# 5. Quantum Theory of Cosmological Perturbations

## a) Review

s-t sketch

inflation

$t_f(k)$

$t_i(k)$

$t > t_f(k)$ Newtonian theory

$t_i(k) < t < t_f(k)$ classical GR

$t < t_i(k)$ quantum generation

$$ds^2 = a^2[(1+2\phi)d\eta^2 - (1-2\phi)d\underline{x}^2] \qquad \phi(\underline{x}, t) \quad a(t)$$

$$\varphi = \varphi_0(\eta) + \delta\varphi(\eta, \underline{x})$$

$$\phi'' + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi' - \nabla^2\phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\varphi_0''}{\varphi_0'}\right)\phi = 0$$

$$\mathcal{H} = \frac{a'}{a} \qquad ' = \frac{\partial}{\partial\eta}$$

Sub-Hubble : oscillations (damped)

Super-Hubble : growing

## b) Action

$$S = \int d^4x \sqrt{-g}\left[-\frac{1}{16\pi G}R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]$$

Idea : * expand about classical FRWL background

$\varphi_0(\eta), a(\eta)$

* linearize EoM

$\Leftrightarrow$ expand action to quadratic order in $\phi, \delta\varphi$

* include constraints

$\phi$ & $\delta\varphi$ dependent $\to$ single DoF

* action must be action of a free scalar field

with time dependent mass

$$S = S_0(a, \varphi_0) + S_2(\phi, \delta\varphi; a, \varphi_0)$$

$$\boxed{S_2 = \frac{1}{2}\int d^4x \left[ v'^2 - v_{,i} v_{,i} + \frac{z''}{z} v^2 \right]}$$

$v$ some combination of $\phi$ and $\delta\varphi$

$z$ some fct of background

relativistic $\Rightarrow c_s^2 = 1$

$$v = a\left[\delta\varphi + \frac{\varphi_0'}{\mathcal{H}}\phi\right]$$

$$z = \frac{a\varphi_0'}{\mathcal{H}}$$

$$v'' - \nabla^2 v - \frac{z''}{z} v = 0$$

(Veq) $$\boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0}$$

slow rolling: $3H\dot{\varphi}_0 = -V'$

$3H^2\dot{\varphi}_0 = -V'H$

$\parallel$

$3m_{pl}^{-2} V \dot{\varphi}_0$

$$\left.\right\} \dot{\varphi}_0 = -\underbrace{\frac{V'}{V}}_{\sim \text{const}} m_{pl}^2 H$$

$\Rightarrow \varphi_0' \sim \mathcal{H}$

$\Rightarrow z \sim a$ if EoS constant in time

$\Rightarrow \frac{z''}{z} \sim \mathcal{H}^2 \, O(1)$

depends on equation of state

oscillations of $v$ on sub-Hubble

squeezing of $v$ on super-Hubble $v \sim z$

Interpretation:

$v \sim$ curv. pert in comoving gauge $(\delta\varphi = 0)$

$$v_k = z\zeta_k + O(k^2)$$

$$\zeta = \phi + \frac{2}{3}\frac{\mathcal{H}^{-1}\phi' + \phi}{1+w}$$

late times: $\zeta = O(1)\,\phi$ $\quad z \sim a$

$\Rightarrow \phi \sim a^{-1} v$

c) Quantization

Note: This is an improvement over scalar QFT in curved space-time

scalar QFT in curved space-time
e.g. R. Wald
Birrell & Davies
} violates constraint eq, wrong dynamics

quantum theory of cosm. pert. — consistent with constr. eq, correct dynamics

QFT in Minkowski ($m=0$)

$$\hat{v}(\underline{x},t) = \int d^3k \left[ V_k^*(\eta) e^{ikx} (2\pi)^{-3/2} a_k + V_k(\eta) e^{-ikx} (2\pi)^{-3/2} a_k^+ \right]$$

$a_k$: annihilation, $a_k^+$: creation

$$v(\underline{x},t) \to \hat{v}(\underline{x},t)$$

$$\left.\begin{aligned} [a_{\underline{k}}, a_{\underline{k}'}] &= [a_{\underline{k}}^+, a_{\underline{k}'}^+] = 0 \\ [a_{\underline{k}}, a_{\underline{k}'}^+] &= \delta^3(\underline{k}-\underline{k}') \end{aligned}\right\} \text{CCR}$$

$V_k(\eta) \sim e^{ik\eta}$ $\quad k = |\underline{k}|$ $\quad$ positive frequency

vacuum: $a_k |0\rangle = 0 \quad \forall \underline{k}$

QFT in Curved Space-time

* no unique space-time splitting
  → no unique $|0\rangle$
* modes not pure positive/negative frequency
  → no unique $|0\rangle$

QFT in FRWL

* unique space-time splitting given by FRWL background
* modes not pure positive/negative frequency
  → no unique $|0\rangle$

