Phys 514 - Assignment 6
Solutions

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1. The metric for the three-sphere in coordinates (ψ, θ, ϕ) is

$$ds^2 = d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{1}$$

a) Calculate the Christoffel symbols
b) Calculate the Riemann tensor, Ricci tensor, and Ricci scalar.

Solution

a) Recall that

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

And with our metric, the nonzero metric elements are

$$g_{\psi\psi} = 1 \qquad g_{\theta\theta} = \sin^2\psi \qquad g_{\phi\phi} = \sin^2\psi\sin^2\theta$$

We will approach this systematically, by check the upper index of the connection coefficients first.

$$\Gamma^{\psi}_{\mu\nu} = \frac{1}{2}g^{\psi\psi}(\partial_\mu g_{\nu\psi} + \partial_\nu g_{\psi\mu} - \partial_\psi g_{\mu\nu})$$

The first two terms are 0 since derivatives of $g_{\psi\psi}$ vanish. This gives us two choices for the last term, $\mu = \nu = \theta$ and $\mu = \nu = \phi$. These yield the coefficients

$$\Gamma^{\psi}_{\theta\theta} = -\sin\psi\cos\psi$$
$$\Gamma^{\psi}_{\phi\phi} = -\sin\psi\cos\psi\sin^2\theta$$

That's it for the upper ψ index. Lets move onto θ. We have

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2}g^{\theta\theta}(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\theta\mu} - \partial_\theta g_{\mu\nu})$$

Lets start with the last term. This is only nonzero if $\mu = \nu = \phi$. This yields the symbol

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

This takes care of the ϕ indices. The only other way to get a nonzero symbol is by $\mu = \theta$ or $\nu = \theta$. In the first instance, we get

$$\Gamma^{\theta}_{\theta\nu} = \frac{1}{2} g^{\theta\theta} (\partial_\theta g_{\nu\theta} + \partial_\nu g_{\theta\theta} - \partial_\theta g_{\theta\nu})$$

The first term and the final term are 0 by inspection. The middle term is nonzero if $\nu = \psi$. By the symmetry of the symbols we get

$$\Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} = \cot\psi$$

Finally, lets look at the last upper index. It is

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2} g^{\phi\phi} (\partial_\mu g_{\nu\phi} + \partial_\nu g_{\phi\mu} - \partial_\phi g_{\mu\nu})$$

The final term is always 0 since no part of the metric is ϕ dependent. Now, either μ or ν must be ϕ to get a nonzero symbol. The derivative will alternate between the symmetric indices, and so a straightforward computation yields the final four symbols

$$\begin{aligned} \Gamma^{\phi}_{\psi\phi} &= \Gamma^{\phi}_{\phi\psi} = \cot\psi \\ \Gamma^{\phi}_{\theta\phi} &= \Gamma^{\phi}_{\phi\theta} = \cot\theta \end{aligned}$$

To recap, the nonzero Christoffel symbols here are

$$\begin{aligned} \Gamma^{\psi}_{\theta\theta} &= -\sin\psi\cos\psi \\ \Gamma^{\psi}_{\phi\phi} &= -\sin\psi\cos\psi\sin^2\theta \\ \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} &= \cot\psi \\ \Gamma^{\phi}_{\psi\phi} = \Gamma^{\phi}_{\phi\psi} &= \cot\psi \\ \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \cot\theta \end{aligned}$$

b) The Riemann tensor is defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2}$$

The techniques to find the Riemann tensor were illustrated in one of the solutions for assignment 4, and so we will compute as many as my energy will allow for these Christoffel symbols. We recall the useful expression

$$R_{\rho\sigma\mu\nu} = g_{\lambda\rho} R^{\lambda}_{\sigma\mu\nu}$$

Which allows us to fully exploit the symmetries

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \qquad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \qquad R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \qquad R_{\rho[\sigma\mu\nu]} = 0$$

With this in mind, lets start by computing the upper ψ index Riemann tensor indices. Since $g_{\psi\psi} = 1$, we don't have to do any conversion between the two forms. We get

$$R_{\psi\sigma\mu\nu} = \partial_\mu \Gamma^\psi_{\nu\sigma} - \partial_\nu \Gamma^\psi_{\mu\sigma} + \Gamma^\psi_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\psi_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

Now, to keep the first term nonzero, we note that we have three options. First, we can set $\nu = \sigma = \theta$ and $\mu = \psi$ to find

$$\begin{aligned} R_{\psi\theta\psi\theta} &= \partial_\psi(-\sin\psi\cos\psi) - \partial_\theta \Gamma^\psi_{\psi\theta} + \Gamma^\psi_{\psi\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^\psi_{\theta\lambda}\Gamma^\lambda_{\psi\theta} \\ &= \sin^2\psi - \cos^2\psi + \cos^2\psi \\ &= \sin^2\psi \end{aligned}$$

The middle two terms are 0 in the top line. By the symmetries, we can write a few more terms

$$R_{\psi\theta\psi\theta} = \sin^2\psi \qquad R_{\theta\psi\psi\theta} = R_{\psi\theta\theta\psi} = -\sin^2\psi \qquad R_{\psi\psi\theta\theta} = 0$$

As our second check, we note that choosing $\nu = \sigma = \phi$ and $\mu = \psi$ yield a nonzero first term as well. This expression is

$$\begin{aligned} R_{\psi\phi\psi\phi} &= \partial_\psi(-\sin\psi\cos\psi\sin^2\theta) - \partial_\phi \Gamma^\psi_{\psi\phi} + \Gamma^\psi_{\psi\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\psi_{\phi\lambda}\Gamma^\lambda_{\psi\phi} \\ &= \sin^2\theta(\sin^2\psi - \cos^2\psi) + \cos^2\psi\sin^2\theta \\ &= \sin^2\theta\sin^2\psi \end{aligned}$$

The symmetries then yield

$$R_{\psi\phi\psi\phi} = \sin^2\theta\sin^2\psi \qquad R_{\phi\psi\psi\phi} = R_{\psi\phi\phi\psi} = -\sin^2\theta\sin^2\psi \qquad R_{\psi\psi\phi\phi} = 0$$

Finally, the last option to keep the first term nonzero is $\nu = \sigma = \phi$ with $\mu = \theta$ This yields

$$\begin{aligned} R_{\psi\phi\theta\phi} &= \partial_\theta(-\sin\psi\cos\psi\sin^2\theta) - \partial_\phi \Gamma^\psi_{\theta\phi} + \Gamma^\psi_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\psi_{\phi\lambda}\Gamma^\lambda_{\theta\phi} \\ &= -2\sin\psi\cos\psi\sin\theta\cos\theta + \sin\psi\cos\psi\sin\theta\cos\theta + \sin\psi\cos\psi\sin\theta\cos\theta \\ &= 0 \end{aligned}$$

So this one vanishes, as well as all its forms related by symmetries.

Now we move to the second term, $\partial_\nu \Gamma^\psi_{\mu\sigma}$. We have multiple choices again, so first, set $\mu = \sigma = \theta$ and $\nu = \psi$. This means we would be computing $R_{\psi\theta\theta\psi}$. We already know what this is from above! Its simply $-\sin^2\theta\sin^2\psi$, thus no computation is necessary. Moving on, we can choose $\mu = \sigma = \phi$ and $\nu = \rho$. This is the computation of $R_{\psi\phi\phi\psi}$ which is also done. Finally, the last thing that keeps the second term nonzero is $\mu = \sigma = \phi$ with $\nu = \theta$. This computes $R_{\psi\phi\phi\theta}$, which is 0 by an antisymmetric transformation of our previous result. Thus, we didn't have to perform any explicit calculation for this term!

