji0} &= -a\ddot{a}\delta_{ij} \qquad & R_{00ij} &= 0 \\
R_{knij} &= a^2\dot{a}^2(\delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn}) \\
R_{knji} &= -a^2\dot{a}^2(\delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn}) \\
R_{nkij} &= -a^2\dot{a}^2(\delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn})
\end{aligned}
$$

Now lets get it back into the proper form for the Riemann tensor. Recall that $R_{\alpha\sigma\mu\nu} = g_{\rho\alpha}R^{\rho}_{\sigma\mu\nu}$, so $R^{\rho}_{\sigma\mu\nu} = g^{\rho\alpha}R_{\alpha\sigma\mu\nu}$. So the 0 components we calculated wont change, as $g^{00} = 1$, and the k components will pick up a factor of $g^{kk} = -a^{-2}$. The nonzero components of the Riemann tensor are thus

$$
\begin{aligned}
R^{0}_{j0i} &= a\ddot{a}\delta_{ij} \\
R^{j}_{00i} &= \frac{\ddot{a}}{a}\delta^{j}_{i} \\
R^{0}_{ji0} &= -a\ddot{a}\delta_{ij} \\
R^{j}_{0i0} &= -\frac{\ddot{a}}{a}\delta^{j}_{i} \\
R^{k}_{nij} &= -\dot{a}^2(\delta^{k}_{j}\delta_{in} - \delta^{k}_{i}\delta_{jn}) \\
R^{k}_{nji} &= \dot{a}^2(\delta^{k}_{j}\delta_{in} - \delta^{k}_{i}\delta_{jn})
\end{aligned}
$$