$V_k(\eta) \sim e^{ik\eta}$ $\quad$ sub-Hubble

$V_k(\eta)$ not oscillatory $\quad$ super-Hubble pick time $\eta_0$

$v_k(\eta) = v_k^{+(0)}(\eta)$ looks positive freq. at $\eta_0$

pick time $\eta_1 > \eta_0$

$v_k^{(0)}(\eta)$ not pure positive freq (super-Hubble)

$v_k^{(1)}(\eta)$ pure pos. freq. at $\eta_1$ $\alpha_k, \beta_k$ <u>Bogoliubov coeff</u>

$$v_k^{(1)+} = \alpha_k v_k^{(0)+} + \beta_k v_k^{(0)}$$

$$\hat{v} = \int d^3k \left[ v_k^{-(0)}(\eta) e^{ikx} (2\pi)^{-3/2} a_k + v_k^{+(0)}(\eta) e^{-ikx} (2\pi)^{-3/2} a_k^+ \right]$$

$$= \int \quad v_k^{-(1)}(\eta)\, b_k \qquad v_k^{+(1)}(\eta)\, b_k^+$$

$|0\rangle_{\eta_0}$ : $a_{\underline{k}} |0\rangle_{\eta_0} = 0 \quad \forall \underline{k}$

$|0\rangle_{\eta_1}$ $b_{\underline{k}} |0\rangle_{\eta_1} = 0 \quad \forall \underline{k}$

$|0\rangle_{\eta_0}$ does <u>not</u> look like vacuum at $\eta_1$

$$\langle 0_{\eta_0} | \hat{N}_k | 0_{\eta_0} \rangle = |\beta_{\underline{k}}|^2$$

<u>Parker particle production</u> L. Parker 1968

sub-Hubble : $v_k$ oscillates $\rightarrow \beta_k = 0$

super-Hubble : $v_k \sim z \nearrow \rightarrow \beta_k \neq 0$

$\beta_k \sim z$

$N_k \sim z^2$

$|0\rangle_{\eta_0}$ <u>squeezed vacuum state</u> (super-Hubble)
pure
not thermal

$$v_k(\eta_0) = \frac{1}{\sqrt{2\omega_k}} \qquad v_k'(\eta_0) = \frac{\sqrt{\omega_k}}{\sqrt{2}}$$

sketch s-t inflation

vacuum IC

squeezing

$$v_k(\eta_H(k)) = \frac{1}{\sqrt{2\omega_k}} \qquad v_k'(\eta_H(k)) = \sqrt{\omega_k/2}$$

d) Application to Inflation

$$\mathcal{R} = z^{-1} v = \phi + \delta\varphi \frac{H}{\varphi_0'}$$

$$\underline{P_{\mathcal{R}}(k,t)} = k^3 \mathcal{R}_k^2(t) = k^3 z^{-2}(t)\,|v_k(t)|^2$$

$$= k^3 z^{-2}(t) \left(\frac{z(t)}{z(t_H(k))}\right)^2 \underbrace{|v_k(t_H(k))|^2}_{|v_k(t_i)|^2 = \frac{1}{2\omega_k}}$$

$$= \underbrace{k^3 a^{-2}(t_H(k))}_{\left(\frac{H}{k}\right)^2} \underbrace{\left(\frac{a}{z}(t_H(k))\right)^2}_{\frac{1}{(\dot\varphi_0/H)^2} \;\; \varepsilon m_{pl}^2} \frac{1}{2\underbrace{\omega_k}_{k}} = \underline{\frac{1}{\varepsilon}\left(\frac{H}{m_{pl}}\right)^2}$$

e) Quantum Theory of Gravitational Waves

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + h_{ij})\,dx^i dx^j\right]$$

$$h_{ij}(\underline{x},\eta) = h_+(\eta,x)\,e^+_{ij} + h_\times(\eta,x)\,e^\times_{ij}$$

$$S^{(2)} = \int d^4x \, \frac{a^2}{2}\left[h'^2 - (\nabla h)^2\right]$$

$$h_k'' + 2\frac{a'}{a} h_k' + k^2 h_k = 0$$

$$u_k = a\, h_k\, m_{pl}$$

(u eq) $$\boxed{u_k'' + \left(k^2 - \frac{a''}{a}\right) u_k = 0}$$

same as (v eq) except $z \to a$

sub-Hubble oscillations

super-Hubble squeezing

$$u_k(\eta_H(k)) = \frac{1}{\sqrt{2k}} \qquad u_k'(\eta_H(k)) = \sqrt{k/2}$$

(sketch)

$$\underline{P_h(k,t)} = k^3 |h(k,t)|^2 = k^3 a^{-2}(t) |u(k,t)|^2 m_{pl}^{-2}$$

$$= k^3 a^{-2}(t) \left(\frac{a(t)}{a(t_H(k))}\right)^2 |u(k,t_H)|^2 m_{pl}^{-2}$$

$$= k^3 \underbrace{a^{-2}(t_H(k))^2}_{\left(\frac{H}{k}\right)^2} (2k)^{-1} m_{pl}^{-2} = \underline{\left(\frac{H}{m_{pl}}\right)^2}$$

tensor to scalar ratio

$$r = \frac{P_h}{P_R} \sim \varepsilon .$$