Moving now to the third term,

$$\Gamma^\psi_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} = \Gamma^\psi_{\mu\psi}\Gamma^\psi_{\nu\sigma} + \Gamma^\psi_{\mu\theta}\Gamma^\theta_{\nu\sigma} + \Gamma^\psi_{\mu\phi}\Gamma^\phi_{\nu\sigma} = 0 + \Gamma^\psi_{\mu\theta}\Gamma^\theta_{\nu\sigma} + \Gamma^\psi_{\mu\phi}\Gamma^\phi_{\nu\sigma}$$

We have options here, first, we can set $\mu = \theta$ and $\nu = \sigma = \phi$. This will mean we compute $R_{\psi\phi\theta\phi}$ We have computed this already, and its 0. Next option, set $\mu = \theta$, $\nu = \theta$, and $\sigma = \psi$. This computation is of $R_{\psi\psi\theta\theta}$, also 0. Our next choice is $\mu = \phi$, $\nu = \psi$, $\sigma = \phi$. This is computing $R_{\psi\phi\phi\psi} = -\sin^2\theta\sin^2\psi$ from symmetry. Finally, we can also make the choice $\mu = \phi$, $\nu = \theta$, $\sigma = \phi$. This yields $R_{\psi\phi\phi\theta}$, which is 0 once again by symmetry.

Lets move onto the final term, expanded it is

$$\Gamma^{\psi}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} = \Gamma^{\psi}_{\nu\psi}\Gamma^{\psi}_{\mu\sigma} + \Gamma^{\psi}_{\nu\theta}\Gamma^{\theta}_{\mu\sigma} + \Gamma^{\psi}_{\nu\phi}\Gamma^{\phi}_{\mu\sigma} = 0 + \Gamma^{\psi}_{\nu\theta}\Gamma^{\theta}_{\mu\sigma} + \Gamma^{\psi}_{\nu\phi}\Gamma^{\phi}_{\mu\sigma}$$

We have four choices again, so lets speed through them. First, choose $\nu = \theta$, $\mu = \sigma = \phi$. This corresponds to computing $R_{\psi\phi\phi\theta}$, again 0. Next choose $\nu = \theta$, $\mu = \theta$, $\sigma = \psi$. This computes $R_{\psi\psi\theta\theta} = 0$. Next choice would be $\mu = \phi$, $\nu = \psi$, $\sigma = \phi$. This computes $R_{\psi\phi\phi\psi}$, already computed above. Finally, choose $\nu = \phi$, $\mu = \theta$, $\sigma = \phi$. This computes $R_{\psi\phi\theta\phi} = 0$.

This concludes the computation of the Riemann symbols for the upper index of ψ. As you can see, (and I'm sure you know after doing it yourself) it is a bit of a slow process. I will quote the results of the other nonzero Riemann tensor elements here, but rest assured they come from the exact same process as above but for upper indices θ and ϕ instead. Our nonzero elements are

$$\begin{aligned}
R_{\psi\theta\psi\theta} &= R_{\theta\psi\theta\psi} = \sin^2\psi & \qquad R_{\theta\psi\psi\theta} &= R_{\psi\theta\theta\psi} = -\sin^2\psi \\
R_{\psi\phi\psi\phi} &= R_{\phi\psi\phi\psi} = \sin^2\theta\sin^2\psi & \qquad R_{\phi\psi\psi\phi} &= R_{\psi\phi\phi\psi} = -\sin^2\theta\sin^2\psi \\
R_{\theta\phi\theta\phi} &= R_{\phi\theta\phi\theta} = \sin^2\theta\sin^4\psi & \qquad R_{\phi\theta\theta\phi} &= R_{\theta\phi\phi\theta} = -\sin^2\theta\sin^4\psi
\end{aligned}$$

To get back to the proper form, we take the transformation $R^{\rho}_{\sigma\mu\nu} = g^{\rho\lambda}R_{\lambda\sigma\mu\nu}$, where we have

$$g^{\psi\psi} = 1 \qquad g^{\theta\theta} = \frac{1}{\sin^2\psi} \qquad g^{\phi\phi} = \frac{1}{\sin^2\psi\sin^2\theta}$$

With these transforms, we get our final Riemann tensor elements.

$$\begin{aligned}
R^{\psi}_{\theta\psi\theta} &= \sin^2\psi & R^{\psi}_{\theta\theta\psi} &= -\sin^2\psi & R^{\psi}_{\phi\psi\phi} &= \sin^2\theta\sin^2\psi & R^{\psi}_{\phi\phi\psi} &= -\sin^2\theta\sin^2\psi \\
R^{\theta}_{\psi\theta\psi} &= 1 & R^{\theta}_{\psi\psi\theta} &= -1 & R^{\theta}_{\phi\theta\phi} &= \sin^2\theta\sin^2\psi & R^{\theta}_{\phi\phi\theta} &= -\sin^2\theta\sin^2\psi \\
R^{\phi}_{\psi\phi\psi} &= 1 & R^{\phi}_{\psi\psi\phi} &= -1 & R^{\phi}_{\theta\phi\theta} &= \sin^2\psi & R^{\phi}_{\theta\theta\phi} &= -\sin^2\psi
\end{aligned}$$

The Ricci tensor is determined by a contraction of the Riemann tensor, $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ Since we know this is a symmetric tensor in three dimensions, we can calculate the six independent components individually without much work.

$$\begin{aligned}
R_{\psi\psi} &= R^{\psi}_{\psi\psi\psi} + R^{\theta}_{\psi\theta\psi} + R^{\phi}_{\psi\phi\psi} \\
&= 0 + 1 + 1 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
R_{\theta\theta} &= R^{\psi}_{\theta\psi\theta} + R^{\theta}_{\theta\theta\theta} + R^{\phi}_{\theta\phi\theta} \\
&= \sin^2\psi + 0 + \sin^2\psi \\
&= 2\sin^2\psi
\end{aligned}$$

$$
\begin{aligned}
R_{\phi\phi} &= R^{\psi}_{\phi\psi\phi} + R^{\theta}_{\phi\theta\phi} + R^{\phi}_{\phi\phi\phi} \\
&= \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta \\
&= 2\sin^2\psi\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
R_{\psi\theta} &= R^{\psi}_{\psi\psi\theta} + R^{\theta}_{\psi\theta\theta} + R^{\phi}_{\psi\phi\theta} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
R_{\psi\phi} &= R^{\psi}_{\psi\psi\phi} + R^{\theta}_{\psi\theta\phi} + R^{\phi}_{\psi\phi\phi} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\phi} &= R^{\psi}_{\theta\psi\phi} + R^{\theta}_{\theta\theta\phi} + R^{\phi}_{\theta\phi\phi} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

So the independent components of our Ricci tensor are

$$
\begin{aligned}
R_{\psi\psi} &= 1 \\
R_{\theta\theta} &= 2\sin^2\psi \\
R_{\phi\phi} &= 2\sin^2\psi\sin^2\theta \\
R_{\psi\theta} &= R_{\psi\phi} = R_{\theta\phi} = 0
\end{aligned}
$$

Our Ricci scalar is defined as $R = g^{\mu\nu}R_{\mu\nu}$. Using the inverse metric written above, this yields

$$
\begin{aligned}
R &= g^{\psi\psi}R_{\psi\psi} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\
&= 2 + 2 + 2 \\
&= 6
\end{aligned}
$$

2. Do the same calculation using the tetrad basis.

Solution

Recall our line element from the previous problem

$$ds^2 = d\psi^2 + \sin^2\psi d\theta + \sin^2\psi \sin^2\theta d\phi^2$$

To make use of the tetrad formalism, we would like to have a line element that looks like

$$ds^2 e^a e^b \delta_{ab}$$

Where a, b run from 1 to 3. This allows us to define our tetrad basis. It will be

$$e^\psi = d\psi \qquad e^\theta = \sin\psi d\theta \qquad e^\phi = \sin\psi \sin\theta d\phi$$

Note that we are now in a noncoordinate basis. The appendix J of the book provides a good background on the application of the tetrad formalism, so refer to it if you are having any confusion. We wish to compute the Riemann tensor, which by equation $J.29$ is

$$R^a_b = \mathrm{d}\omega^{\mathrm{a}}_{\mathrm{b}} + \omega^{\mathrm{a}}_{\mathrm{c}} \wedge \omega^{\mathrm{c}}_{\mathrm{b}} \tag{3}$$

If we expect to find this, we had better start by first computing the spin connection, ω. Note that the Riemann tensor in the above expression has been expressed in a basis of one forms, so $R^a_b = R^a_{b\mu\nu}\mathrm{dx}^\mu \mathrm{dx}^\nu$. We can determine the spin connection by the expression

$$\mathrm{de}^{\mathrm{a}} = \mathrm{e}^{\mathrm{b}} \wedge \omega^{\mathrm{a}}_{\mathrm{b}}$$

Note that the spin connections are antisymmetric (see appendix J), so $\omega^a_a = 0$. Clearly we need the differential of our basis to compute the spin connections, but at least that we can do! This basis is

$$\begin{aligned}
\mathrm{de}^\psi &= 0 \\
\mathrm{de}^\theta &= \cos\psi d\psi \wedge d\theta \\
\mathrm{de}^\phi &= \cos\psi \sin\theta d\psi \wedge d\phi + \sin\psi \cos\theta d\theta \wedge d\phi
\end{aligned}$$

Lets write down our three equations now, and deduce the elements of the spin connection.

$$\begin{aligned}
0 &= \sin\psi d\theta \wedge \omega^\psi_\theta + \sin\psi \sin\theta d\phi \wedge \omega^\psi_\phi \\
\cos\psi d\psi \wedge d\theta &= d\psi \wedge \omega^\theta_\psi + \sin\psi \sin\theta d\phi \wedge \omega^\theta_\phi \\
\cos\psi \sin\theta d\psi \wedge d\phi + \sin\psi \cos\theta d\theta \wedge d\phi &= d\psi \wedge \omega^\phi_\psi + \sin\psi d\theta \wedge \omega^\phi_\theta
\end{aligned}$$

From the third line, comparing the left and right hand sides, we can see that $\omega^\phi_\theta = \cos\theta d\phi$ and $\omega^\phi_\psi = \cos\psi \sin\theta d\phi$. Now from the second line we can see that $\omega^\theta_\psi = \cos\psi d\theta$. Since the spin connection is a 3×3 antisymmetric object, there are only three independent components, which we have found. For clarity, they are

$$\begin{aligned}\omega^{\phi}_{\theta} &= \cos\theta d\phi\\ \omega^{\phi}_{\psi} &= \cos\psi\sin\theta d\phi\\ \omega^{\theta}_{\psi} &= \cos\psi d\theta\end{aligned}$$

In order to compute the Riemann tensor, we must take the differential of these objects as well. They are

$$\begin{aligned}\mathrm{d}\omega^{\phi}_{\theta} &= -\sin\theta d\theta\wedge d\phi\\ \mathrm{d}\omega^{\phi}_{\psi} &= -\sin\psi\sin\theta d\psi\wedge d\phi + \cos\psi\cos\theta d\theta\wedge d\phi\\ \mathrm{d}\omega^{\theta}_{\psi} &= -\sin\psi d\psi\wedge d\theta\end{aligned}$$

From here, we can finally compute the Riemann tensor, $R^a_b = \mathrm{d}\omega^{\mathrm{a}}_{\mathrm{b}} + \omega^{\mathrm{a}}_{\mathrm{c}}\wedge\omega^{\mathrm{c}}_{\mathrm{b}}$. We note this object is also antisymmetric in a and b, so we only need to compute the three independent components once again.

$$\begin{aligned}R^{\psi}_{\theta} &= \mathrm{d}\omega^{\psi}_{\theta} + \omega^{\psi}_{\phi}\wedge\omega^{\phi}_{\theta}\\ &= \sin\psi d\psi\wedge d\theta - \cos\theta\sin\theta\cos\psi d\phi\wedge d\phi\\ &= \sin\psi d\psi\wedge d\theta\end{aligned}$$

$$\begin{aligned}R^{\psi}_{\phi} &= \mathrm{d}\omega^{\psi}_{\phi} + \omega^{\psi}_{\theta}\wedge\omega^{\theta}_{\phi}\\ &= \sin\psi\sin\theta d\psi\wedge d\phi - \cos\psi\cos\theta d\theta\wedge d\phi + \cos\theta\cos\psi d\theta\wedge d\phi\\ &= \sin\psi\sin\theta d\psi\wedge d\phi\end{aligned}$$

$$\begin{aligned}R^{\theta}_{\phi} &= \mathrm{d}\omega^{\theta}_{\phi} + \omega^{\theta}_{\psi}\wedge\omega^{\psi}_{\phi}\\ &= \sin\theta d\theta\wedge d\phi - \cos^2\psi\sin\theta d\theta\wedge d\phi\\ &= \sin\theta(1-\cos^2\psi)d\theta\wedge d\phi\\ &= \sin\theta\sin^2\psi d\theta\wedge d\phi\end{aligned}$$

Our Riemann tensor is thus

$$\begin{aligned}R^{\psi'}_{\theta'} &= \sin\psi d\psi\wedge d\theta\\ R^{\psi'}_{\phi'} &= \sin\psi\sin\theta d\psi\wedge d\phi\\ R^{\theta'}_{\phi'} &= \sin\theta\sin^2\psi d\theta\wedge d\phi\end{aligned}$$

Where I have introduced the prime notation to specify that these are in the noncoordinate basis. We wish to switch back to the coordinate basis now, since the difficult part of finding the tensor is taken care of. The expression to do so is (equation $J.49$ in the book)

$$R^{\rho}_{\sigma\mu\nu} = e^{\rho}_{a}e^{b}_{\sigma}R^{a}_{b\mu\nu}$$

Recall that we had expressed everything on a basis of one-forms, so we had $e^a = e^a_\mu dx^\mu$. This means that the e^b_σ is the coefficient attached to the differential in our tetrad coordinates, and e^ρ_a the reciprocal (since the metric is diagonal). Note also that the differentials in the wedge product are our basis one-forms, μ and ν, so for example, $R^{\psi'}_{\theta'\psi\theta} = \sin\psi d\psi \wedge d\theta$. Lets compute the upper ψ Riemann metric elements now.

$$\begin{aligned} R^{\psi}_{\sigma\mu\nu} &= e^{\psi}_{\psi'} e^{b}_{\sigma} R^{\psi'}_{b\mu\nu} \\ &= e^{\psi}_{\psi'} e^{\psi'}_{\sigma} R^{\psi'}_{\psi'\mu\nu} + e^{\psi}_{\psi'} e^{\theta'}_{\sigma} R^{\psi'}_{\theta'\mu\nu} + e^{\psi}_{\psi'} e^{\phi'}_{\sigma} R^{\psi'}_{\phi'\mu\nu} \end{aligned}$$

We note that since the tetrad basis is diagonal in the coordinate basis (no cross-terms), we only have three nonzero, diagonal e terms. Since $R^{\psi'}_{\psi'} = 0$, this yields the two Riemann tensor symbols from the last two terms above as

$$\begin{aligned} R^{\psi}_{\theta\mu\nu} &= e^{\psi}_{\psi'} e^{\theta'}_{\theta} R^{\psi'}_{\theta'\mu\nu} \\ &= (1)(\sin\psi)(\sin\psi d\psi \wedge d\theta) \\ R^{\psi}_{\theta\psi\theta} &= \sin^2\psi \end{aligned}$$

For one of them. The other given by the ϕ term

$$\begin{aligned} R^{\psi}_{\phi\mu\nu} &= e^{\psi}_{\psi'} e^{\phi'}_{\phi} R^{\psi'}_{\phi'\mu\nu} \\ &= (1)(\sin\psi\sin\theta)(\sin\psi\sin\theta d\psi \wedge d\phi) \\ R^{\psi}_{\phi\psi\phi} &= \sin^2\psi\sin^2\theta \end{aligned}$$

Moving onto upper θ terms yields

$$\begin{aligned} R^{\theta}_{\psi\mu\nu} &= e^{\theta}_{\theta'} e^{\psi'}_{\psi} R^{\theta'}_{\psi'\mu\nu} \\ &= \left(\frac{1}{\sin\psi}\right)(1)(-\sin\psi d\psi \wedge d\theta) \\ R^{\theta}_{\psi\psi\theta} &= -1 \end{aligned}$$

and

$$\begin{aligned} R^{\theta}_{\phi\mu\nu} &= e^{\theta}_{\theta'} e^{\phi'}_{\phi} R^{\theta'}_{\phi'\mu\nu} \\ &= \left(\frac{1}{\sin\psi}\right)(\sin\psi\sin\theta)(\sin\theta\sin^2\psi d\theta \wedge d\phi) \\ R^{\theta}_{\phi\theta\phi} &= \sin^2\psi\sin^2\theta \end{aligned}$$

Lastly, we compute the upper ϕ components

$$\begin{aligned} R^{\phi}_{\psi\mu\nu} &= e^{\phi}_{\phi'} e^{\psi'}_{\psi} R^{\phi'}_{\psi'\mu\nu} \\ &= \left(\frac{1}{\sin\psi\sin\theta}\right)(1)(-\sin\psi\sin\theta d\psi \wedge d\phi) \\ R^{\phi}_{\psi\psi\phi} &= -1 \end{aligned}$$

and finally

$$
\begin{aligned}
R^{\phi}_{\theta\mu\nu} &= e^{\phi}_{\phi'} e^{\theta'}_{\theta} R^{\phi'}_{\theta'\mu\nu} \\
&= \left(\frac{1}{\sin\psi\sin\theta}\right)(\sin\psi)(-\sin\theta\sin^2\psi d\theta\wedge d\phi) \\
R^{\phi}_{\theta\theta\phi} &= -\sin^2\psi
\end{aligned}
$$

Putting all these symbols together, and exploiting the antisymmetry of the final two indices yields a full Riemann tensor of

$$
\begin{array}{llll}
R^{\psi}_{\theta\psi\theta} = \sin^2\psi & R^{\psi}_{\theta\theta\psi} = -\sin^2\psi & R^{\psi}_{\phi\psi\phi} = \sin^2\theta\sin^2\psi & R^{\psi}_{\phi\phi\psi} = -\sin^2\theta\sin^2\psi \\
R^{\theta}_{\psi\theta\psi} = 1 & R^{\theta}_{\psi\psi\theta} = -1 & R^{\theta}_{\phi\theta\phi} = \sin^2\theta\sin^2\psi & R^{\theta}_{\phi\phi\theta} = -\sin^2\theta\sin^2\psi \\
R^{\phi}_{\psi\phi\psi} = 1 & R^{\phi}_{\psi\psi\phi} = -1 & R^{\phi}_{\theta\phi\theta} = \sin^2\psi & R^{\phi}_{\theta\theta\phi} = -\sin^2\psi
\end{array}
$$

Exactly as we had in the previous problem, thus the Ricci tensor and scalar are obviously the same as before.

3. Consider 3-dimensional (i.e. $2+1$) gravity.
a) How many degrees of freedom are in the Riemann tensor?
b) How many degrees of freedom are in the Ricci tensor?
c) The Riemann tensor can be decomposed into the Ricci tensor and the Weyl tensor (see textbook, Page 130). Using this fact, how many degrees of freedom are in the Weyl tensor?
d) Are there gravity waves (gravity waves are fluctuations of space-time without associated matter perturbations)?

Solution

a) The first the realize when approaching this problem, is that the number of degrees of freedom (or analogously, the number of independent components) is the same in either $R^{\rho}_{\sigma\mu\nu}$ and $R_{\rho\sigma\mu\nu}$. The latter expression has its symmetries manifest, so lets consider how many degrees of freedom are in that object.

Please refer to pg 128 of the textbook for a great derivation on the number of independent degrees of freedom for the Riemann tensor in n dimensional spacetime. The end result is

$$\text{D.O.F.} = \frac{1}{12}\text{n}^2(\text{n}^2 - 1)$$

For 3 dimensional spacetime, we get that the Riemann tensor has 6 degrees of freedom.

b) The Ricci tensor is a symmetric 2 tensor. In three dimensional spacetime, the Ricci tensor has 9 elements, and since the tensor is symmetric, this leaves 6 independent degrees of freedom.

c) The Weyl tensor in n dimensions is defined as

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R$$

The easiest way to see the number of degrees of freedom in the Weyl tensor is to note the decomposition. The question states that you can decompose the Riemann tensor into the Weyl tensor and the Ricci tensor. Since we know from above that the Riemann tensor has 6 degrees of freedom, and the Ricci tensor also has 6 degrees of freedom, this leaves 0 degrees of freedom for the Weyl tensor to possess. Thus, the Weyl tensor has no degrees of freedom and must vanish in $2+1$ dimensions.

$$C_{\rho\sigma\mu\nu} = 0$$

d) Since the Weyl tensor is 0, let us rewrite the Riemann tensor in terms of the Ricci tensor and scalar. We get (in $n = 3$ dimensions)

$$R_{\rho\sigma\mu\nu} = 2(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) - g_{\rho[\mu}g_{\nu]\sigma}R$$

Now, in vacuum we know that Einstein's equations reduce down to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$
$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$$

Now we recall that the Ricci scalar is defined as $R = g^{\mu\nu}R_{\mu\nu}$. Multiply both sides of the above equation by $g^{\mu\nu}$ to find

$$g^{\mu\nu} R_{\mu\nu} = \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R$$
$$R = \frac{3}{2} R$$

Where we have used the fact $g^{\mu\nu} g_{\mu\nu} = 3$ in three dimensions, by definition. This is clearly only satisfied if $R = 0$. This implies that $R_{\mu\nu} = 0$ as well. Now, since the Riemann tensor is defined by the Ricci tensor and scalar, it too must vanish.

The Riemann tensor is a measure of curvature. Having it vanish implies that there is no curvature, so $g_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric. Recall that gravitational waves can be investigated by perturbing a Minkowski metric, such that

$$g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$$

Where $h_{\mu\nu}$ are small perturbations ontop of a Minkowski background. Since we have just shown that the metric goes exactly to a Minkowski one, there can be no extra perturbations, and so it is impossible to get gravitational waves within this theory.

4. Find the solution to Einstein's field equations for a massive point particle (at rest) in $2+1$ dimensional gravity. Hint: Write the stress-tensor for a point particle, make an ansatz for the metric making use of the symmetries of the problem (diagonal, static, polar symmetry), and then find the resulting Riemann tensor. Then solve for the metric.

Solution

The stress energy tensor for a point particle is $T^{00} = m\delta^2(\vec{x})$ in $2+1$ dimensions.

As an ansatz, we take our metric to be

$$ds^2 = e^{2A(r)}dt^2 - dr^2 - e^{2B(r)}d\phi^2$$

Where we have eliminated the parameter attached to the r variable in the same method as used in Carroll, page 194. A, B are arbitrary functions of r, and we will drop the r brackets in what follows. To find the Riemann tensor, we make use of the tetrad formalism. Our tetrad basis is defined as

$$\theta^t = e^A dt \qquad \theta^r = dr \qquad \theta^\phi = e^B d\phi$$

Their differentials are

$$\mathrm{d}\theta^t = A'e^A dr \wedge dt \qquad \mathrm{d}\theta^r = 0 \qquad \mathrm{d}\theta^\phi = B'e^B dr \wedge d\phi$$

Where $A' = dA/dr$. To deduce the spin connections, we must solve $\mathrm{d}\theta^a = \theta^b \wedge \omega^a_b$. Noting the antisymmetry of ω, we can write down our three equations

$$\begin{aligned}
\mathrm{d}\theta^t &= A'e^A dr \wedge dt = dr \wedge \omega^t_r + e^B d\phi \wedge \omega^t_\phi \\
\mathrm{d}\theta^r &= 0 = e^A dt \wedge \omega^\phi_t + e^B d\phi \wedge \omega^r_\phi \\
\mathrm{d}\theta^\phi &= B'e^B dr \wedge d\phi = e^A dt \wedge \omega^\phi_t + dr \wedge \omega^\phi_r
\end{aligned}$$

From these three equations, we can deduce the elements of the spin connection and its exterior derivative

$$\begin{aligned}
\omega^t_r &= A'e^A dt & \mathrm{d}\omega^t_r &= e^A(A'' + A'^2)dr \wedge dt \\
\omega^t_\phi &= 0 & \mathrm{d}\omega^t_\phi &= 0 \\
\omega^r_\phi &= -B'e^B d\phi & \mathrm{d}\omega^r_\phi &= -e^B(B'' + B'^2)dr \wedge d\phi
\end{aligned}$$

To find the elements of the Riemann tensor, we use $R^a_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b$. Doing so yields

$$\begin{aligned}
R^{t'}_{r'} &= e^A(A'' + A'^2)dr \wedge dt \\
R^{t'}_{\phi'} &= -A'B'e^{A+B} dt \wedge d\phi \\
R^{r'}_{\phi'} &= -e^B(B'' + B'^2)dr \wedge d\phi
\end{aligned}$$

Now we can find the nonzero components of the Riemann tensor in a coordinate basis, using the expression $R^\rho_{\sigma\mu\nu} = \theta^\rho_a \theta^b_\sigma R^a_{b\mu\nu}$, where we make note that

$$\theta^\rho_a = \mathrm{diag}(e^{-A}, 1, e^{-B}) \qquad \theta^b_\sigma = \mathrm{diag}(e^A, 1, e^B)$$

So lets compute each element.

$$
\begin{aligned}
R^t_{r\mu\nu} &= \theta^t_{t'}\theta^{r'}_r R^{t'}_{r'\mu\nu} \\
&= (e^{-A})(1)(A'' + A'^2)e^A dr \wedge dt) \\
R^t_{rtr} &= -A'' - A'^2
\end{aligned}
$$

$$
\begin{aligned}
R^t_{\phi\mu\nu} &= \theta^t_{t'}\theta^{\phi'}_\phi R^{t'}_{\phi'\mu\nu} \\
&= (e^{-A})(e^B)(-A'B')e^{A+B} dt \wedge d\phi \\
R^t_{\phi t\phi} &= -e^{2B}A'B'
\end{aligned}
$$

$$
\begin{aligned}
R^r_{\phi\mu\nu} &= \theta^r_{r'}\theta^{\phi'}_\phi R^{r'}_{\phi'\mu\nu} \\
&= (1)(e^B)(-(B'' + B'^2)e^B dr \wedge d\phi) \\
R^r_{\phi r\phi} &= -e^{2B}(B'' + B'^2)
\end{aligned}
$$

Where the other symbols are related by symmetries and the metric. Lets compute the Ricci tensor. The metric and inverse metric are

$$
g_{\mu\nu} = \text{diag}(e^{2A}, -1, -e^{2B}) \qquad g^{\mu\nu} = \text{diag}(e^{-2A}, -1, -e^{-2B})
$$

So our Ricci tensor is

$$
\begin{aligned}
R_{tt} &= R^t_{ttt} + R^r_{trt} + R^\phi_{t\phi t} \\
&= 0 + g_{tt}g^{rr}R^t_{rtr} + g_{tt}g^{\phi\phi}R^t_{\phi t\phi} \\
&= e^{2A}(A'' + A'B' + A'^2)
\end{aligned}
$$

$$
\begin{aligned}
R_{rr} &= R^t_{rtr} + R^r_{rrr} + R^\phi_{r\phi r} \\
&= R^t_{rtr} + 0 + g_{rr}g^{\phi\phi}R^r_{\phi r\phi} \\
&= -(A'' + A'^2 + B'' + B'^2)
\end{aligned}
$$

$$
\begin{aligned}
R_{\phi\phi} &= R^t_{\phi t\phi} + R^r_{\phi r\phi} + R^\phi_{\phi\phi\phi} \\
&= R^t_{\phi t\phi} + R^r_{\phi r\phi} + 0 \\
&= -e^{2B}(A'B' + B'' + B'^2)
\end{aligned}
$$

And our Ricci scalar

$$
\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\phi\phi}R_{\phi\phi} \\
&= 2(A'' + A'^2 + B'' + B'^2 + A'B')
\end{aligned}
$$

Calculate Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

$$\begin{aligned}G_{tt} &= e^{2A}(A'' + A'B' + A'^2) - e^{2A}(A'' + A'^2 + B'' + B'^2 + A'B') \\ &= -e^{2A}(B'' + B'^2)\end{aligned}$$

$$\begin{aligned}G_{rr} &= -(A'' + A'^2 + B'' + B'^2) - (-1)(A'' + A'^2 + B'' + B'^2 + A'B') \\ &= A'B'\end{aligned}$$

$$\begin{aligned}G_{\phi\phi} &= -e^{2B}(A'B' + B'' + B'^2) - (-e^{2B})(A'' + A'^2 + B'' + B'^2 + A'B') \\ &= e^{2B}(A'' + A'^2)\end{aligned}$$

From the stress tensor, we know that the G_{rr} and $G_{\phi\phi}$ components must be zero, for r nonzero, this implies that $A' = 0$. We can also set A to 0. This satisfies the two lower equations. Now, the first equation is

$$G_{tt} = 8\pi G T_{tt} = 8\pi G g^{tt} g^{tt} T^{tt}$$

With $A = 0$ we have $g_{tt} = g^{tt} = 1$, so we must solve

$$-B'' - B'^2 = 8\pi m G \delta^2(\vec{r})$$

We note that the elemental area in polar coordinates is $e^B dr d\phi$, and so we can write the normalization of the delta function as

$$\int_0^\infty \int_0^{2\pi} \delta^2(\vec{r}) e^B dr d\phi = 1$$

Now, define a new variable $\Lambda = e^B$ so that $\Lambda'' = (B'' + B'^2)e^B$. The temporal part of the Einstein tensor becomes

$$\Lambda'' e^{-B} = -8\pi m G \delta^2(\vec{r})$$

Clearly from this, $\delta^2(\vec{r})e^B = -\Lambda''/8\pi m G$ so from the normalization we have

$$\begin{aligned}1 &= -\int_0^\infty \int_0^{2\pi} \frac{\Lambda''}{8\pi m G} dr d\phi \\ &= -\frac{1}{4mG}(\Lambda'(\infty) - \Lambda'(0))\end{aligned}$$

So

$$\Lambda'(\infty) = \Lambda'(0) - 4mG$$

Since Λ' is constant, we can say that $\Lambda = Dr$. Integrating not to infinity but to r yields

$$\Lambda'(r) - \Lambda'(0) = -4mG$$

So Λ' will be discontinuous across the origin. so we can take $\Lambda(r) = dr$ for $r \neq 0$ and $\Lambda'(0) = 1$ to find $\Lambda(r) = (1 - 4mG)r$. With this, we have solved the metric. It is

$$ds^2 = dt^2 - dr^2 - (1 - 4mG)^2 r^2 d\phi^2$$

5. In class I sketched the derivation of the Einstein tensor for the spherically symmetric metric

$$ds^2 = e^{2a(r)}dt^2 - [e^{2b(r)}dr^2 + r^2 d\Omega^2] \tag{4}$$

using the tetrad formalism. Complete the derivation.

Solution

This will follow the same structure as problem 2. The full metric is (for clarity)

$$ds^2 = e^{2a(r)}dt^2 - [e^{2b(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$$

Our tetrad basis here is

$$\sigma^t = e^{a(r)}dt \qquad \sigma^r = e^{b(r)}dr \qquad \sigma^\theta = r d\theta \qquad \sigma^\phi = r\sin\theta d\phi$$

The differentials of this basis are thus

$$\begin{aligned}
\mathrm{d}\sigma^{\mathrm{t}} &= a'(r)e^{a(r)}dr \wedge dt \\
\mathrm{d}\sigma^{\mathrm{r}} &= b'(r)e^{b(r)}dr \wedge dr = 0 \\
\mathrm{d}\sigma^{\theta} &= dr \wedge d\theta \\
\mathrm{d}\sigma^{\phi} &= \sin\theta dr \wedge d\phi + r\cos\theta d\theta \wedge d\phi
\end{aligned}$$

Where we note that $a'(r) = da(r)/dr$. Now as before, we need to find the spin connections, ω. The expression is $\mathrm{d}\sigma^{\mathrm{a}} = \sigma^{\mathrm{b}} \wedge \omega^{\mathrm{a}}_{\mathrm{b}}$. Writing the four equations that we get yields

$$\begin{aligned}
\mathrm{d}\sigma^t &= a'e^a dr \wedge dt = e^b dr \wedge \omega^t_r + r d\theta \wedge \omega^t_\theta + r\sin\theta d\phi \wedge \omega^t_\phi \\
\mathrm{d}\sigma^r &= 0 = e^a dt \wedge \omega^r_t + r d\theta \wedge \omega^r_\theta + r\sin\theta d\phi \wedge \omega^r_\phi \\
\mathrm{d}\sigma^\theta &= dr \wedge d\theta = e^a dt \wedge \omega^\theta_t + e^b dr \wedge \omega^\theta_r + r\sin\theta d\phi \wedge \omega^\theta_\phi \\
\mathrm{d}\sigma^\phi &= \sin\theta dr \wedge d\phi + r\cos\theta d\theta \wedge d\phi = e^a dt \wedge \omega^\phi_t + e^b dr \wedge \omega^\phi_r + r d\theta \wedge \omega^\phi_\theta
\end{aligned}$$

Now from these equations, and the fact that $\omega^a_b = -\omega^b_a$, we can deduce the six independent elements of the spin connection. From the last equation, we can easily deduce $\omega^\phi_r = e^{-b}\sin\theta d\phi$ and $\omega^\phi_\theta = \cos\theta d\phi$. From the third equation we find that $\omega^\theta_r = e^{-b}d\theta$. Now, from the first equation we can deduce $\omega^t_r = a'e^{a-b}dt$. That's it for the easy to compute elements of the spin connection. We currently have

$$\begin{aligned}
\omega^\phi_r &= e^{-b}\sin\theta d\phi \\
\omega^\phi_\theta &= \cos\theta d\phi \\
\omega^\theta_r &= e^{-b}d\theta \\
\omega^t_r &= a'e^{a-b}dt
\end{aligned}$$

We are still missing ω^t_θ and ω^t_ϕ. From the first line, ω^t_θ is either 0 or proportional to $d\theta$. From the third line, ω^t_θ is either 0 or proportional to dt. To satisfy both constraints, we must have $\omega^t_\theta = 0$. By the same reasoning, we find that $\omega^t_\phi = 0$, so our full spin connection is characterized by

$$\omega^\phi_r = e^{-b}\sin\theta d\phi$$
$$\omega^\phi_\theta = \cos\theta d\phi$$
$$\omega^\theta_r = e^{-b}d\theta$$
$$\omega^t_r = a'e^{a-b}dt$$
$$\omega^t_\theta = \omega^t_\phi = 0$$

To find the Riemann tensor, we have to now find the differentials of this. They are

$$\mathrm{d}\omega^\phi_r = -b'e^{-b}\sin\theta dr\wedge d\phi + e^{-b}\cos\theta d\theta\wedge d\phi$$
$$\mathrm{d}\omega^\phi_\theta = -\sin\theta d\theta\wedge d\phi$$
$$\mathrm{d}\omega^\theta_r = -b'e^{-b}dr\wedge d\theta$$
$$\mathrm{d}\omega^t_r = a''e^{a-b}dr\wedge dt + a'(a'-b')e^{a-b}dr\wedge dt = (a''+(a')^2-a'b')e^{a-b}dr\wedge dt$$
$$\mathrm{d}\omega^t_\theta = \mathrm{d}\omega^t_\phi = 0$$

The Riemann tensor is defined by

$$R^a_b = \mathrm{d}\omega^{\mathrm{a}}_{\mathrm{b}} + \omega^{\mathrm{a}}_{\mathrm{c}}\wedge\omega^{\mathrm{c}}_{\mathrm{b}}$$

and is antisymmetric, meaning we must compute only the six independent elements once again. Let us begin

$$\begin{aligned} R^t_r &= \mathrm{d}\omega^t_r + \omega^t_\theta\wedge\omega^\theta_r + \omega^t_\phi\wedge\omega^\phi_r \\ &= (a''+(a')^2-a'b')e^{a-b}dr\wedge dt + 0 + 0 \\ &= (a''+(a')^2-a'b')e^{a-b}dr\wedge dt \end{aligned}$$

$$\begin{aligned} R^t_\theta &= \mathrm{d}\omega^t_\theta + \omega^t_r\wedge\omega^r_\theta + \omega^t_\phi\wedge\omega^\phi_\theta \\ &= 0 + a'e^{a-b}dt\wedge(-e^{-b}d\theta) + 0 \\ &= a'e^{a-2b}d\theta\wedge dt \end{aligned}$$

$$\begin{aligned} R^t_\phi &= \mathrm{d}\omega^t_\phi + \omega^t_r\wedge\omega^r_\phi + \omega^t_\theta\wedge\omega^\theta_\phi \\ &= 0 + a'e^{a-b}dt\wedge(-e^{-b}\sin\theta d\phi) + 0 \\ &= a'\sin\theta e^{a-2b}d\phi\wedge dt \end{aligned}$$

$$\begin{aligned} R^r_\theta &= \mathrm{d}\omega^r_\theta + \omega^r_t\wedge\omega^t_\theta + \omega^r_\phi\wedge\omega^\phi_\theta \\ &= b'e^{-b}dr\wedge d\theta + 0 + (-e^{-b}\sin\theta d\phi)\wedge(\cos\theta d\phi) \\ &= b'e^{-b}dr\wedge d\theta \end{aligned}$$

$$
\begin{aligned}
R^r_\phi &= \mathrm{d}\omega^r_\phi + \omega^r_t \wedge \omega^t_\phi + \omega^r_\theta \wedge \omega^\theta_\phi \\
&= b' e^{-b} \sin\theta dr \wedge d\phi - e^{-b} \cos\theta d\theta \wedge d\phi + 0 + (-e^{-b} d\theta) \wedge (-\cos\theta d\phi) \\
&= b' e^{-b} \sin\theta dr \wedge d\phi
\end{aligned}
$$

$$
\begin{aligned}
R^\theta_\phi &= \mathrm{d}\omega^\theta_\phi + \omega^\theta_t \wedge \omega^t_\phi + \omega^\theta_r \wedge \omega^r_\phi \\
&= \sin\theta d\theta \wedge d\phi + 0 + e^{-b} d\theta \wedge (-e^{-b} \sin\theta d\phi) \\
&= \sin\theta (1 - e^{-2b}) d\theta \wedge d\phi
\end{aligned}
$$

Ok, let us list our Riemann tensor elements once more for clarity (in the tetrad basis).

$$
\begin{aligned}
R^{t'}_{r'} &= (a'' + (a')^2 - a'b') e^{a-b} dr \wedge dt \\
R^{t'}_{\theta'} &= a' e^{a-2b} d\theta \wedge dt \\
R^{t'}_{\phi'} &= a' \sin\theta e^{a-2b} d\phi \wedge dt \\
R^{r'}_{\theta'} &= b' e^{-b} dr \wedge d\theta \\
R^{r'}_{\phi'} &= b' e^{-b} \sin\theta dr \wedge d\phi \\
R^{\theta'}_{\phi'} &= \sin\theta (1 - e^{-2b}) d\theta \wedge d\phi
\end{aligned}
$$

We have primed the variables to distinguish between coordinate and tetrad basis now. We now want to switch back to the coordinate basis to compute the rest of our objects. We switch back with the usual expression

$$
\begin{aligned}
R^t_{\sigma\mu\nu} &= e^t_{t'} e^b_\sigma R^{t'}_{b\mu\nu} \\
&= e^t_{t'} e^{t'}_\sigma R^{t'}_{t'\mu\nu} + e^t_{t'} e^{r'}_\sigma R^{t'}_{r'\mu\nu} + e^t_{t'} e^{\theta'}_\sigma R^{t'}_{\theta'\mu\nu} + e^t_{t'} e^{\phi'}_\sigma R^{t'}_{\phi'\mu\nu}
\end{aligned}
$$

Where we have illustrated the formula to solve for an upper t, for clarity. Recall that in our tetrad basis, all the e's are diagonal, explicitly they are

$$
\begin{aligned}
e^{t'}_t &= e^{a(r)} & e^{r'}_r &= e^{b(r)} & e^{\theta'}_\theta &= r & e^{\phi'}_\phi &= r\sin\theta \\
e^t_{t'} &= e^{-a(r)} & e^r_{r'} &= e^{-b(r)} & e^\theta_{\theta'} &= \frac{1}{r} & e^\phi_{\phi'} &= \frac{1}{r\sin\theta}
\end{aligned}
$$

In coordinate basis, our Riemann tensor becomes

$$
\begin{aligned}
R^t_{r\mu\nu} &= e^t_{t'} e^{r'}_r R^{t'}_{r'\mu\nu} \\
&= (e^{-a})(e^b)(a'' + (a')^2 - a'b') e^{a-b} dr \wedge dt \\
R^t_{rrt} &= a'' + (a')^2 - a'b'
\end{aligned}
$$

$$
\begin{aligned}
R^t_{\theta\mu\nu} &= e^t_{t'} e^{\theta'}_\theta R^{t'}_{\theta'\mu\nu} \\
&= (e^{-a})(r)(a' e^{a-2b} d\theta \wedge dt) \\
R^t_{\theta\theta t} &= r a' e^{-2b}
\end{aligned}
$$

$$
\begin{aligned}
R^t_{\phi\mu\nu} &= e^t_{t'} e^{\phi'}_{\phi} R^{t'}_{\phi'\mu\nu} \\
&= (e^{-a})(r\sin\theta)(a'\sin\theta e^{a-2b} d\phi \wedge dt) \\
R^t_{\phi\phi t} &= r a' \sin^2\theta e^{-2b}
\end{aligned}
$$

$$
\begin{aligned}
R^r_{t\mu\nu} &= e^r_{r'} e^{t'}_{t} R^{r'}_{t'\mu\nu} \\
&= (e^{-b})(e^{a})(-(a'' + (a')^2 - a'b')e^{a-b} dr \wedge dt) \\
R^r_{trt} &= -e^{2(a-b)}(a'' + (a')^2 - a'b')
\end{aligned}
$$

$$
\begin{aligned}
R^r_{\theta\mu\nu} &= e^r_{r'} e^{\theta'}_{\theta} R^{r'}_{\theta'\mu\nu} \\
&= (e^{-b})(r)(b' e^{-b} dr \wedge d\theta) \\
R^r_{\theta r\theta} &= r b' e^{-2b}
\end{aligned}
$$

$$
\begin{aligned}
R^r_{\phi\mu\nu} &= e^r_{r'} e^{\phi'}_{\phi} R^{r'}_{\phi'\mu\nu} \\
&= (e^{-b})(r\sin\theta)(b' e^{-b} \sin\theta dr \wedge d\phi) \\
R^r_{\phi r\phi} &= r b' \sin^2\theta e^{-2b}
\end{aligned}
$$

$$
\begin{aligned}
R^\theta_{t\mu\nu} &= e^\theta_{\theta'} e^{t'}_{t} R^{\theta'}_{t'\mu\nu} \\
&= \left(\frac{1}{r}\right)(e^{a})(-a' e^{a-2b} d\theta \wedge dt) \\
R^\theta_{t\theta t} &= -\frac{1}{r} a' e^{2(a-b)}
\end{aligned}
$$

$$
\begin{aligned}
R^\theta_{r\mu\nu} &= e^\theta_{\theta'} e^{r'}_{r} R^{\theta'}_{r'\mu\nu} \\
&= \left(\frac{1}{r}\right)(e^{b})(-b' e^{-b} dr \wedge d\theta) \\
R^\theta_{rr\theta} &= -\frac{1}{r} b'
\end{aligned}
$$

$$
\begin{aligned}
R^\theta_{\phi\mu\nu} &= e^\theta_{\theta'} e^{\phi'}_{\phi} R^{\theta'}_{\phi'\mu\nu} \\
&= \left(\frac{1}{r}\right)(r\sin\theta)(\sin\theta(1 - e^{-2b}) d\theta \wedge d\phi) \\
R^\theta_{\phi\theta\phi} &= \sin^2\theta(1 - e^{-2b})
\end{aligned}
$$

$$
\begin{aligned}
R^\phi_{t\mu\nu} &= e^\phi_{\phi'} e^{t'}_{t} R^{\phi'}_{t'\mu\nu} \\
&= \left(\frac{1}{r\sin\theta}\right)(e^{a})(-a'\sin\theta e^{a-2b} d\phi \wedge dt) \\
R^\phi_{t\phi t} &= -\frac{1}{r} a' e^{2(a-b)}
\end{aligned}
$$

$$
\begin{aligned}
R^{\phi}_{r\mu\nu} &= e^{\phi}_{\phi'} e^{r'}_{r} R^{\phi'}_{r'\mu\nu} \\
&= \left(\frac{1}{r\sin\theta}\right)(e^{b})(-b' e^{-b}\sin\theta dr\wedge d\phi) \\
R^{\phi}_{rr\phi} &= -\frac{1}{r}b'
\end{aligned}
$$

$$
\begin{aligned}
R^{\phi}_{\theta\mu\nu} &= e^{\phi}_{\phi'} e^{\theta'}_{\theta} R^{\phi'}_{\theta'\mu\nu} \\
&= \left(\frac{1}{r\sin\theta}\right)(r)(-\sin\theta((1-e^{-2b})d\theta\wedge d\phi) \\
R^{\phi}_{\theta\theta\phi} &= -(1-e^{-2b})
\end{aligned}
$$

So our full coordinate Riemann tensor is thus

$$
\begin{aligned}
R^{t}_{rtr} &= -(a''+(a')^2-a'b') & R^{t}_{\theta t\theta} &= -ra'e^{-2b} & R^{t}_{\phi t\phi} &= -ra'\sin^2\theta e^{-2b} \\
R^{t}_{rrt} &= (a''+(a')^2-a'b') & R^{t}_{\theta\theta t} &= ra'e^{-2b} & R^{t}_{\phi\phi t} &= ra'\sin^2\theta e^{-2b} \\
R^{r}_{trt} &= e^{2(a-b)}(a''+(a')^2-a'b') & R^{r}_{\theta r\theta} &= rb'e^{-2b} & R^{r}_{\phi r\phi} &= rb'\sin^2\theta e^{-2b} \\
R^{r}_{ttr} &= -e^{2(a-b)}(a''+(a')^2-a'b') & R^{r}_{\theta\theta r} &= -rb'e^{-2b} & R^{r}_{\phi\phi r} &= -rb'\sin^2\theta e^{-2b} \\
R^{\theta}_{t\theta t} &= -\frac{1}{r}a'e^{2(a-b)} & R^{\theta}_{r\theta r} &= \frac{1}{r}b' & R^{\theta}_{\phi\theta\phi} &= \sin^2\theta(1-e^{-2b}) \\
R^{\theta}_{tt\theta} &= \frac{1}{r}a'e^{2(a-b)} & R^{\theta}_{rr\theta} &= -\frac{1}{r}b' & R^{\theta}_{\phi\phi\theta} &= -\sin^2\theta(1-e^{-2b}) \\
R^{\phi}_{t\phi t} &= -\frac{1}{r}a'e^{2(a-b)} & R^{\phi}_{r\phi r} &= \frac{1}{r}b' & R^{\phi}_{\theta\phi\theta} &= (1-e^{-2b}) \\
R^{\phi}_{tt\phi} &= \frac{1}{r}a'e^{2(a-b)} & R^{\phi}_{rr\phi} &= -\frac{1}{r}b' & R^{\phi}_{\theta\theta\phi} &= -(1-e^{-2b})
\end{aligned}
$$

The Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ so we need to compute the Ricci tensor and scalar. The Ricci tensor is $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$. Lets do so

$$
\begin{aligned}
R_{tt} &= R^{t}_{ttt} + R^{r}_{trt} + R^{\theta}_{t\theta t} + R^{\phi}_{t\phi t} \\
&= 0 + e^{2(a-b)}(a''+(a')^2-a'b') - \frac{1}{r}a'e^{2(a-b)} - \frac{1}{r}a'e^{2(a-b)} \\
&= e^{2(a-b)}\left(a''+(a')^2-a'b'+\frac{2a'}{r}\right)
\end{aligned}
$$

$$
\begin{aligned}
R_{rr} &= R^{t}_{rtr} + R^{r}_{rrr} + R^{\theta}_{r\theta r} + R^{\phi}_{r\phi r} \\
&= -(a''+(a')^2-a'b') + 0 + \frac{1}{r}b' + \frac{1}{r}b' \\
&= -a''-(a')^2+a'b'+\frac{2b'}{r}
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\theta} &= R^{t}_{\theta t\theta} + R^{r}_{\theta r\theta} + R^{\theta}_{\theta\theta\theta} + R^{\phi}_{\theta\phi\theta} \\
&= -ra'e^{-2b} + rb'e^{-2b} + 0 + 1 - e^{-2b} \\
&= 1 + e^{-2b}(r(b'-a')-1)
\end{aligned}
$$

$$
\begin{aligned}
R_{\phi\phi} &= R^t_{\phi t\phi} + R^r_{\phi r\phi} + R^\theta_{\phi\theta\phi} + R^\phi_{\phi\phi\phi} \\
&= -ra'\sin^2\theta e^{-2b} + rb'\sin^2\theta e^{-2b} + \sin^2\theta(1-e^{-2b}) \\
&= \sin^2\theta(1+e^{-2b}(r(b'-a')-1) = \sin^2\theta R_{\theta\theta}
\end{aligned}
$$

Finally, the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$

$$
\begin{aligned}
R &= g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\
&= e^{-2a}e^{2(a-b)}\left(a'' + (a')^2 - a'b' + \frac{2a'}{r}\right) - e^{-2b}\left(-a'' - (a')^2 + a'b' + \frac{2b'}{r}\right) \\
&- \frac{1}{r^2}(1+e^{-2b}(r(b'-a')-1)) - \frac{1}{r^2\sin^2\theta}(\sin^2\theta(1+e^{-2b}(r(b'-a')-1)) \\
&= 2e^{-2b}\left(a'' + (a')^2 - a'b' + \frac{2(a'-b')}{r} + \frac{1-e^{2b}}{r^2}\right)
\end{aligned}
$$

Since $R_{\mu\nu} = 0$ outside the spherically symmetric object, we can solve for the coefficients in the same way as page 196 of Carroll. This yields

$$
e^{2a} = 1 - \frac{R_s}{r} \qquad e^{2b} = \left(1 - \frac{R_s}{r}\right)^{-1}
$$

Where R_s comes from an undetermined constant of integration, and is interpreted as the Schwarzschild radius.

The Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Since we know (and used) the fact that $R_{\mu\nu} = 0$, a straightforward computation in mathematica shows that $R = 0$ as well. The Einstein tensor is thus

$$
G_{\mu\nu} = 0
$$

As expected for the vacuum outside a spherically symmetric source.

6. Derive the Einstein tensor for the metric of the above problem, this time using the coordinate approach. After solving this problem you should be convinced that it is easier to use the tetrad formalism.

Solution

For this problem, we will just compute the Christoffel symbols, showing that they match those given in equation (5.12) of Carroll, and leave the rest as an exercise, as we have gone from symbols to Riemann tensor elements many times in previous problems during the course. The line elements is

$$ds^2 = e^{2A}dt^2 - e^{2B}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

$$g_{tt} = e^{2A} \qquad g_{rr} = -e^{2B} \qquad g_{\theta\theta} = -r^2 \qquad g_{\phi\phi} = -r^2\sin^2\theta$$

The Christoffel symbols are defined by

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

As usual, lets start with an upper t index.

$$\Gamma^{t}_{\mu\nu} = \frac{1}{2}g^{tt}(\partial_\mu g_{\nu t} + \partial_\nu g_{t\mu} - \partial_t g_{\mu\nu})$$

The final term is 0 since there is no time dependence on the metric. In this cas we can take $\mu = r$ and $\nu = t$ (or vice versa) to find

$$\begin{aligned}\Gamma^{t}_{rt} = \Gamma^{t}_{tr} &= \frac{1}{2}e^{-2A}(2A'e^{2A}) \\ &= A'\end{aligned}$$

That's all for this upper index. Next lets look at upper r

$$\Gamma^{r}_{\mu\nu} = \frac{1}{2}g^{rr}(\partial_\mu g_{\nu r} + \partial_\nu g_{r\mu} - \partial_r g_{\mu\nu})$$

There are many options here. For the first two terms, to be nonzero we can set $\mu = \nu = r$ to find

$$\Gamma^{r}_{rr} = \frac{1}{2}g^{rr}\partial_r g_{rr} = \frac{1}{2}(-e^{-2B})(-2B'e^{2B}) = B'$$

Now we get a term for each $\mu = \nu = t, \theta, \phi$ (we already did the case where it was r). In each case, only the last term contributes and the symbols are $\Gamma^{r}_{\mu\mu} = -\frac{1}{2}g^{rr}\partial_r g_{\mu\mu} = \frac{e^{-2B}}{2}\partial_r g_{\mu\mu}$

$$\begin{aligned}\Gamma^{r}_{tt} &= A'e^{2(A-B)} \\ \Gamma^{r}_{\theta\theta} &= -re^{-2B} \\ \Gamma^{r}_{\phi\phi} &= -r\sin^2\theta e^{-2B}\end{aligned}$$

Now lets consider an upper θ. This gives us the equation

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2} g^{\theta\theta} (\partial_\mu g_{\nu\theta} + \partial_\nu g_{\theta\mu} - \partial_\theta g_{\mu\nu})$$

The first two terms are symmetric, and require either $\mu = r$, $\nu = \theta$, or vice versa. This yields

$$\begin{aligned}\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} &= \frac{1}{2} g^{\theta\theta} \partial_r g_{\theta\theta} = -\frac{1}{2} r^{-2}(-2r) \\ &= \frac{1}{r}\end{aligned}$$

For the last term, we require that $\mu = \nu = \phi$ to be nonzero, and so we find

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} r^{-2}(-2r^2 \sin\theta\cos\theta) = -\sin\theta\cos\theta$$

Now lets move onto the last terms, with upper index ϕ

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2} g^{\phi\phi} (\partial_\mu g_{\nu\phi} + \partial_\nu g_{\phi\mu} - \partial_\phi g_{\mu\nu})$$

The last term is always 0. The first two terms are nonzero for $\mu = r$, $\nu = \phi$ AND $\mu = \theta$, $\nu = \phi$ (and vice versa). This leads to the symbols

$$\begin{aligned}\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} &= \frac{1}{2} g^{\phi\phi} \partial_r g_{\phi\phi} = \frac{1}{2}(-r^{-2}\sin^{-2}\theta)(-2r\sin^2\theta) \\ &= \frac{1}{r} \\ \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \frac{1}{2} g^{\phi} \partial_\theta g_{\phi\phi} = \frac{1}{2}(-r^{-2}\sin^{-2}\theta)(-2r^2\sin\theta\cos\theta) \\ &= \cot\theta\end{aligned}$$

To recap, our symbols were

$$\begin{aligned}\Gamma^{t}_{rt} = \Gamma^{t}_{tr} &= A' \\ \Gamma^{r}_{rr} &= B' \\ \Gamma^{r}_{tt} &= A' e^{2(A-B)} \\ \Gamma^{r}_{\theta\theta} &= -re^{-B} \\ \Gamma^{r}_{\phi\phi} &= -r\sin^2\theta e^{-2B} \\ \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} &= \frac{1}{r} \\ \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} &= \frac{1}{r} \\ \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \cot\theta\end{aligned}$$

These 9 independent symbols match (5.12) of Carroll. From here it is straightforward (and very tedious) to go to the Riemann tensor, and eventually to the Ricci tensor and scalar. If you have any specific questions on those steps, feel free to email me at bryce.cyr@mail.mcgill.ca and I can help you out, but I won't be continuing the full derivation here.